Chapter 4

One-Dimensional Elements

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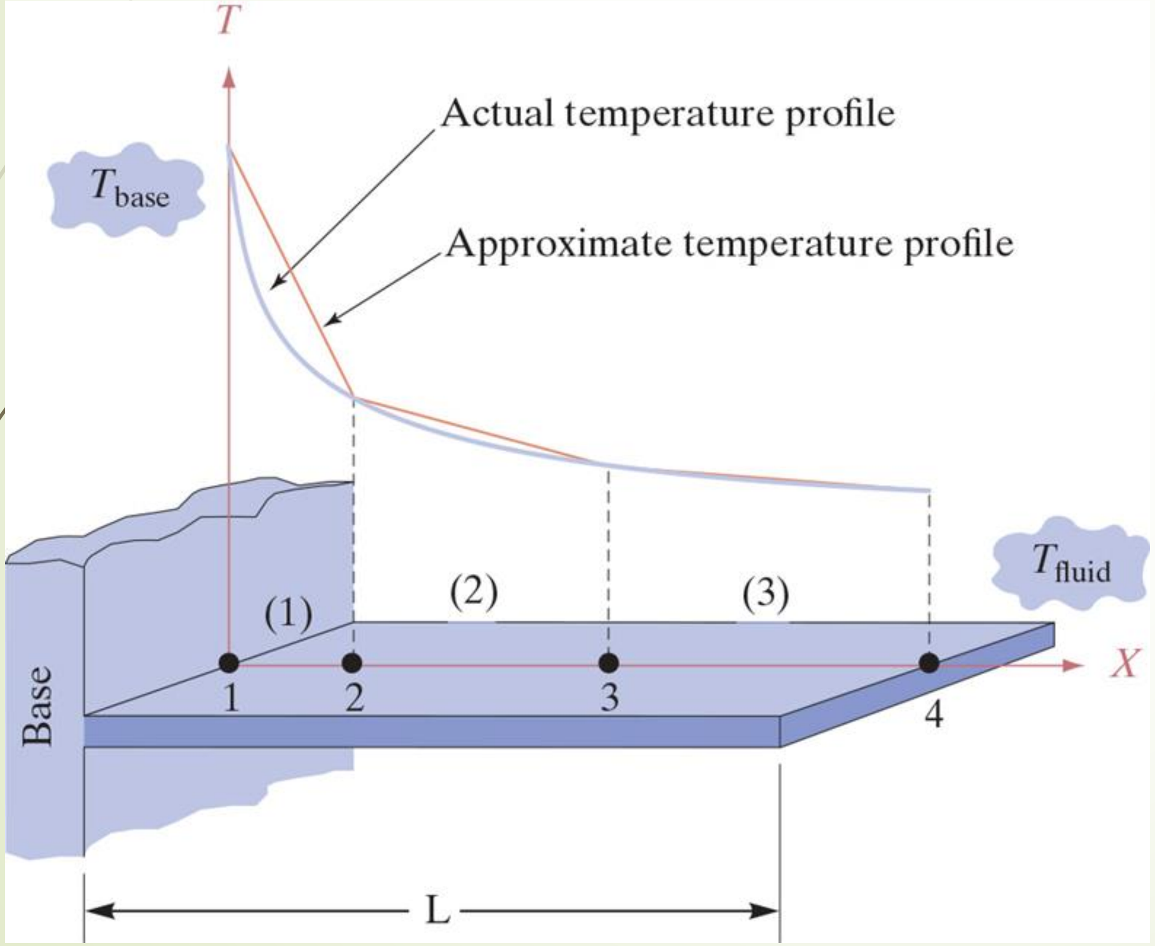
Objectives

- The objectives of this chapter are to introduce the concepts of one dimensional elements and shape functions and their properties. The idea of local and nature coordinate systems will also be presented.

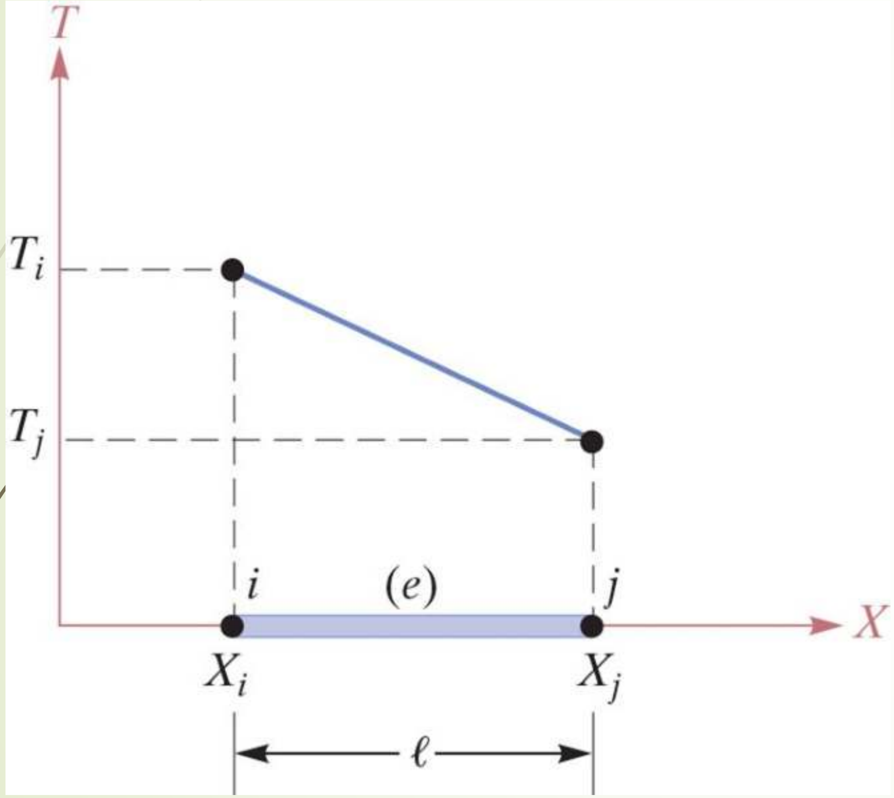
Temperature distribution for a fin of uniform cross section



The temperature distribution along the element may be interpolated or approximated using a linear function us depected in the figure. The linear temperature distribution for a typical element may be expressed as:

$$T^e = c_1 + c_2 X$$

Linear approximation of temperature distribution for an element.



$$c_1 = \frac{T_i X_j - T_j X_i}{X_j - X_i}$$

$$c_2 = \frac{T_j - T_i}{X_j - X_i}$$

Solving for the unknowns c_1 and c_2

$$c_1 = \frac{T_i X_j - T_j X_i}{X_j - X_i}$$

$$c_2 = \frac{T_j - T_i}{X_j - X_i}$$

The element`s temperature distribution in terms of its nodal values is:

$$T^e = \frac{T_i X_j - T_j X_i}{X_j - X_i} + \frac{T_j - T_i}{X_j - X_i} X$$

- Grouping the T_i terms together and T_j terms together, we obtain:

$$T^e = \left(\frac{X_j - X}{X_j - X_i}\right)T_i + \left(\frac{X_j - X}{X_j - X_i}\right)T_j$$

- We now define the shape function S_i and S_j according to equations :

$$S_i = \frac{X_j - X}{X_j - X_i} = \frac{X_j - X}{l}$$

$$S_j = \frac{X - X_i}{X_j - X_i} = \frac{X - X_i}{l}$$

- L is the length of the element so:
- $$T^e = S_i T_i + S_j T_j$$
- $$T^e = \begin{bmatrix} S_i & S_j \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \end{Bmatrix}$$ for a structural example $$u^e = \begin{bmatrix} S_i & S_j \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

