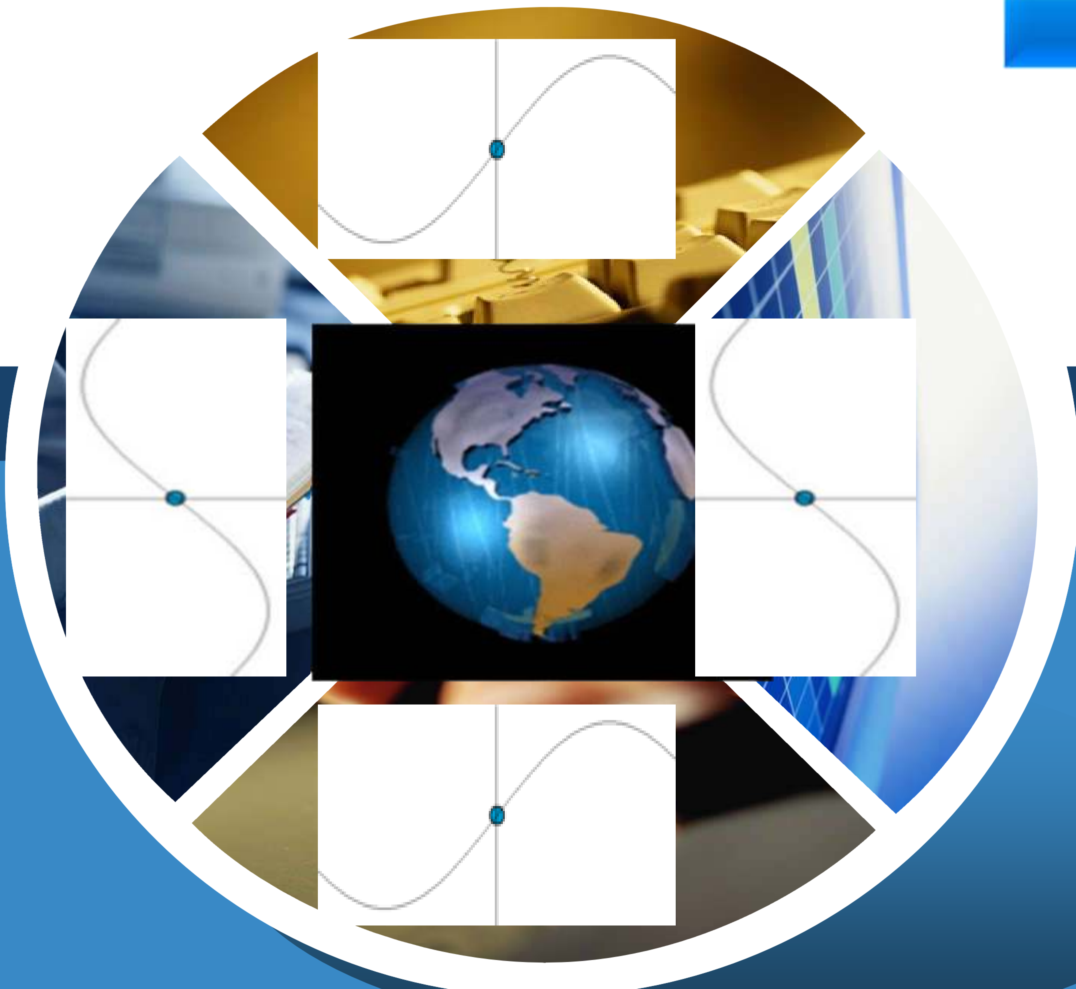
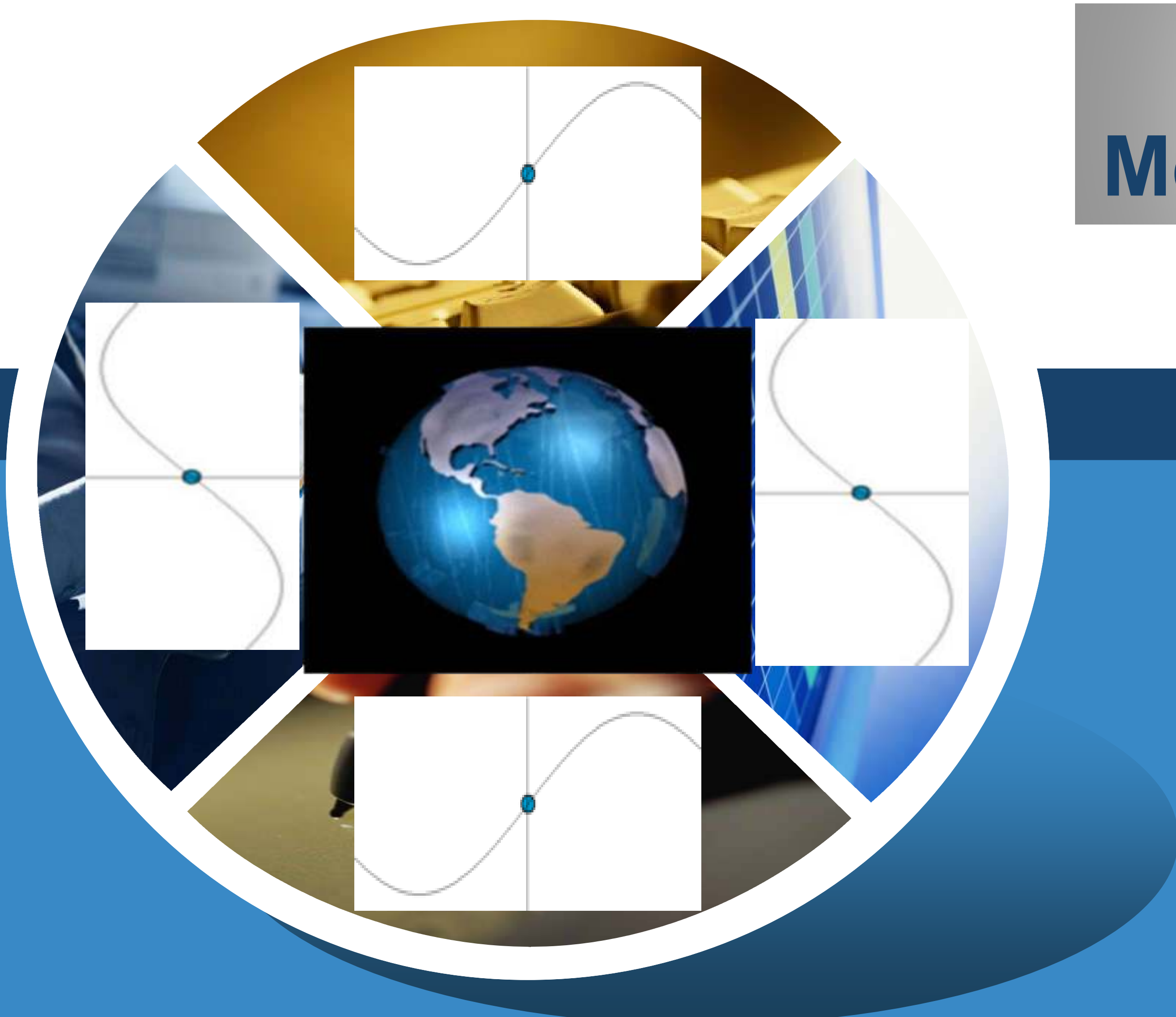
ME 413 System Dynamics & Control

