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Department of Electrical Engineering
Electromagnetics (2) - (8022320)
Term 1: 2021/2022
Solution Homework 10

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Do not submit this homework. There will be a quiz from this homework on Wednesday, 17 Nov 2021.

Topics covered in this week:

- Voltages and currents on transmission line:

$$V_s(z) = V_0^+[e^{-\gamma z} + \Gamma e^{\gamma z}]$$
$$I_s(z) = \frac{V_0^+}{Z_0}[e^{-\gamma z} - \Gamma e^{\gamma z}]$$

- Max and min voltage and current on the line:

$$V_{max} = V_0^+[1 + |\Gamma|] \qquad V_{min} = V_0^+[1 - |\Gamma|]$$
$$I_{max} = \frac{V_0^+}{Z_0}[1 + |\Gamma|] \qquad I_{min} = \frac{V_0^+}{Z_0}[1 - |\Gamma|]$$

- Impedance on transmission line:

$$Z(l) = \frac{Z_L + Z_0 \tanh \gamma l}{Z_0 + Z_L \tanh \gamma l} Z_0 \qquad \text{if loss-less} \qquad Z(l) = \frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l} Z_0$$

- Max and min impedance on the line:

$$Z_{max} = \frac{V_{max}}{I_{min}} = Z_0 \frac{1+|\Gamma|}{1-|\Gamma|} = Z_0(SWR) \qquad Z_{min} = \frac{V_{min}}{I_{max}} = Z_0 \frac{1-|\Gamma|}{1+|\Gamma|} = \frac{Z_0}{SWR}$$

- Characteristic impedance of loss less line:

$$Z_0 = \sqrt{Z_{short}\, Z_{open}}$$

- Smith chart:
  - Marking normalized impedance on the Smith chart
  - Finding reflection co-efficient
  - Finding standing wave ratio (SWR)

Q1. A transmission line with characteristic impedance $Z_0 = 100\,\Omega$ has a load of $Z_L = 50\,\Omega$. Load voltage is 50 volt. Calculate

a. Maximum and minimum voltage on the line
b. Location of voltage maxima and minima on the line
c. The maximum and minimum impedance on the line. Where do these occure?
d. Power transmitted to the load.

Solution:

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{50-100}{50+100} = \frac{-50}{150} = -\frac{1}{3} = 0.3333\angle 180° \qquad SWR = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+0.333}{1-0.333} = 2$$

a. $V_s(z) = V_0^+(e^{-\gamma z} + \Gamma e^{\gamma z}) \quad \Rightarrow \quad V_s(0) = V_0^+(1+\Gamma) = V_L = 50 \quad \Rightarrow \quad V_0^+ = \frac{50}{1+\Gamma} = \frac{50}{1-\frac{1}{3}} = 75$

$$V_{max} = V_0^+(1+|\Gamma|) = 75\left(1 + \frac{1}{3}\right) = 100 \text{ volt}$$
$$V_{min} = V_0^+(1-|\Gamma|) = 75\left(1 - \frac{1}{3}\right) = 50 \text{ volt}$$

b. Location of voltage maxima:

$$z_{max} = -\frac{\phi}{2\beta} - n\frac{\pi}{\beta} = -\frac{\phi + 2n\pi}{2\beta} = -\frac{\pi + 2n\pi}{2(2\pi)}\lambda = -\frac{1+2n}{4}\lambda = -\frac{\lambda}{4}, -\frac{3\lambda}{4}, -\frac{5\lambda}{4}, \ldots$$

Location of voltage minima:

$$z_{min} = z_{max} + \frac{\lambda}{4} = 0, -\frac{\lambda}{2}, -\lambda, \dots$$

c. $Z_{max} = Z_0(SWR) = 100(2) = 200\ \Omega$ $\qquad Z_{min} = \frac{Z_0}{SWR} = \frac{100}{2} = 50\ \Omega$

d. $P_L = \frac{|V_0^+|^2}{2Z_0}[1 - |\Gamma_L|^2] = \frac{75^2}{2(100)}\left[1 - \left|\frac{1}{3}\right|^2\right] = 25\ W$

Q2. A transmission line with characteristic impedance $Z_0 = 50\ \Omega$ has a load of $Z_L = 100\ \Omega$. Load voltage is 50 volt. Calculate

a. Maximum and minimum voltage on the line
b. Location of voltage maxima and minima on the line
c. The maximum and minimum impedance on the line. Where do these occure?
d. Power transmitted to the load.