Properties of shape functions

- The shape functions possess unique properties that are important for us understand because they simplify the evaluation of certain integrals when we are deriving the conductance or stiffness matrices. One of the inherent properties of a shape function is that it has a value of unity at its corresponding node and has a value of zero at the other adjacent node.
- **Demonstration**

$$S_i\Big|_{X=X_i} = \frac{X_j - X}{l}\Big|_{X=X_i} = \frac{X_j - X_i}{l} = 1$$

$$S_i\Big|_{X=X_j} = \frac{X_j - X}{l}\Big|_{X=X_j} = \frac{X_j - X_j}{l} = 0$$

- Also evaluating S_j at $X = X_i$ and $X = X_j$

$$S_j\Big|_{X=X_i} = \frac{X - X_i}{l}\Big|_{X=X_i} = \frac{X_i - X_i}{l} = 0$$

$$S_j\Big|_{X=X_j} = \frac{X - X_i}{l}\Big|_{X=X_j} = \frac{X_j - X_i}{l} = 1$$

- Where u_i and u_j represent the deflections of nodes i and j of an arbitrary element (e) it should be clear by now that we can represent the spatial variation of any unknown variable over a given element by using shape functions and the corresponding nodal values. Thus, in general, we can write:

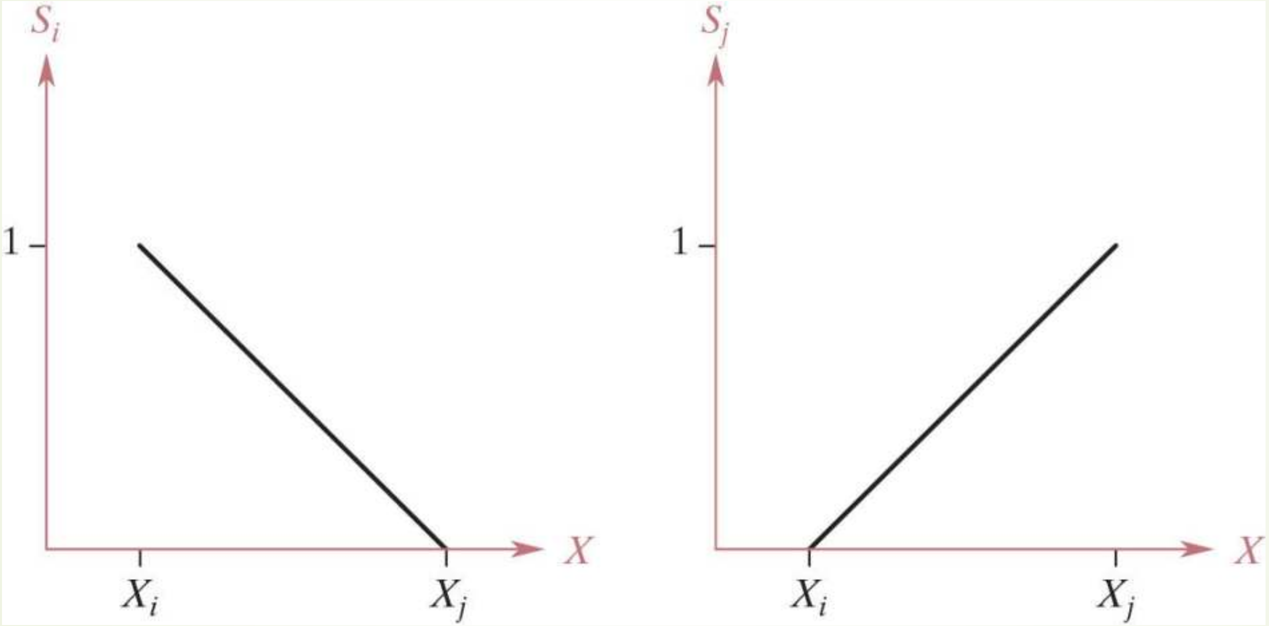
$$\psi^e = \begin{bmatrix} S_i & S_j \end{bmatrix} \begin{Bmatrix} \psi_i \\ \psi_j \end{Bmatrix}$$

- *Where* ψ_i and ψ_j *represent the nodal values of the unknown variable, such the temperature, or deflection, or velocity, ..*

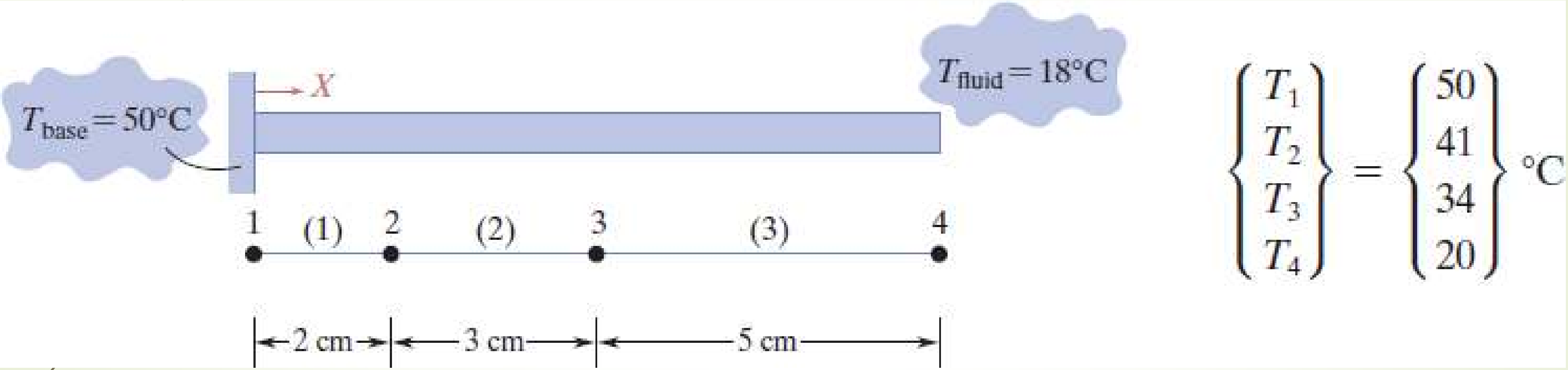
- It can also be readily shown that for linear shape functions, the sum of the derivatives with respect to X is zero. That is,

$$\frac{d}{dX}\left(\frac{X_j - X}{X_j - X_i}\right) + \frac{d}{dX}\left(\frac{X - X_i}{X_j - X_i}\right) = -\frac{1}{X_j - X_i} + \frac{1}{X_j - X_i} = 0$$

Linear shape functions.



Example1 :The nodal temperatures and their corresponding positions along the fin in example 5.1.



We have used linear one dimensional elements to approximate the temperature distribution along a fin. the nodal temperatures and their corresponding positions are shown in the figure Bellow. What is the temperature of the fin at X=4cm and X=8cm.

a. The temperature of the fin at $X = 4$ cm is represented by element (2);

$$T^{(2)} = S_2^{(2)} T_2 + S_3^{(2)} T_3 = \frac{X_3 - X}{\ell} T_2 + \frac{X - X_2}{\ell} T_3$$