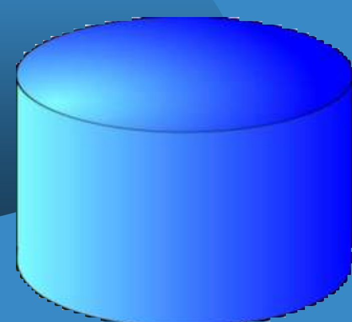


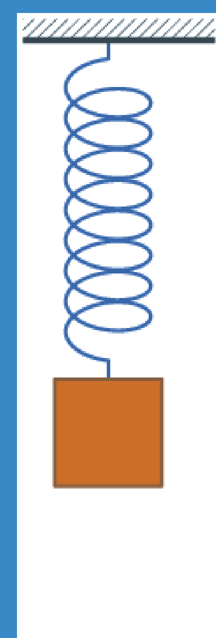
Dr. A. Aziz Bazoune
Mechanical Engineering Department
King Fahd University of Petroleum & Minerals

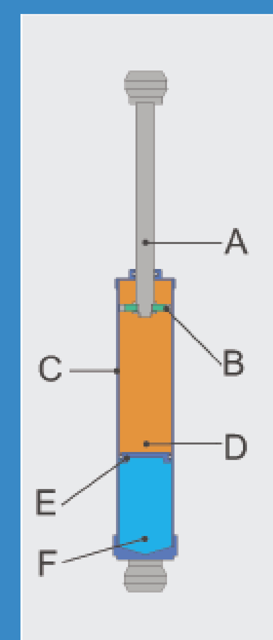


Chapter 3
Mechanical Systems

Mechanical Systems







Turn OFF

Mobile Phones

CHAPTER CONTENTS

3-1 Introduction

3-2 Mechanical Elements

3-3 Mathematical Modeling of Simple Mechanical Systems

3-4 Work, Energy, and Power

Example Problems and Solutions

Problems

CHAPTER OBJECTIVES

When you have finished this chapter, you should be able to:

1. Identify the **mechanical elements** (**Mass, Spring and Damper**) for translational and rotational motion.
2. Model systems containing ideal spring elements and ideal damper elements.
3. Obtain the equivalent spring, mass and dampers.
4. Obtain Equations of Motion (EOM) for systems having spring and damper elements.
5. Apply Newton's second law to obtain EOM.
6. Apply energy methods to obtain EOM.
7. Obtain the free and forced response of mass-spring-damper systems.
8. Utilize MATLAB to assist in the response analysis.

Introduction

Mass

- The ***mass*** of a body is the quantity of matter in it that is assumed to be constant.
- Mass is the property of a body that gives it inertia, that is, resistance to starting and stopping.

The unit of mass is kilogram [kg] in SI units

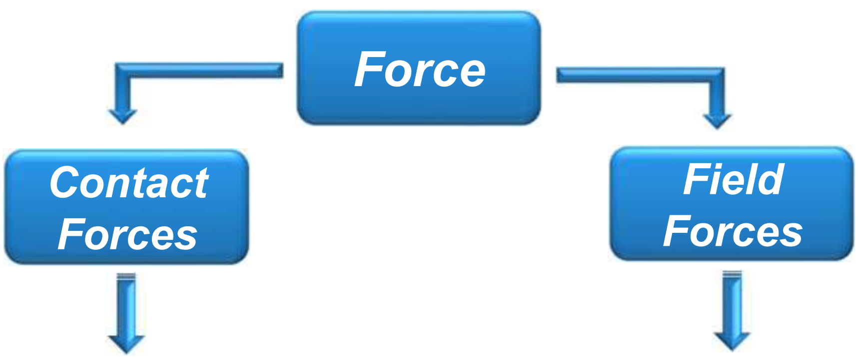
Weight

- A body is attracted by the earth, and the magnitude of the force that the earth exerts on the body is called its ***weight***.

The unit of weight is Newton [N] in SI units

Force

- ***Force*** can be defined as the cause that tends to produce a change in motion of a body on which it acts.



Contact Forces: come into direct contact with a body.

Field Forces: Act on a body but do not come into contact with it. Example: gravitational and magnetic forces.