Solution:

$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{100-50}{100+50} = \frac{50}{150} = \frac{1}{3} = 0.3333\angle 0°$ $\qquad SWR = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+0.333}{1-0.333} = 2$

a. $V_s(z) = V_0^+(e^{-\gamma z} + \Gamma e^{\gamma z}) \quad \Rightarrow \quad V_s(0) = V_0^+(1+\Gamma) = 50 \quad \Rightarrow \quad V_0^+ = \frac{50}{1+\Gamma} = \frac{50}{1+\frac{1}{3}} = 37.5$ volt

$$V_{max} = V_0^+(1 + |\Gamma|) = 37.5\left(1 + \frac{1}{3}\right) = 50 \text{ volt}$$

$$V_{min} = V_0^+(1 - |\Gamma|) = 37.5\left(1 - \frac{1}{3}\right) = 25 \text{ volt}$$

b. Location of voltage maxima:

$$z_{max} = -\frac{\phi}{2\beta} - n\frac{\pi}{\beta} = -\frac{\phi + 2n\pi}{2\beta} = -\frac{0 + 2n\pi}{2(2\pi)}\lambda = -\frac{n}{2}\lambda = 0, -\frac{\lambda}{2}, -\lambda, -\frac{3\lambda}{2}, \dots$$

Location of voltage minima:

$$z_{min} = z_{max} + \frac{\lambda}{4} = -\frac{\lambda}{4}, -\frac{3\lambda}{4}, -\frac{5\lambda}{4} \dots$$

c. $Z_{max} = Z_0(SWR) = 50(2) = 100\ \Omega$ $\qquad Z_{min} = \frac{Z_0}{SWR} = \frac{50}{2} = 25\ \Omega$

d. $P_L = \frac{|V_0^+|^2}{2Z_0}[1 - |\Gamma_L|^2] = \frac{37.5^2}{2(50)}\left[1 - \left|\frac{1}{3}\right|^2\right] = 12.5\ W$

Q3. A lossless line has $Z_0 = 300\ \Omega,\ l = 2\ m, and\ v_p = 2.5 \times 10^8$ m/s. Let $Z_L = 300\ \Omega$. Source voltage is $60\angle 0°$ volt at 100 MHz and source impedance is 300 Ω.

a. Find voltage, current and power at the input of the line and at the load.
b. If load impedance becomes 150 Ω, find the new values of part (a).
c. Find voltage maxima, minima and their locations on the line.

Solution:

a. Since the line is matched $(Z_0 = Z_L)$, reflection co-efficient is zero; standing wave ratio is unity (no standing waves; only travelling waves).

Wavelength on the line $= \lambda = \frac{2.5\times 10^8}{100\times 10^6} = 2.5\ m\ and\ \beta = \frac{2\pi}{\lambda} = \frac{2\pi}{2.5} = 0.8\pi\ rad/m.$

Attenuation constant is zero, electrical length of line is $\beta l = 1.6\pi\ rad = 288°, or\ 0.8\lambda$.

Impedance at the input of the line $Z_{in} = Z_0 = 300\ \Omega$

Using voltage divider rule, voltage at the input of the line $V_{in} = 30\angle 0°$ volt

Voltage at the load $V_L = 30\angle -288°$.

Current at the input of the line $I_{in} = \frac{30}{300}\angle 0° = 0.1\angle 0°$ Amp and at the load $I_L = 0.1\angle -288°$ Amp.