$$T = \frac{5 - 4}{3}(41) + \frac{4 - 2}{3}(34) = 36.3°\text{C}$$

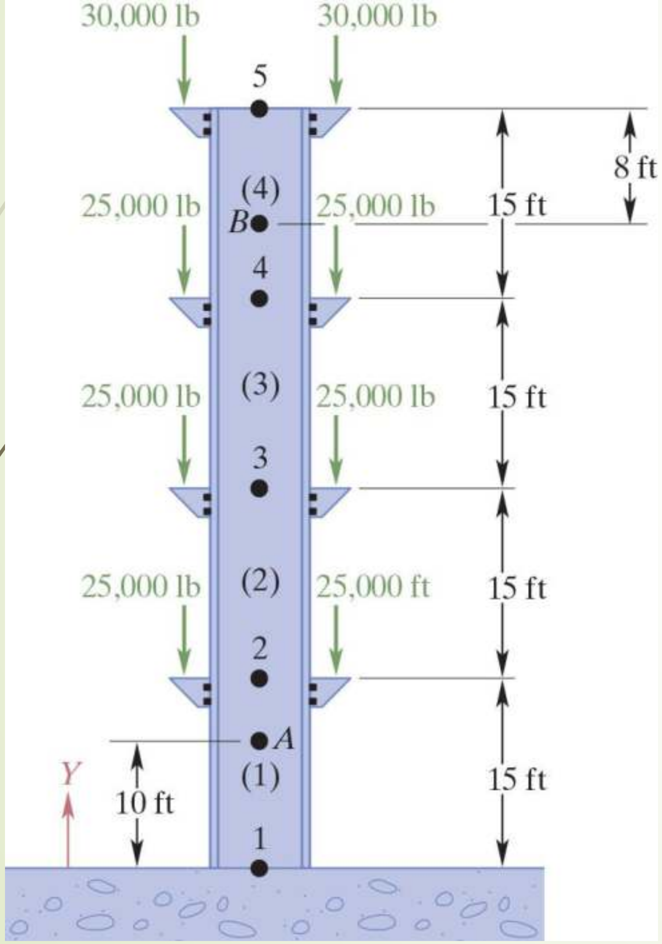
b. The temperature of the fin at $X = 8$ cm is represented by element (3);

$$T^{(3)} = S_3^{(3)} T_3 + S_4^{(3)} T_4 = \frac{X_4 - X}{\ell} T_3 + \frac{X - X_3}{\ell} T_4$$

$$T = \frac{10 - 8}{5}(34) + \frac{8 - 5}{5}(20) = 25.6°\text{C}$$

For this example, note the difference between $S_3^{(2)}$ and $S_3^{(3)}$.

Example2

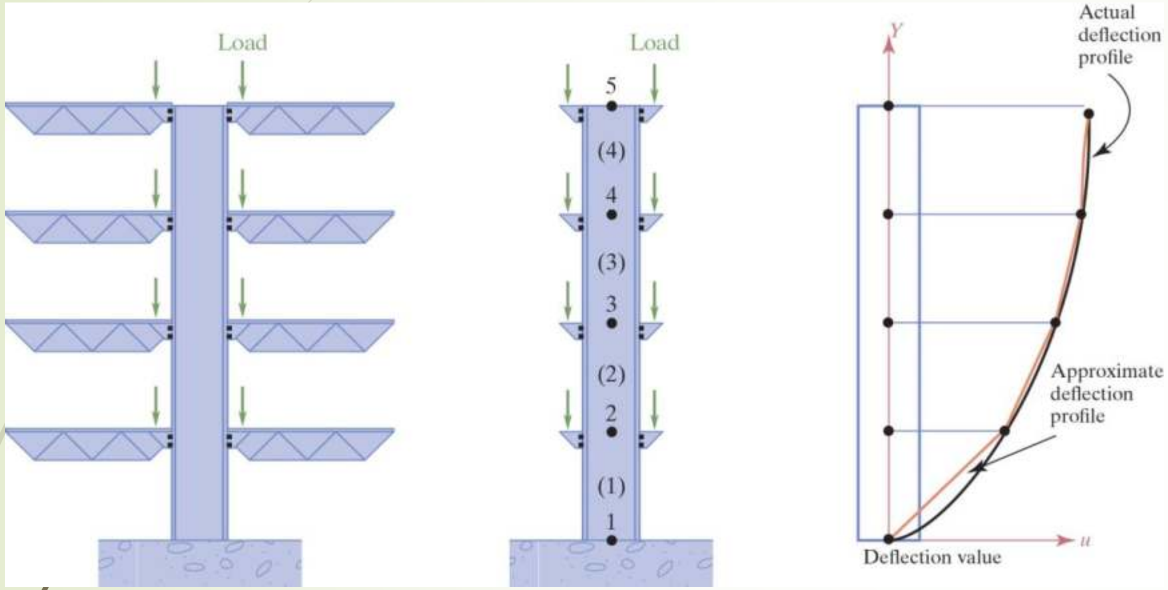


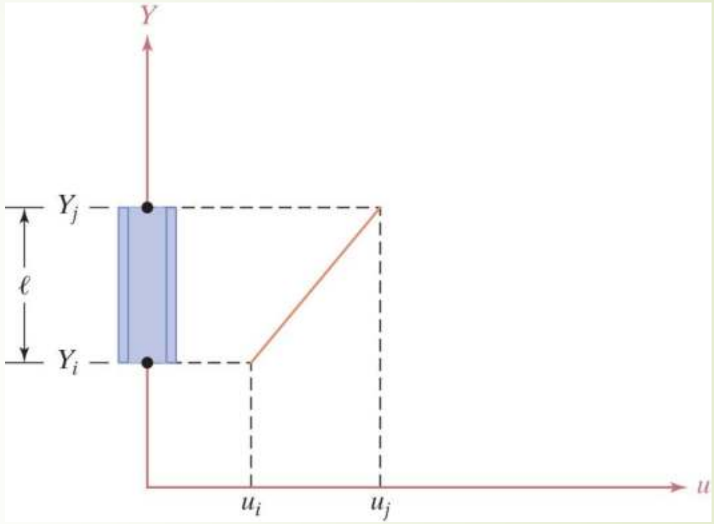
Consider a four-story building with steel columns. One column is subjected to the loading shown in Figure 3.6. Under axial loading assumption and using linear elements, the vertical displacements of the column at various floor–column connection points were determined to be

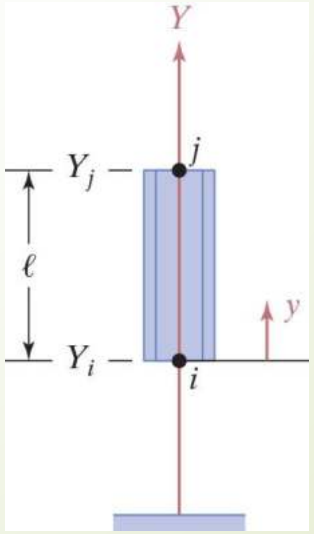
$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.03283 \\ 0.05784 \\ 0.07504 \\ 0.08442 \end{Bmatrix} \text{ in.}$$

The modulus of elasticity of $E = 29 \times 10^6$ lb/in^2, and area of $A = 39.7$ in^2 were used in the calculations. A detailed analysis of this problem is given in Chapter 4. For now, given

Figure 5.1 Deflection of a steel column subject to floor loading.







the nodal displacement values, we are interested in determining the deflections of points A and B.

(a) Using the global coordinate Y, the displacement of point A is represented by element (1):

$$u^{(1)} = S_1^{(1)} u_1 + S_2^{(1)} u_2 = \frac{Y_2 - Y}{\ell} u_1 + \frac{Y - Y_1}{\ell} u_2$$

$$u = \frac{15 - 10}{15}(0) + \frac{10 - 0}{15}(0.03283) = 0.02188 \text{ in.}$$

(b) The displacement of point B is represented by element (4):

$$u^{(4)} = S_4^{(4)} u_4 + S_5^{(4)} u_5 = \frac{Y_5 - Y}{\ell} u_4 + \frac{Y - Y_4}{\ell} u_5$$

$$u = \frac{60 - 52}{15}(0.07504) + \frac{52 - 45}{15}(0.08442) = 0.07941 \text{ in.}$$