The unit of force is Newton [N] in SI units

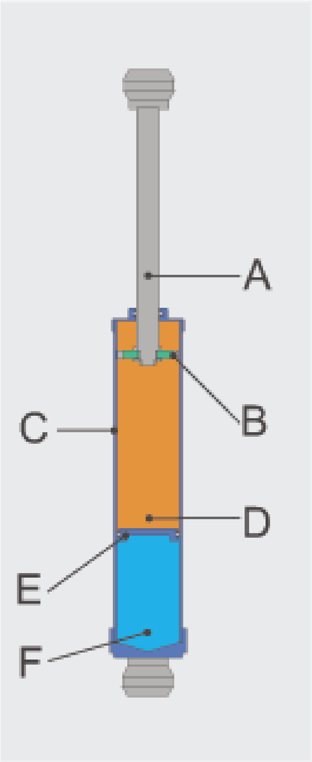
Mechanical Elements

Any mechanical system consists of mechanical elements. There are three types of basic elements in mechanical systems:

Mass or (inertia) Element

Spring Element

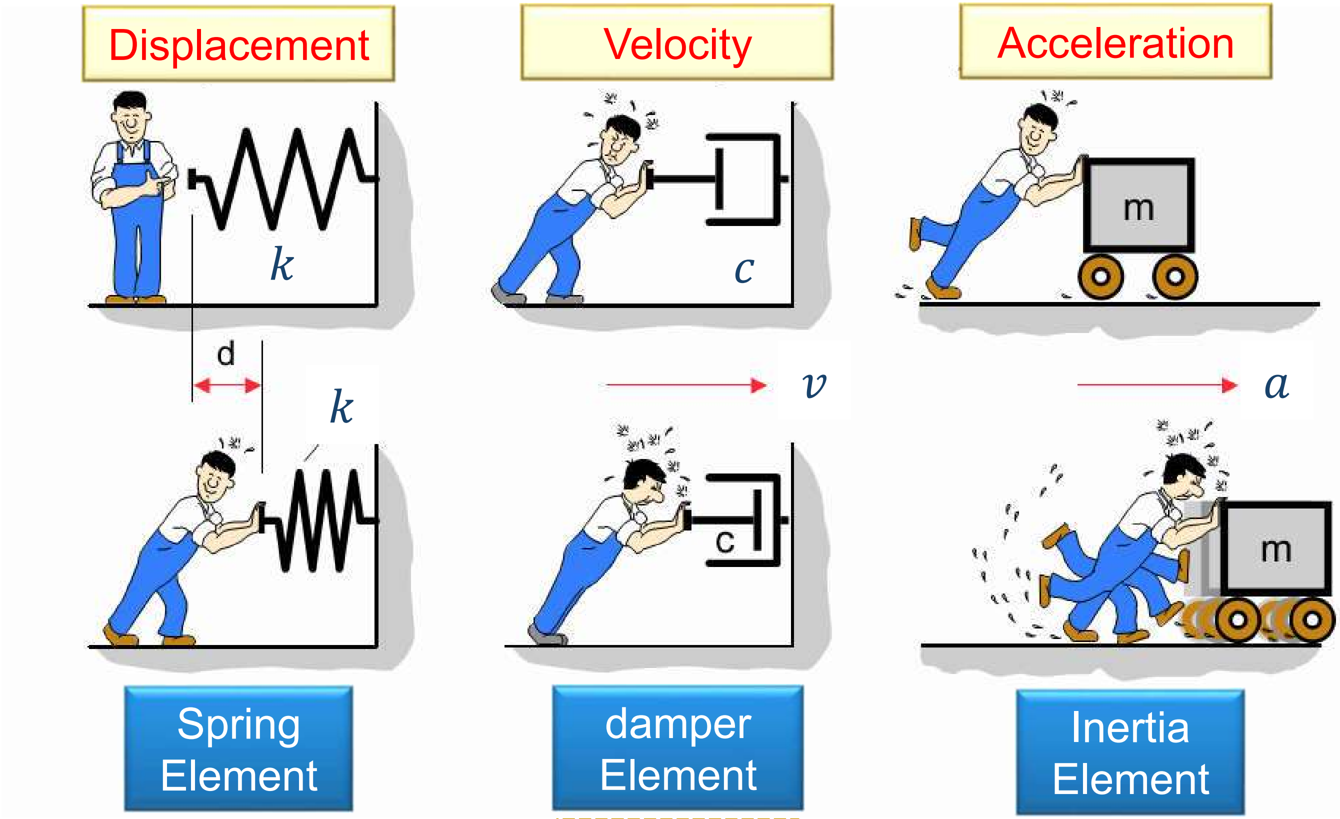
Damper Element or Dashpot



Translational Elements

- All mechanical systems contain the three basic elements: spring, damper, and mass (inertia element).

- When each of these in turn is exposed to a constant force they react with a constant displacement, a constant velocity and a constant acceleration, respectively.

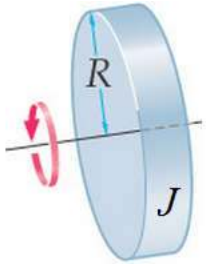


$$F = kd \qquad F = cv \qquad F = ma$$

- Similarly we can obtain the same relationships for rotational (torsional) elements.

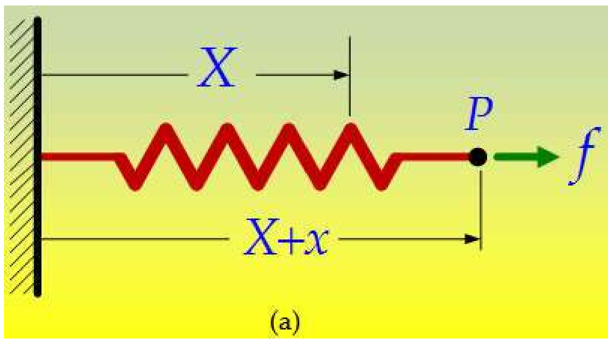
Mass or Inertia Elements

- ***Mass* and *inertia elements*.** Inertia may be defined as the change in force (torque) required to make a unit change in acceleration (angular acceleration).