Power delivered at the input of the line must all be delivered to the load

$$P_{in} = P_L = \frac{1}{2}V_{in}I_{in}\cos\theta = \frac{1}{2}(30)(0.1) = 1.5 \text{ Watt.}$$

b. Load impedance $Z_L = 150\ \Omega$. Reflection co-efficient $\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{150-300}{150+300} = -\frac{1}{3}$. Standing wave ratio SWR = $\frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+\frac{1}{3}}{1-\frac{1}{3}} = 2$.

Impedance at the input of the line $Z_{in} = Z(2m) = Z_0\frac{Z_L + jZ_0\tan\beta l}{Z_0 + jZ_L\tan\beta l} = 300\frac{150 + j300\tan 288}{300 + j150\tan 288} = 300\frac{1 + j2\tan 288}{2 + j\tan 288} = 300\frac{1 - j6.1554}{2 - j3.0777} = 300\frac{6.2361\angle -80.77°}{3.6705\angle -56.98°} = 509.69\angle -23.79° = (466.38 - j205.6)\ \Omega$

Current at the input of the line

$I_{in} = \frac{60\angle 0°}{300 + (466.38 - j205.6)} = \frac{60\angle 0°}{766.38 - j205.6} = \frac{60\angle 0°}{793.48\angle -15.02°} = 0.0756\angle 15.02°$ Amp.

Voltage at the input of the line $V_{in} = (0.0756\angle 15.02°)(509.69\angle -23.79°) = 38.53\angle -8.77°$ volt

Voltage at the load is a bit complicated:

Recall $V_s(z) = V_0^+\left[e^{-j\beta z} + \Gamma e^{j\beta z}\right]$.

Hence $V_0^+ = \frac{V_s(z)}{e^{-j\beta z} + \Gamma e^{j\beta z}} = \frac{V_{in}}{e^{-j(-2)\beta} + \Gamma e^{j(-2)\beta}} = \frac{38.53\angle -8.77°}{e^{j1.6\pi} - \frac{1}{3}e^{-j1.6\pi}} = \frac{38.53\angle -8.77°}{(0.309 - j0.951) - (0.103 + j0.317)}$

$= \frac{38.53\angle -8.77°}{1.2846\angle 80.77°} = 30\angle 72°$ volt

Now $V_L = V_s(0) = V_0^+\left[e^{-j\beta(0)} + \Gamma e^{j\beta(0)}\right] = 30\angle 72°\left[1 - \frac{1}{3}\right] = 20\angle 72°$ volt

$I_L = \frac{20}{150}\angle 72° = 0.133\angle 72°$ Amp

$P_{in} = P_L = \frac{1}{2}V_{in}I_{in}\cos\theta = \frac{1}{2}(38.53)(0.0756)\cos(15.02° + 8.77°) = 1.33$ Watt.

c. $V_{s,max} = |V_0^+|[1 + |\Gamma|] = 30\left[1 + \frac{1}{3}\right] = 40$ volt

$V_{s,min} = |V_0^+|[1 - |\Gamma|] = 30\left[1 - \frac{1}{3}\right] = 20$ volt

$z_{max} = -\frac{\phi}{2\beta} - n\frac{\pi}{\beta} = -\frac{\pi}{2(0.8\pi)} - n\frac{\pi}{0.8\pi} = -0.625 - 1.25n = -0.625\ m, -1.875\ m$

$z_{min} = \frac{\lambda}{4} + z_{max} = 0.625 - 0.625,\ 0.625 - 1.875 = 0\ m, -1.25\ m$

Q4. Mark the following load impedances (given in Ohms) on the Smith chart ($Z_0 = 50\ \Omega$):

$$Z_L = 30 + j40, 25 - j45, j15, -j25, 400 + j800, 50, SC, OC$$

Solution:

Z_L	Inductive 30+j40	Capacitive 25-j45	Pure inductive j15	Pure capacitive -j25	Large 400+j800	Matched 50+j0	Short circuit 0+j0	Open circuit $\infty \pm j\infty$
$z_L = Z_L/Z_0$	0.6+j0.8	0.5-j0.9	0+j0.3	0-j0.5	8+j16	1+j0	0+j0	$\infty \pm j\infty$
	P0	P1	P2	P3	P4	P5	P6	P7