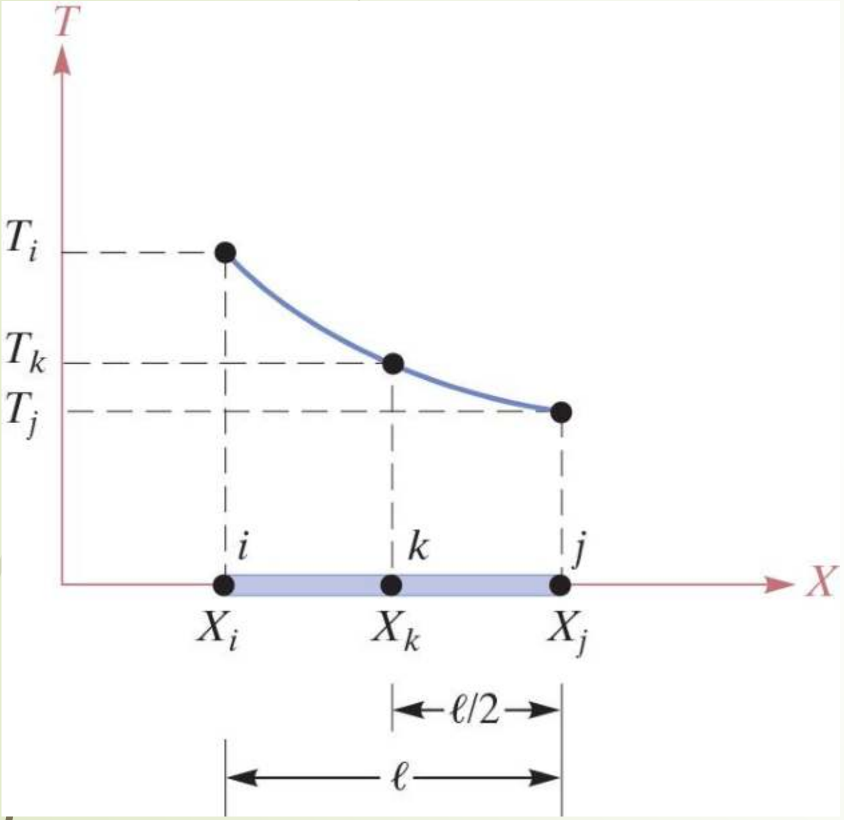
QUADRATIC ELEMENTS

We can increase the accuracy of our finite element findings either by increasing the number of linear elements used in the analysis or by using higher order interpolation functions. For example, we can employ a quadratic function to represent the spatial variation of an unknown variable. Using a quadratic function instead of a linear function requires

that we use three nodes to define an element. We need three nodes to define an element because in order to fit a quadratic function, we need three points. The third point can be created by placing a node, such as node *k*, in the middle of an element, as shown in Figure 5.5. Referring to the previous example of a fin, using quadratic approximation, the temperature distribution for a typical element can be represented by

$$T^{(e)} = c_1 + c_2X + c_3X^2 \tag{5.18}$$

Quadratic approximation of the temperature distribution for an element.



and the nodal values are

$$T = T_i \quad \text{at} \quad X = X_i \qquad (5.19)$$
$$T = T_k \quad \text{at} \quad X = X_k$$
$$T = T_j \quad \text{at} \quad X = X_j$$

Three equations and three unknowns are created upon substitution of the nodal values into Eq. (5.18):

$$T_i = c_1 + c_2 X_i + c_3 X_i^2 \qquad (5.20)$$
$$T_k = c_1 + c_2 X_k + c_3 X_k^2$$
$$T_j = c_1 + c_2 X_j + c_3 X_j^2$$

Solving for c_1, c_2, and c_3 and rearranging terms leads to the element's temperature distribution in terms of the nodal values and the shape functions:

$$T^{(e)} = S_i T_i + S_j T_j + S_k T_k \qquad (5.21)$$

In matrix form, the above expression is

$$T^{(e)} = [S_i \quad S_j \quad S_k] \begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix} \qquad (5.22)$$

where the shape functions are

$$S_i = \frac{2}{\ell^2}(X - X_j)(X - X_k) \tag{5.23}$$

$$S_j = \frac{2}{\ell^2}(X - X_i)(X - X_k)$$

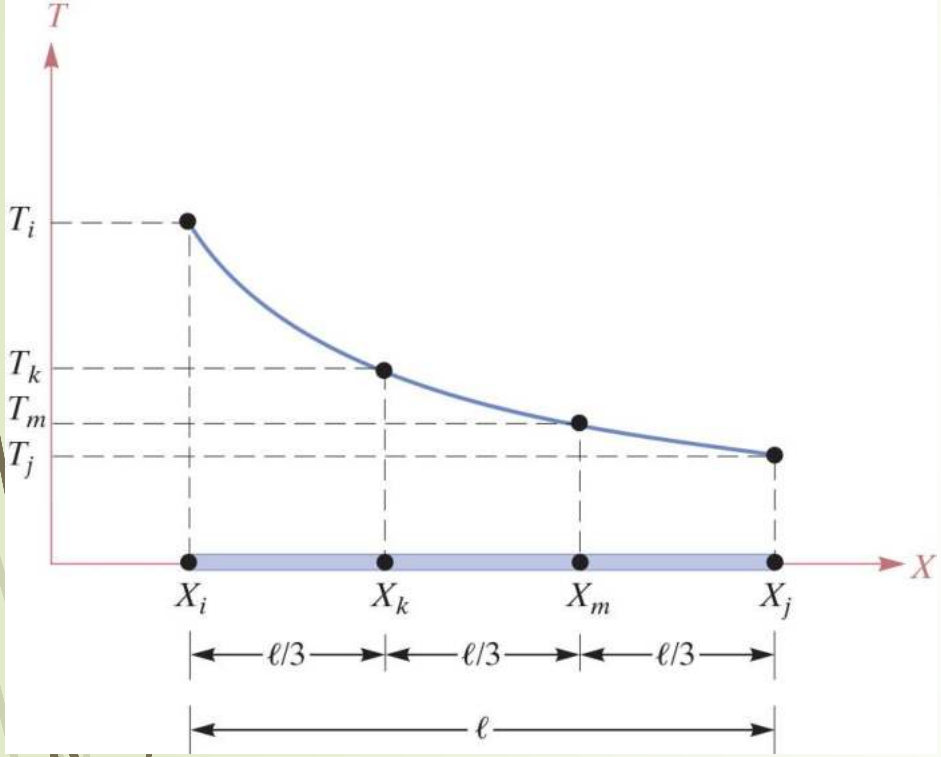
$$S_k = \frac{-4}{\ell^2}(X - X_i)(X - X_j)$$

In general, for a given element the variation of any parameter Ψ in terms of its nodal values may be written as

$$\Psi^{(e)} = [S_i \quad S_j \quad S_k]\begin{Bmatrix} \Psi_i \\ \Psi_j \\ \Psi_k \end{Bmatrix} \tag{5.24}$$

It is important to note here that the quadratic shape functions possess properties similar to those of the linear shape functions; that is, (1) a shape function has a value of unity at its corresponding node and a value of zero at the other adjacent node, and (2) if we sum up the shape functions, we will again come up with a value of unity. The main difference between linear shape functions and quadratic shape functions is in their derivatives. The derivatives of the quadratic shape functions with respect to X are not constant.

5.3 CUBIC ELEMENTS



The quadratic interpolation functions offer good results in finite element formulations . However, if additional accuracy is needed, we can resort to even higher order interpolation functions, such as third order polynomials. Thus, we can use cubic functions to represent the spatial variation of a given variable. Utilizing a cubic function instead of a quadratic function requires that we use four nodes to define an element. We need four nodes to define an element because in order to fit a third order polynomial, we need four points. The element is divided into three equal lengths. The placement of the four nodes is depicted in the figure.