Motion		Symbol	Units	Sketch
Translation	inertia *(mass)* = $\dfrac{\textit{change in force}}{\textit{change in acceleration}}$	m or M	$[\frac{N}{m/s^2}]$ or [kg]	
Rotation	inertia *(mass moment of inertia)* = $\dfrac{\textit{change in torque}}{\textit{change in angular acceleration}}$	J or I	$[\frac{N \cdot m}{rad/s^2}]$ or $[kg \cdot m^2]$	

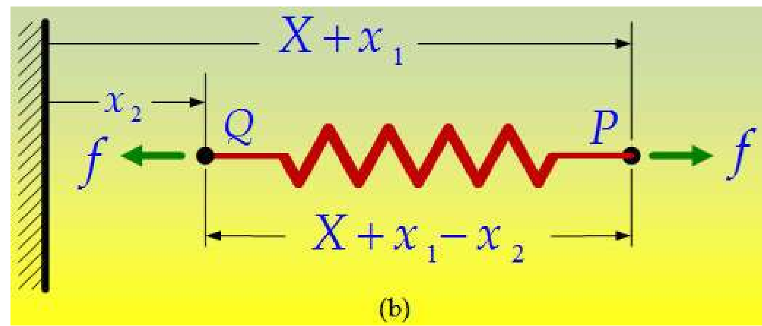
Spring Elements: Translational Motion

- *Ideal spring* : a mechanical element that is generally assumed to have *negligible mass and damping*.
- A *linear spring* can be deformed by external force or torque such that the deformation is directly proportional to the force or torque applied to the element.
- *Linear Spring force Relationship* is given by (Hook's Law) :



(a) One end of the spring is deflected

$$F = kx \qquad (3.1)$$



(b) both ends of the spring are deflected.

$$F = k\left(x_1 - x_2\right) \qquad (3.2)$$

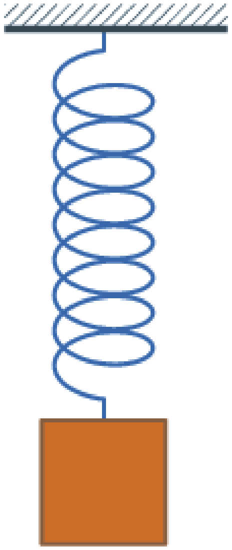
F = spring force,

X = free length of the spring.

x = elongation of the spring (displacement of one end with respect to the other) and

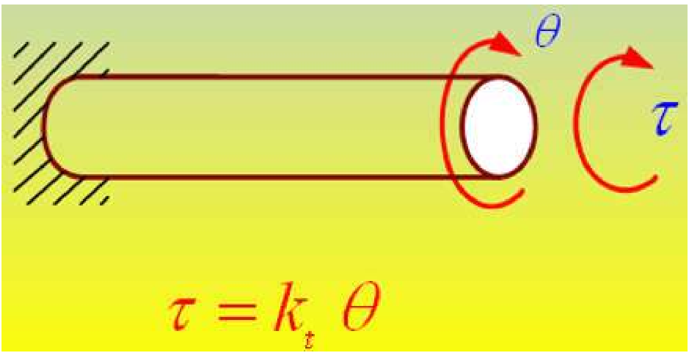
k = spring stiffness or spring constant, and has units of [force/displacement] = [N/m] in SI units.

Examples of Translating Spring Elements



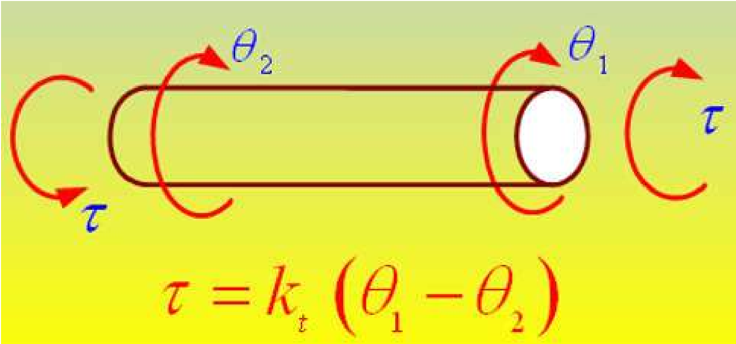
Spring Elements: Torsional Motion

- The *angular displacement* of the free end is θ .
- The torque in the torsional spring is



(a) A torque τ is applied at one end of torsional spring and the other end is fixed;

$$\tau = k_t \theta \quad (3.3)$$



(b) a torque τ is applied at both ends.

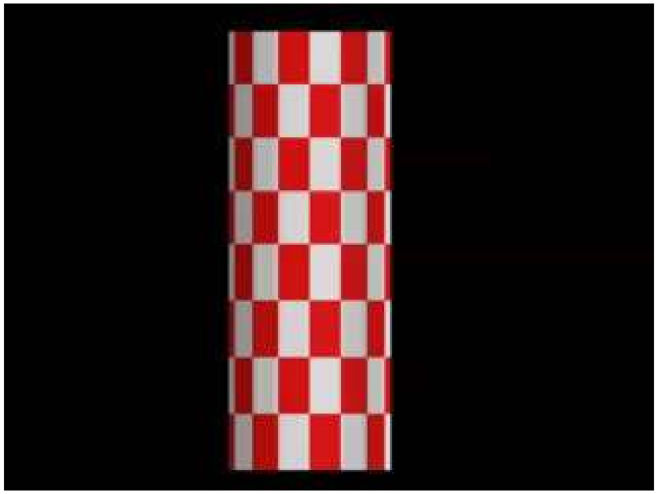
$$\tau = k_t(\theta_1 - \theta_2) \quad (3.2)$$

τ = spring torque,

θ = angular displacement and

k_t= spring stiffness or spring constant, and has units of [Torque/ang. Disp.] = [N-m/rad] in SI units.

Examples of Torsional Spring Elements

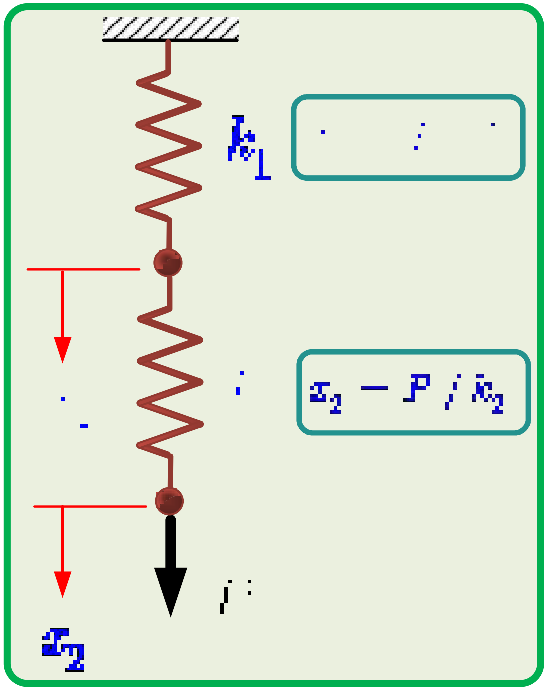


http://www.esm.psu.edu/courses/emch13d/design/animation/animation.htm

Equivalent Springs: Summary

Springs in series:

Force is common - flexibilities add

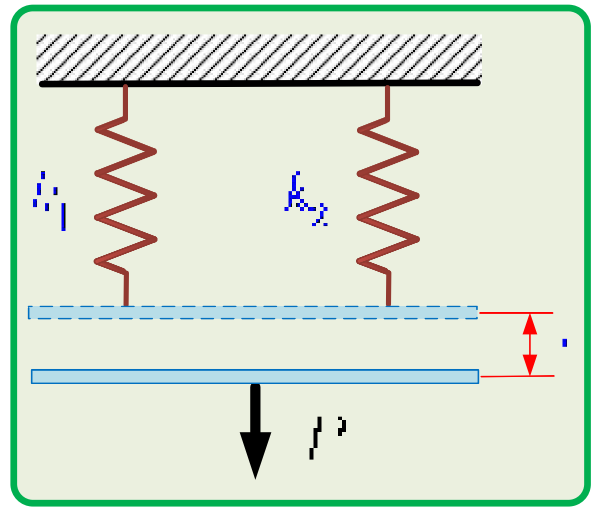


$$x = x_1 + x_2 = \left(1/k_1 + 1/k_2\right)P$$
$$= (1/k_{eq})P$$

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$$

Springs in parallel:

Displacement is common - stiffnesses add

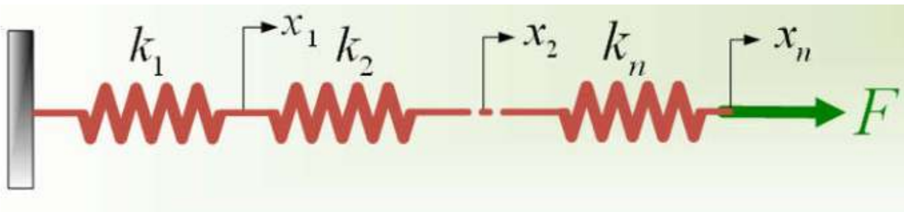


$$F = k_1 x + k_2 x$$
$$= (k_1 + k_2)x = k_{eq} x$$

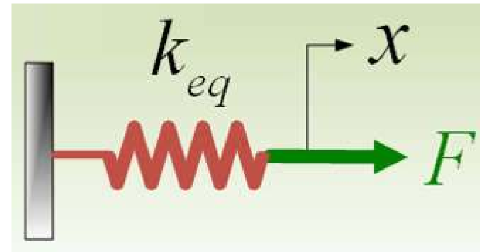
$$k_{eq} = k_1 + k_2$$

Equivalent Springs: Summary

Springs in series:



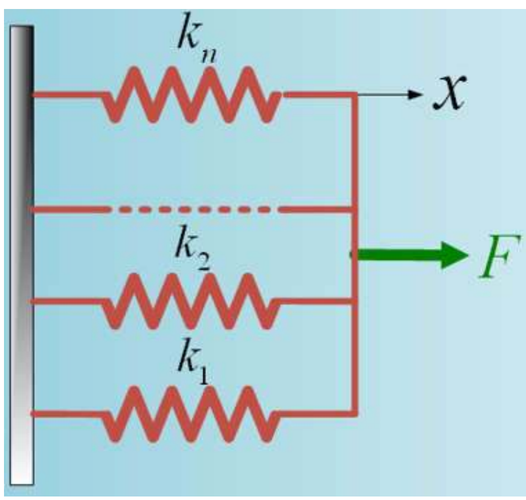
Equivalent to:



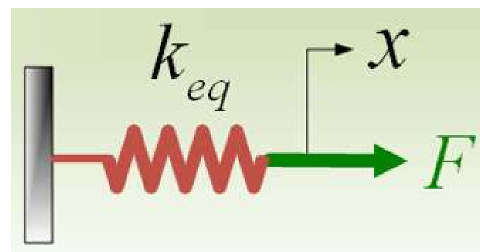
In general

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n} = \sum_{i=1}^{n} \frac{1}{k_i}$$

Springs in Parallel:



Equivalent to:



In general

$$k_{eq} = k_1 + k_2 + \cdots + k_n = \sum_{i=1}^{n} k_i$$

Compliance or Mechanical Impedance

- Spring constants indicate stiffness; a large value of k corresponds to a hard spring, a small value of k to a soft spring.
- The reciprocal of the spring constant k is called *compliance* or *mechanical capacitance* C.
- Compliance, $C = 1/k$. Compliance indicates the softness of a spring.

Practical Spring versus Ideal Spring

- All practical springs have inertia and damping.
- An ideal spring has neither mass nor damping (internal friction) and will obey the linear force displacement law.