Referring to the previous example of a fin, using cubic approximation, the temperature distribution for a typical element can be represented by:

$$T^e = c_1 + c_2 X + c_3 X^2 + c_4 X^3$$

and the nodal values are

$$\begin{aligned} T &= T_i \quad \text{at} \quad X = X_i \\ T &= T_k \quad \text{at} \quad X = X_k \\ T &= T_m \quad \text{at} \quad X = X_m \\ T &= T_j \quad \text{at} \quad X = X_j \end{aligned} \tag{5.26}$$

Four equations and four unknowns are created upon substitution of the nodal values into Eq. (5.25). Solving for c_1, c_2, c_3, and c_4 and rearranging terms leads to the element's temperature distribution in terms of the nodal values and the shape functions:

$$T^{(e)} = S_i T_i + S_j T_j + S_k T_k + S_m T_m \tag{5.27}$$

In matrix form, the above expression is

$$T^{(e)} = [S_i \quad S_j \quad S_k \quad S_m] \begin{Bmatrix} T_i \\ T_j \\ T_k \\ T_m \end{Bmatrix} \tag{5.28}$$

where the shape functions are

$$S_i = -\frac{9}{2\ell^3}(X - X_j)(X - X_k)(X - X_m) \qquad (5.29)$$

$$S_j = \frac{9}{2\ell^3}(X - X_i)(X - X_k)(X - X_m)$$

$$S_k = \frac{27}{2\ell^3}(X - X_i)(X - X_j)(X - X_m)$$

$$S_m = -\frac{27}{2\ell^3}(X - X_i)(X - X_j)(X - X_k)$$

It is worth noting that when the order of the interpolating function increases, it is necessary to employ **Lagrange interpolation functions** instead of taking the above approach to obtain the shape functions. The main advantage the Lagrange method offers is that using it, we do not have to solve a set of equations simultaneously to obtain the unknown coefficients of the interpolating function. Instead, we represent the shape functions in terms of the products of three linear functions. For cubic interpolating

functions, the shape function associated with each node can be represented in terms of the product of three linear functions. For a given node—for example, *i*—we select the functions such that their product will produce a value of zero at other nodes—namely, *j*, *k*, and *m*—and a value of unity at the given node, *i*. Moreover, the product of the functions must produce linear and nonlinear terms similar to the ones given by a general third-order polynomial function.

To demonstrate this method, let us consider node *i*, with the global coordinate X_i. First, the functions must be selected such that when evaluated at nodes *j*, *k*, and *m*, the outcome is a value of zero. We select

$$S_i = a_1(X - X_j)(X - X_k)(X - X_m) \tag{5.30}$$

which satisfies the above condition. That is, if you substitute for $X = X_j$, or $X = X_k$, or $X = X_m$, the value of S_i is zero. We then evaluate a_1 such that when the shape function S_i is evaluated at node $i(X = X_i)$, it will produce a value of unity:

$$1 = a_1(X_i - X_j)(X_i - X_k)(X_i - X_m) = a_1(-\ell)\left(-\frac{\ell}{3}\right)\left(-\frac{2\ell}{3}\right)$$

Solving for a_1, we get

$$a_1 = -\frac{9}{2\ell^3}$$

and substituting into Eq. (5.30), we have

$$S_i = -\frac{9}{2\ell^3}(X - X_j)(X - X_k)(X - X_m)$$

The other shape functions are obtained in a similar fashion. Keeping in mind the explanation offered above, we can generate shape functions of an $(N - 1)$-order polynomial directly from the Lagrange polynomial formula:

$$S_K = \prod_{M=1}^{N} \frac{X - X_M \text{ omitting } (X - X_K)}{X_K - X_M \text{ omitting } (X_K - X_K)} = \frac{(X - X_1)(X - X_2)\cdots(X - X_N)}{(X_K - X_1)(X_K - X_2)\cdots(X_K - X_N)} \qquad (5.31)$$

Note that in order to accommodate any order polynomial representation in Eq. (5.31) numeral values are assigned to the nodes and the subscripts of the shape functions.

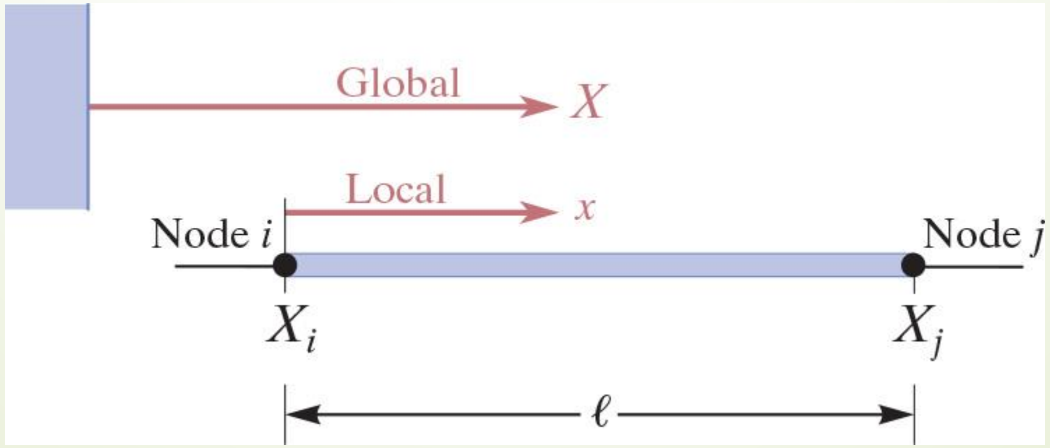
In general, using a cubic interpolation function, the variation of any parameter Ψ in terms of its nodal values may be written as

$$\Psi^{(e)} = [S_i \quad S_j \quad S_k \quad S_m] \begin{Bmatrix} \Psi_i \\ \Psi_j \\ \Psi_k \\ \Psi_m \end{Bmatrix}$$

Once again, note that the cubic shape functions possess properties similar to those of the linear and the quadratic shape functions; that is, (1) a shape function has a value of unity at its corresponding node and a value of zero at the other adjacent node, and (2) if we sum up the shape functions, we will come up with a value of unity. However, note that taking the spatial derivative of cubic shape functions will produce quadratic results.

5.4 GLOBAL, LOCAL, AND NATURAL COORDINATES

Most often, in finite element modeling, it is convenient to use several frames of reference, as we briefly discussed in Chapters 3 and 4. We need a global coordinate system to represent the location of each node, orientation of each element, and to apply boundary conditions and loads (in terms of their respective global components). Moreover, the solution, such as nodal displacements, is generally represented with respect to the global directions. On the other hand, we need to employ local and natural coordinates because they offer certain advantages when we construct the geometry or compute integrals. The advantage becomes apparent particularly when the integrals contain products of shape functions. For one-dimensional elements, the relationship between a global coordinate X and a local coordinate x is given by $X = X_i + x$, as shown in Figure 5.7.



The relationship between a global coordinate *X* and a local coordinate *x*.

Substituting for X in terms of the local coordinate x in Eqs. (5.8) and (5.9), we get

$$S_i = \frac{X_j - X}{\ell} = \frac{X_j - (X_i + x)}{\ell} = 1 - \frac{x}{\ell} \qquad (5.32)$$

$$S_j = \frac{X - X_i}{\ell} = \frac{(X_i + x) - X_i}{\ell} = \frac{x}{\ell} \qquad (5.33)$$

where the local coordinate x varies from 0 to ℓ; that is $0 \le x \le \ell$.

One-Dimensional Linear Natural Coordinates

Natural coordinates are basically local coordinates in a dimensionless form. It is often necessary to use numerical methods to evaluate integrals for the purpose of calculating elemental stiffness or conductance matrices. Natural coordinates offer the convenience of having −1 and 1 for the limits of integration. For example, if we let

$$\xi = \frac{2x}{\ell} - 1$$

where x is the local coordinate, then we can specify the coordinates of node i as −1 and node j by 1. This relationship is shown in Figure 5.8.