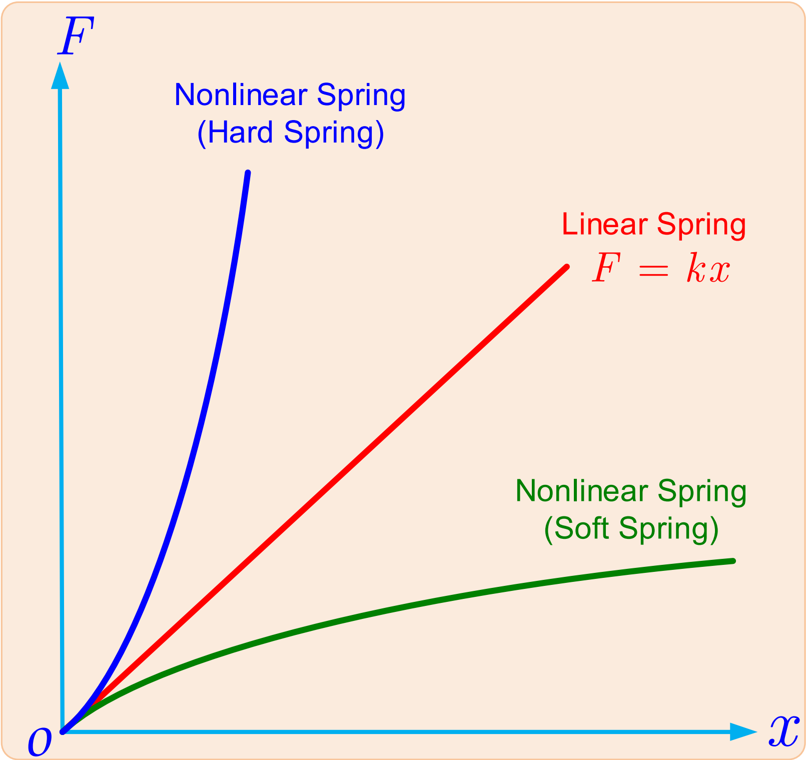
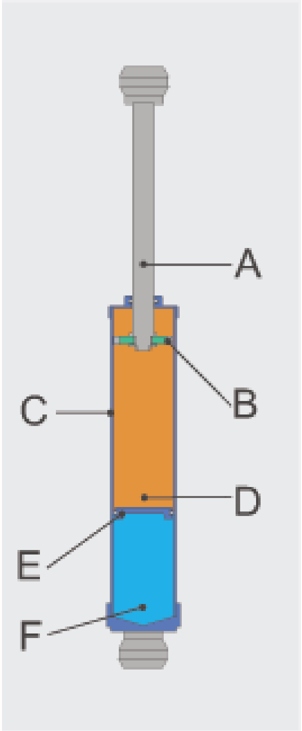


Figure 3-3 Force-displacement characteristic curves for linear and nonlinear springs.

https://phet.colorado.edu/sims/mass-spring-lab/mass-spring-lab_en.html

- A damper is a mechanical element that dissipates energy in the form of heat instead of storing it.
- The figure below shows a translational damper, or a dashpot that consists of a piston and an-oil-filled cylinder.
- Any relative motion between the piston rod and the cylinder is resisted by oil because oil must flow around the piston (or through orifices provided in the piston) from one side to the other.



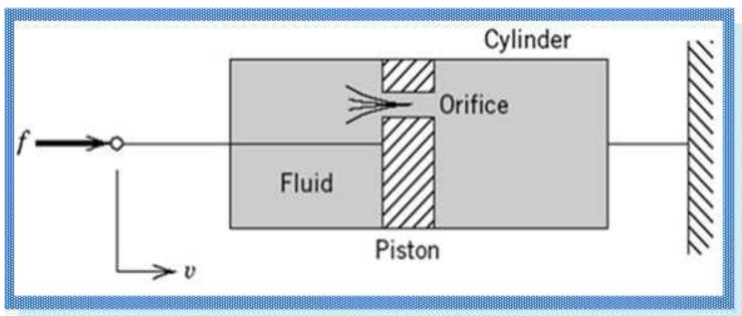
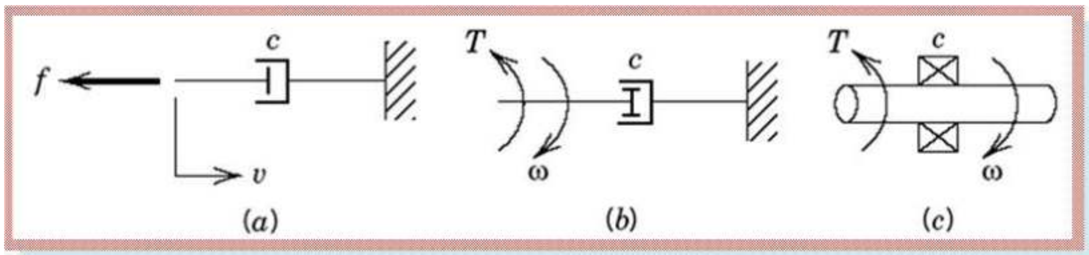


Figure 3-4(a) shows a schematic diagram of translational damper

Shock absorber components with internal reservoir.
- the rod (A),
- the piston with seals (B),
- the cylinder (C),
- the oil reservoir (D),
- the floating piston (E), and
- the air chamber (F).

Damping Elements

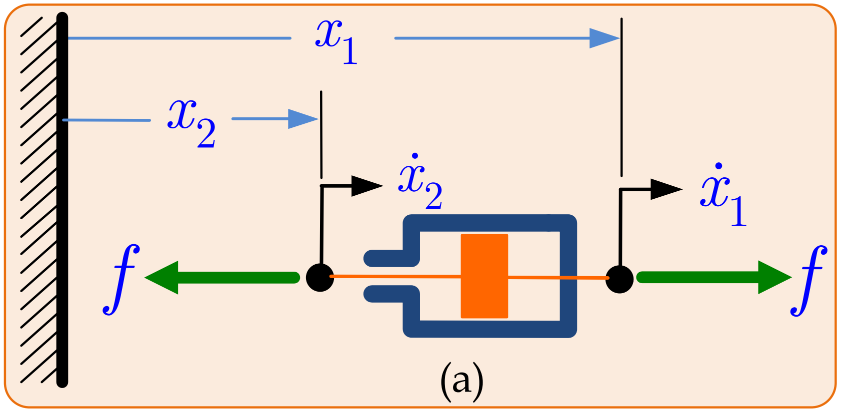
- In many practical systems, the mechanical energy of a system is gradually *converted* into heat or sound.
- The mechanism by which the vibrational energy is gradually converted into heat or sound is known as damping.
- An ***ideal damper*** is assumed to have ***neither mass nor elasticity***, and damping force exists only if there is ***relative velocity*** between the two ends of the damper.
- The symbols shown in the figure are the general symbol for a damping element.