We can obtain the natural linear shape functions by substituting for x in terms of ξ into Eqs. (5.32) and (5.33). This substitution yields

$$S_i = \frac{1}{2}(1 - \xi) \qquad (5.34)$$

$$S_j = \frac{1}{2}(1 + \xi) \qquad (5.35)$$

Natural linear shape functions possess the same properties as linear shape functions; that is, a shape function has a value of unity at its corresponding node and has a value of

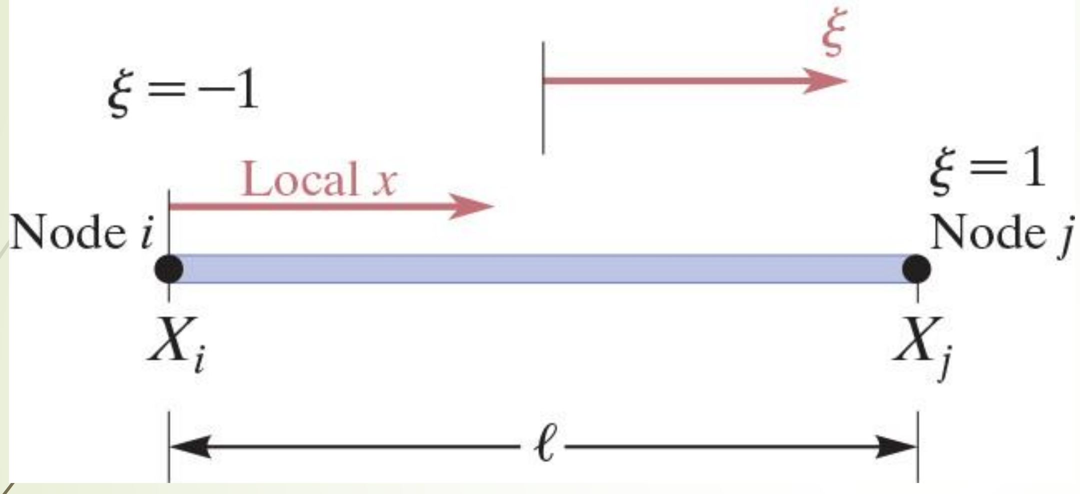


Figure 5.8 The relationship between the local coordinate *x* and the natural coordinate ξ.

zero at the adjacent node in a given element. As an example, the temperature distribution over an element of a one-dimensional fin may expressed by

$$T^{(e)} = S_i T_i + S_j T_j = \frac{1}{2}(1 - \xi)T_i + \frac{1}{2}(1 + \xi)T_j \qquad (5.36)$$

It is clear that at $\xi = -1$, $T = T_i$ and at $\xi = 1$, $T = T_j$.

EXAMPLE 5.3

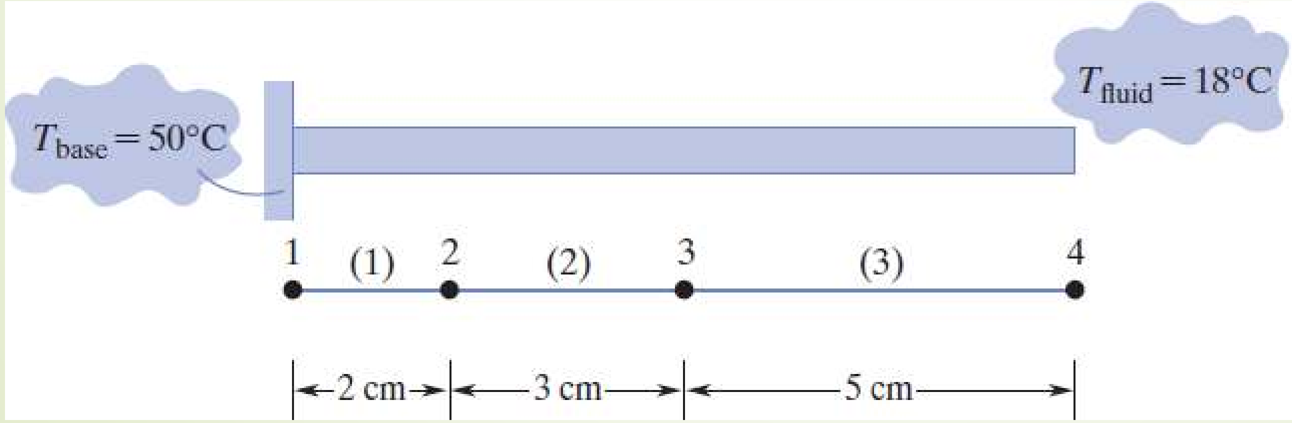
Determine the temperature of the fin in Example 5.1 at the global location $X = 8$ cm using local coordinates. Also determine the temperature of the fin at the global location $X = 7.5$ cm using natural coordinates.

a. Using local coordinates, we find that the temperature of the fin at $X = 8$ cm is represented by element (3) according to the equation

$$T^{(3)} = S_3^{(3)}T_3 + S_4^{(3)}T_4 = \left(1 - \frac{x}{\ell}\right)T_3 + \frac{x}{\ell}T_4$$

Note that element (3) has a length of 5 cm, and the location of a point 8 cm from the base is represented by the local coordinate $x = 3$:

$$T = \left(1 - \frac{3}{5}\right)(34) + \frac{3}{5}(20) = 25.6°\text{C}$$



$$\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 50 \\ 41 \\ 34 \\ 20 \end{Bmatrix} °\text{C}$$

b. Using natural coordinates, we find that the temperature of the fin at $X = 7.5$ cm is represented by element (3) according to the equation

$$T^{(3)} = S_3^{(3)}T_3 + S_4^{(3)}T_4 = \frac{1}{2}(1 - \xi)T_3 + \frac{1}{2}(1 + \xi)T_4$$

Because the point with the global coordinate $X = 7.5$ cm is located in the middle of element (3), the natural coordinate of this point is given by $\xi = 0$:

$$T^{(3)} = \frac{1}{2}(1 - 0)(34) + \frac{1}{2}(1 + 0)(20) = 27°\text{C}$$

One-Dimensional Natural Quadratic and Cubic Shape Functions

The natural one-dimensional quadratic and cubic shape functions can be obtained in a way similar to the method discussed in the previous section. The quadratic natural shape functions are

$$S_i = -\frac{1}{2}\xi(1 - \xi) \tag{5.37}$$

$$S_j = \frac{1}{2}\xi(1 + \xi) \tag{5.38}$$

$$S_k = (1 + \xi)(1 - \xi) \tag{5.39}$$

The natural one-dimensional cubic shape functions are

$$S_i = \frac{1}{16}(1 - \xi)(3\xi + 1)(3\xi - 1) \tag{5.40}$$

$$S_j = \frac{1}{16}(1 + \xi)(3\xi + 1)(3\xi - 1) \tag{5.41}$$

$$S_k = \frac{9}{16}(1 + \xi)(\xi - 1)(3\xi - 1) \tag{5.42}$$

$$S_m = \frac{9}{16}(1 + \xi)(1 - \xi)(3\xi + 1) \tag{5.43}$$

EXAMPLE 5.4

Evaluate the integral $\int_{X_i}^{X_j} S_j^2 dX$ using (a) global coordinates and (b) local coordinates.

a. Using global coordinates, we obtain

$$\int_{X_i}^{X_j} S_j^2 dX = \int_{X_i}^{X_j} \left(\frac{X - X_i}{\ell} \right)^2 dX = \frac{1}{3\ell^2}(X - X_i)^3 \Big|_{X_i}^{X_j} = \frac{\ell}{3}$$

b. Using local coordinates, we obtain

$$\int_{X_i}^{X_j} S_j^2 dX = \int_0^{\ell} \left(\frac{x}{\ell} \right)^2 dx = \frac{x^3}{3\ell^2} \Big|_0^{\ell} = \frac{\ell}{3}$$

This simple example demonstrates that local coordinates offer a simple way to evaluate integrals containing products of shape functions.

Table 5.1 One-dimensional shape functions (1 of 2).

Interpolation function	In terms of global coordinate X $X_i \leq X \leq X_j$	In terms of local coordinate x $0 \leq x \leq \ell$	In terms of natural coordinate ξ $-1 \leq \xi \leq 1$
Linear	$S_i = \dfrac{X_j - X}{\ell}$	$S_i = 1 - \dfrac{x}{\ell}$	$S_i = \dfrac{1}{2}(1 - \xi)$
	$S_j = \dfrac{X - X_i}{\ell}$	$S_j = \dfrac{x}{\ell}$	$S_j = \dfrac{1}{2}(1 + \xi)$
Quadratic	$S_i = \dfrac{2}{\ell^2}(X - X_j)(X - X_k)$	$S_i = \left(\dfrac{x}{\ell} - 1\right)\left(2\left(\dfrac{x}{\ell}\right) - 1\right)$	$S_i = -\dfrac{1}{2}\xi(1 - \xi)$
	$S_j = \dfrac{2}{\ell^2}(X - X_i)(X - X_k)$	$S_j = \left(\dfrac{x}{\ell}\right)\left(2\left(\dfrac{x}{\ell}\right) - 1\right)$	$S_j = \dfrac{1}{2}\xi(1 + \xi)$
	$S_k = \dfrac{-4}{\ell^2}(X - X_i)(X - X_j)$	$S_k = 4\left(\dfrac{x}{\ell}\right)\left(1 - \left(\dfrac{x}{\ell}\right)\right)$	$S_k = (1 - \xi)(1 + \xi)$

Interpolation function	In terms of global coordinate X $X_i \le X \le X_j$	In terms of local coordinate x $0 \le x \le \ell$	In terms of natural coordinate ξ $-1 \le \xi \le 1$
Cubic	$S_i = -\frac{9}{2\ell^3}(X - X_j)(X - X_k)(X - X_m)$	$S_i = \frac{1}{2}\left(1 - \frac{x}{\ell}\right)\left(2 - 3\left(\frac{x}{\ell}\right)\right)\left(1 - 3\left(\frac{x}{\ell}\right)\right)$	$S_i = \frac{1}{16}(1 - \xi)(3\xi + 1)(3\xi - 1)$
	$S_j = \frac{9}{2\ell^3}(X - X_i)(X - X_k)(X - X_m)$	$S_j = \frac{1}{2}\left(\frac{x}{\ell}\right)\left(2 - 3\left(\frac{x}{\ell}\right)\right)\left(1 - 3\left(\frac{x}{\ell}\right)\right)$	$S_j = \frac{1}{16}(1 + \xi)(3\xi + 1)(3\xi - 1)$
	$S_k = \frac{27}{2\ell^3}(X - X_i)(X - X_j)(X - X_m)$	$S_k = \frac{9}{2}\left(\frac{x}{\ell}\right)\left(2 - 3\left(\frac{x}{\ell}\right)\right)\left(1 - \left(\frac{x}{\ell}\right)\right)$	$S_k = \frac{9}{16}(1 + \xi)(\xi - 1)(3\xi - 1)$
	$S_m = -\frac{27}{2\ell^3}(X - X_i)(X - X_j)(X - X_k)$	$S_m = \frac{9}{2}\left(\frac{x}{\ell}\right)\left(3\left(\frac{x}{\ell}\right) - 1\right)\left(1 - \left(\frac{x}{\ell}\right)\right)$	$S_m = \frac{9}{16}(1 + \xi)(1 - \xi)(3\xi + 1)$

5.6 NUMERICAL INTEGRATION: GAUSS–LEGENDRE QUADRATURE

As we discussed earlier, natural coordinates are basically local coordinates in a dimensionless form. Moreover, most finite element programs perform element numerical integration by Gaussian quadratures, and as the limit of integration, they use an interval from -1 to 1. This approach is taken because when the function being integrated is known, the Gauss–Legendre formulae offer a more efficient way of evaluating an integral as compared to other numerical integration methods such as the trapezoidal method. Whereas the trapezoidal method or Simpson's method can be used to evaluate integrals dealing with discrete data (see Problem 24), the Gauss–Legendre method is based on the evaluation of a known function at nonuniformally spaced points to compute the integral. The two-point Gauss–Legendre formula is developed next in this section. The basic goal behind the Gauss–Legendre formulae is to represent an integral in terms of the sum of the product of certain weighting coefficients and the value of the function at some selected points. So, we begin with

$$I = \int_a^b f(x)dx = \sum_{i=1}^{n} w_i f(x_i) \tag{5.44}$$

Next, we must ask (1) How do we determine the value of the weighting coefficients, represented by the w_i's? (2) Where do we evaluate the function, or in other words, how do we select these points (x_i)? We begin by changing the limits of integration from a to b to -1 to 1 with the introduction of the variable λ such that

$$x = c_0 + c_1\lambda$$

Matching the limits, we get

$$a = c_0 + c_1(-1)$$
$$b = c_0 + c_1(1)$$

and solving for c_0 and c_1, we have

$$c_0 = \frac{(b + a)}{2}$$

and

$$c_1 = \frac{(b - a)}{2}$$

Therefore,

$$x = \frac{(b + a)}{2} + \frac{(b - a)}{2}\lambda \qquad (5.45)$$

and

$$dx = \frac{(b - a)}{2}d\lambda \qquad (5.46)$$

Thus, using Eqs. (5.45) and (5.46), we find that any integral in the form of Eq. (5.44) can be expressed in terms of an integral with its limits at −1 and 1:

$$I = \int_{-1}^{1} f(\lambda)\, d\lambda = \sum_{i=1}^{n} w_i f(\lambda_i) \tag{5.47}$$

The two-point Gauss–Legendre formulation requires the determination of two weighting factors w_1 and w_2 and two sampling points λ_1 and λ_2 to evaluate the function at these points. Because there are four unknowns, four equations are created using Legendre polynomials $(1, \lambda, \lambda^2, \lambda^3)$ as follows:

$$w_1 f(\lambda_1) + w_2 f(\lambda_2) = \int_{-1}^{1} 1\, d\lambda = 2$$

$$w_1 f(\lambda_1) + w_2 f(\lambda_2) = \int_{-1}^{1} \lambda\, d\lambda = 0$$

$$w_1 f(\lambda_1) + w_2 f(\lambda_2) = \int_{-1}^{1} \lambda^2\, d\lambda = \frac{2}{3}$$

$$w_1 f(\lambda_1) + w_2 f(\lambda_2) = \int_{-1}^{1} \lambda^3\, d\lambda = 0$$

The above equations lead to the equations

$$w_1(1) + w_2(1) = 2$$
$$w_1(\lambda_1) + w_2(\lambda_2) = 0$$

$$w_1(\lambda_1)^2 + w_2(\lambda_2)^2 = \frac{2}{3}$$
$$w_1(\lambda_1)^3 + w_2(\lambda_2)^3 = 0$$

Solving for w_1, w_2, λ_1, and λ_2, we have $w_1 = w_2 = 1, \lambda_1 = -0.577350269$, and $\lambda_2 = 0.577350269$. The weighting factors and the 2, 3, 4, and 5 sampling points for Gauss–Legendre formulae are given in Table 5.2. Note that as the number of sampling points increases, so does the accuracy of the calculations. As you will see in Chapter 7, we can readily extend the Gauss–Legendre quadrature formulation to two- or three dimensional problems.