(a) Translational damper; (b) torsional (or rotational) damper (c) rotational damping in bearings

http://www.acoustics.salford.ac.uk/feschools/waves/shm4.htm

Translation



(a) Translational damper

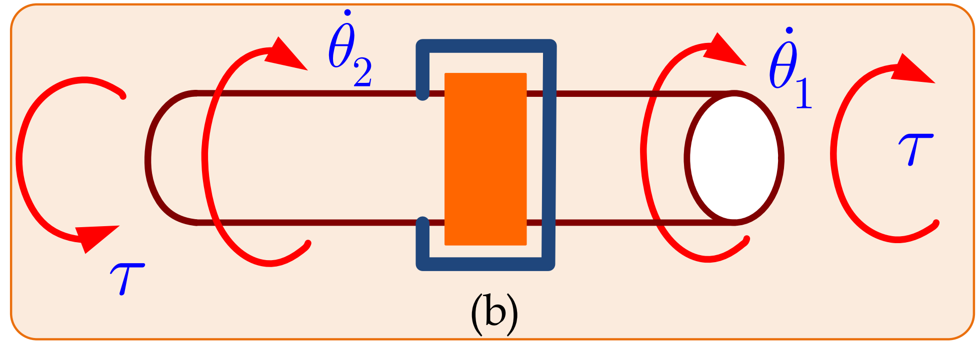
$$F = b\left(\dot{x}_1 - \dot{x}_2\right) = b\dot{x} \qquad (3.5)$$

F = damping force,

$x = (x_1 \quad x_2)$= velocity difference = net velocity

b = viscous friction coefficient, [force/velocity] = [N·s/m] in SI units.

Rotation



(b) Torsional or rotational damper.

$$\tau = b_t\left(\dot{\theta}_1 - \dot{\theta}_2\right) = b_t\dot{\theta} \qquad (3.6)$$

τ = damping torque,

$\theta = (\theta_1 \quad \theta_2)$= angular velocity difference = net angular velocity.

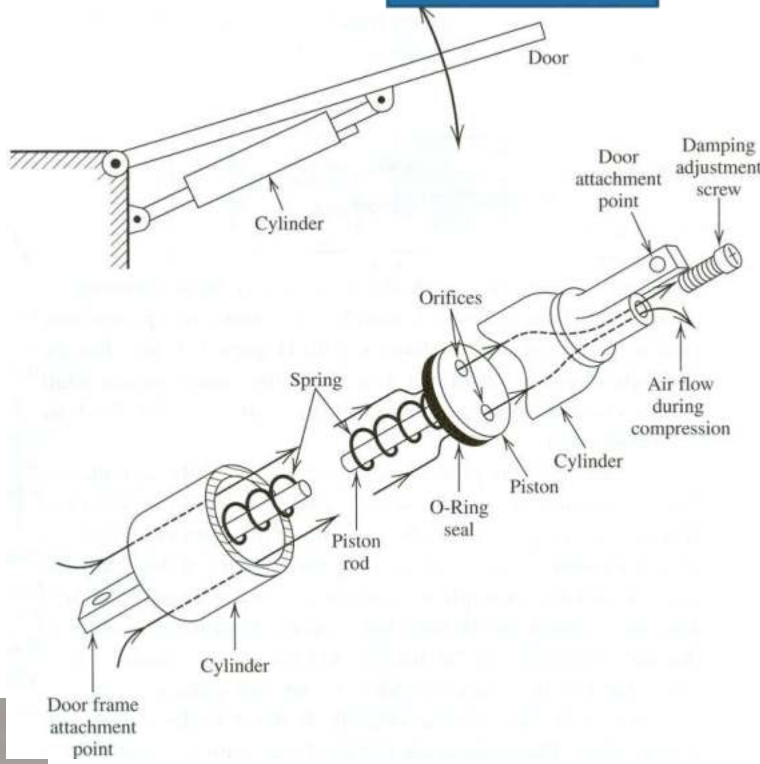
b_t = viscous friction coefficient, [torque/angular velocity] = [N·m·s/rad] in SI units.

Examples of Common Uses of Dashpots

Door Stoppers

Vehicle Suspension

Door Closer



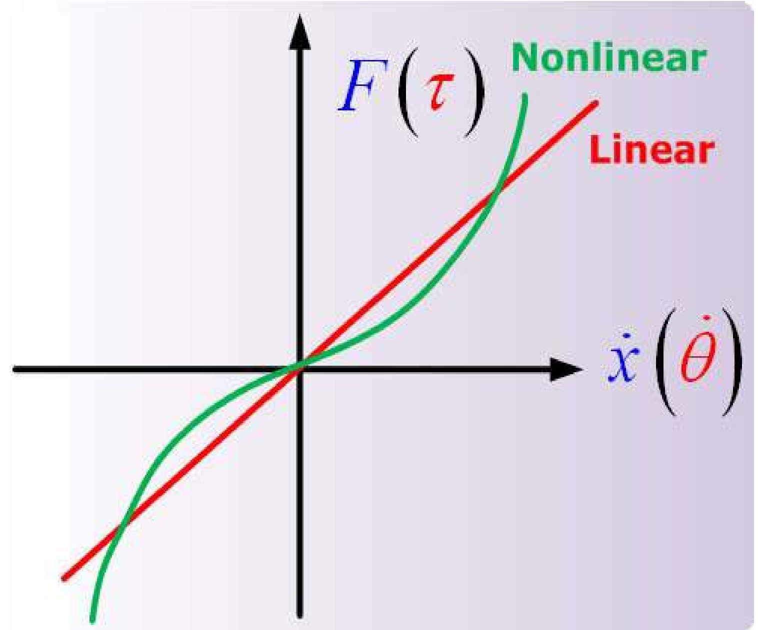
Bridge Suspension

Flyover Suspension

- A damper is an element that provides resistance in mechanical motion, and, as such, its effect on the dynamic behavior of a mechanical system is similar to that of an electrical resistor on the dynamic behavior of an electrical system.
- Consequently, a damper is often referred to as a ***mechanical resistance element*** and the viscous friction coefficient as the ***mechanical resistance***.