TABLE 5.2 Weighting factors and sampling points for Gauss–Legendre formulae

Points (n)	Weighting factors (w_i)	Sampling points (λ_i)
2	$w_1 = 1.00000000$	$\lambda_1 = -0.577350269$
	$w_2 = 1.00000000$	$\lambda_2 = 0.577350269$
3	$w_1 = 0.55555556$	$\lambda_1 = -0.774596669$
	$w_2 = 0.88888889$	$\lambda_2 = 0$
	$w_3 = 0.55555556$	$\lambda_3 = 0.774596669$
4	$w_1 = 0.3478548$	$\lambda_1 = -0.861136312$
	$w_2 = 0.6521452$	$\lambda_2 = -0.339981044$
	$w_3 = 0.6521452$	$\lambda_3 = 0.339981044$
	$w_4 = 0.3478548$	$\lambda_4 = 0.861136312$
5	$w_1 = 0.2369269$	$\lambda_1 = -0.906179846$
	$w_2 = 0.4786287$	$\lambda_2 = -0.538469310$
	$w_3 = 0.5688889$	$\lambda_3 = 0$
	$w_4 = 0.4786287$	$\lambda_4 = 0.538469310$
	$w_5 = 0.2369269$	$\lambda_5 = 0.906179846$

TABLE 5.1 One-dimensional shape functions

Interpolation function	In terms of global coordinate X $X_i \le X \le X_j$	In terms of local coordinate x $0 \le x \le \ell$	In terms of natural coordinate ξ $-1 \le \xi \le 1$
Linear	$S_i = \dfrac{X_j - X}{\ell}$	$S_i = 1 - \dfrac{x}{\ell}$	$S_i = \dfrac{1}{2}(1 - \xi)$
	$S_j = \dfrac{X - X_i}{\ell}$	$S_j = \dfrac{x}{\ell}$	$S_j = \dfrac{1}{2}(1 + \xi)$
Quadratic	$S_i = \dfrac{2}{\ell^2}(X - X_j)(X - X_k)$	$S_i = \left(\dfrac{x}{\ell} - 1\right)\left(2\left(\dfrac{x}{\ell}\right) - 1\right)$	$S_i = -\dfrac{1}{2}\xi(1 - \xi)$
	$S_j = \dfrac{2}{\ell^2}(X - X_i)(X - X_k)$	$S_j = \left(\dfrac{x}{\ell}\right)\left(2\left(\dfrac{x}{\ell}\right) - 1\right)$	$S_j = \dfrac{1}{2}\xi(1 + \xi)$
	$S_k = \dfrac{-4}{\ell^2}(X - X_i)(X - X_j)$	$S_k = 4\left(\dfrac{x}{\ell}\right)\left(1 - \left(\dfrac{x}{\ell}\right)\right)$	$S_k = (1 - \xi)(1 + \xi)$
Cubic	$S_i = -\dfrac{9}{2\ell^3}(X - X_j)(X - X_k)(X - X_m)$	$S_i = \dfrac{1}{2}\left(1 - \dfrac{x}{\ell}\right)\left(2 - 3\left(\dfrac{x}{\ell}\right)\right)\left(1 - 3\left(\dfrac{x}{\ell}\right)\right)$	$S_i = \dfrac{1}{16}(1 - \xi)(3\xi + 1)(3\xi - 1)$
	$S_j = \dfrac{9}{2\ell^3}(X - X_i)(X - X_k)(X - X_m)$	$S_j = \dfrac{1}{2}\left(\dfrac{x}{\ell}\right)\left(2 - 3\left(\dfrac{x}{\ell}\right)\right)\left(1 - 3\left(\dfrac{x}{\ell}\right)\right)$	$S_j = \dfrac{1}{16}(1 + \xi)(3\xi + 1)(3\xi - 1)$
	$S_k = \dfrac{27}{2\ell^3}(X - X_i)(X - X_j)(X - X_m)$	$S_k = \dfrac{9}{2}\left(\dfrac{x}{\ell}\right)\left(2 - 3\left(\dfrac{x}{\ell}\right)\right)\left(1 - \left(\dfrac{x}{\ell}\right)\right)$	$S_k = \dfrac{9}{16}(1 + \xi)(\xi - 1)(3\xi - 1)$
	$S_m = -\dfrac{27}{2\ell^3}(X - X_i)(X - X_j)(X - X_k)$	$S_m = -\dfrac{9}{2}\left(\dfrac{x}{\ell}\right)\left(3\left(\dfrac{x}{\ell}\right) - 1\right)\left(1 - \left(\dfrac{x}{\ell}\right)\right)$	$S_m = \dfrac{9}{16}(1 + \xi)(1 - \xi)(3\xi + 1)$

EXAMPLE 5.5

Evaluate the integral $I = \int_2^6 (x^2 + 5x + 3)\, dx$ using the Gauss–Legendre two-point sampling formula.

This integral is simple and can be evaluated analytically, leading to the solution $I = 161.333333333$. The purpose of this example is to demonstrate the Gauss–Legendre procedure. We begin by changing the variable x to λ by using Eq. (5.45). So, we obtain

$$x = \frac{(b + a)}{2} + \frac{(b - a)}{2}\lambda = \frac{(6 + 2)}{2} + \frac{(6 - 2)}{2}\lambda = 4 + 2\lambda$$

and

$$dx = \frac{(b - a)}{2}\, d\lambda = \frac{(6 - 2)}{2}\, d\lambda = 2\, d\lambda$$

Thus, the integral I can be expressed in terms of λ:

$$I = \int_2^6 \overbrace{(x^2 + 5x + 3)}^{f(x)} dx = \int_{-1}^{1} \overbrace{(2)[(4 + 2\lambda)^2 + 5(4 + 2\lambda) + 3)]}^{f(\lambda)} d\lambda$$

Using the Gauss–Legendre two-point formula and Table 5.2, we compute the value of the integral I from

$$I \cong w_1 f(\lambda_1) + w_2 f(\lambda_2)$$

From Table 5.2, we find that $w_1 = w_2 = 1$, and evaluating $f(\lambda)$ at $\lambda_1 = -0.577350269$ and $\lambda_2 = 0.577350269$, we obtain

$$f(\lambda_1) = (2)[[4 + 2(-0.577350269)]^2 + 5(4 + 2(-0.577350269) + 3)] = 50.6444526769$$

$$f(\lambda_2) = (2)[[4 + 2(0.577350269)]^2 + 5(4 + 2(0.577350269) + 3)] = 110.688880653$$

$$I = (1)(50.6444526769) + (1)110.688880653 = 161.33333333$$

EXAMPLE 5.6

Evaluate the integral $\int_{X_i}^{X_j} S_j^2 dX$ in Example 5.4 using the Gauss–Legendre two-point formula.

Recall from Eq. (5.35) that $S_j = \frac{1}{2}(1 + \xi)$ and by differentiating the relationship between the local coordinate x and the natural coordinate ξ (i.e., $\xi = \frac{2x}{\ell} - 1 \Rightarrow d\xi = \frac{2}{\ell} dx$) we find $dx = \frac{\ell}{2} d\xi$. Also note that for this problem, $\xi = \lambda$. So,

$$I = \int_{X_i}^{X_j} S_j^2 dX = \int_{X_i}^{X_j} \left(\frac{X - X_i}{\ell}\right)^2 dX = \int_0^{\ell} \left(\frac{x}{\ell}\right)^2 dx = \frac{\ell}{2} \int_{-1}^{1} \left[\frac{1}{2}(1 + \xi)\right]^2 d\xi$$

Using the Gauss–Legendre two-point formula and Table 5.2, we compute the value of the integral I from

$$I \cong w_1 f(\lambda_1) + w_2 f(\lambda_2)$$

From Table 5.2, we find that $w_1 = w_2 = 1$, and evaluating $f(\lambda)$ at $\lambda_1 = -0.577350269$ and $\lambda_2 = 0.577350269$, we obtain

$$f(\xi_1) = \frac{\ell}{2}\left[\frac{1}{2}(1 + \xi_1)\right]^2 = \frac{\ell}{2}\left[\frac{1}{2}(1 - 0.577350269)\right]^2 = 0.022329099389\ell$$

$$f(\xi_2) = \frac{\ell}{2}\left[\frac{1}{2}(1 + \xi_2)\right]^2 = \frac{\ell}{2}\left[\frac{1}{2}(1 + 0.577350269)\right]^2 = 0.31100423389\ell$$

$$I = (1)(0.022329099389\ell) + (1)(0.31100423389\ell) = 0.333333333\ell$$

Note that the above result is identical to the results of Example 5.4.