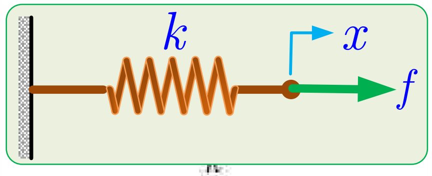
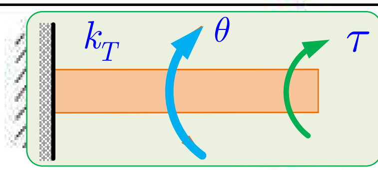
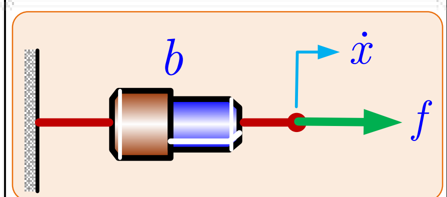
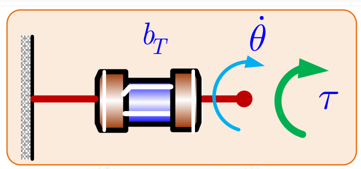
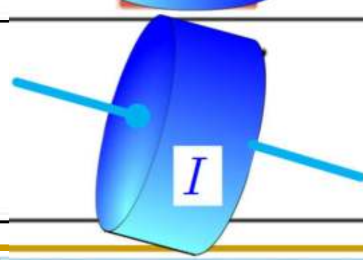
Practical Dampers versus Ideal Dampers

- All practical dampers produce inertia and spring effects.
- An ideal damper is massless and springless, dissipates all energy, and obeys the linear force-velocity law (or linear torque-angular velocity law).



- Friction that obeys a linear law is called ***linear friction***, whereas friction that does not is described as ***nonlinear***.
- Examples of ***nonlinear*** friction include static friction, sliding friction, and square-law friction. Square law-friction occurs when a solid body moves in a fluid medium.

http://www.mwit.ac.th/~physicslab/applet_04/fun@learning/JAVA/dho/dho2.html

Element		Symbol	Energy	Mathematical Model	
				t-domain	s-domain (zero I. C's)
Spring (Stiffness)	Translation		Storage	$f(t) = kx$	$F(s) = kX(s)$
	Rotation		Storage	$\tau(t) = k_T\theta$	$T(s) = k_T\Theta(s)$
Damper (Damping)	Translation		Dissipation	$f(t) = bx$	$F(s) = bs\,X(s)$
	Rotation		Dissipation	$\tau(t) = b_T\theta$	$T(s) = b_Ts\,\Theta(s)$
Mass (Inertia)	Translation		Storage	$f(t) = mx$	$F(s) = ms^2X(s)$
	Rotation		Storage	$\tau(t) = I\theta$	$T(s) = Is^2\Theta(s)$

OBJECTIVES

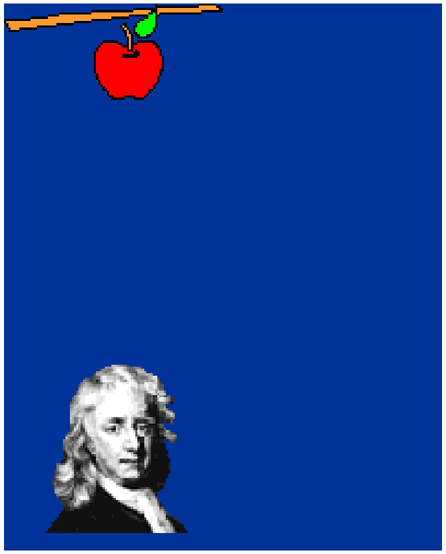
When you have finished this chapter, you should be able to:

1. Apply Newton's Second Law (NSL) to obtain Equations of Motion (EOM).

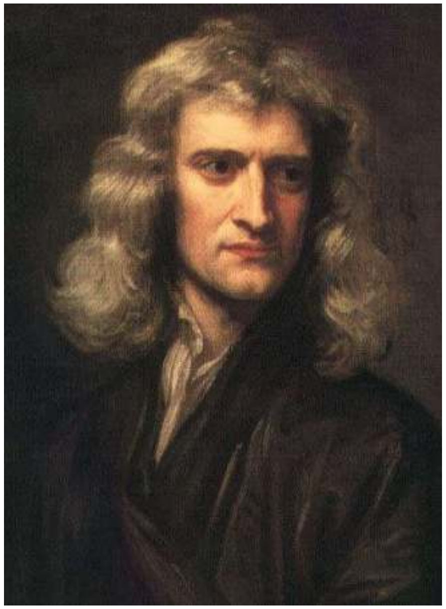
Mathematical Modeling of Mechanical Systems

- **A mathematical model** of any mechanical system can be developed by applying Newton's laws to the system.

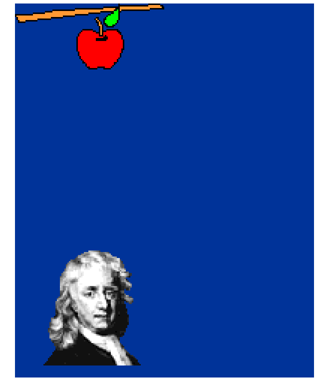
- **Rigid Body** When any real body is accelerated, internal elastic deflections are always present. If these internal deflections are negligibly small relative to the gross motion of the entire body, the body is called ***rigid body***. ***Thus, a rigid body does not deform.***



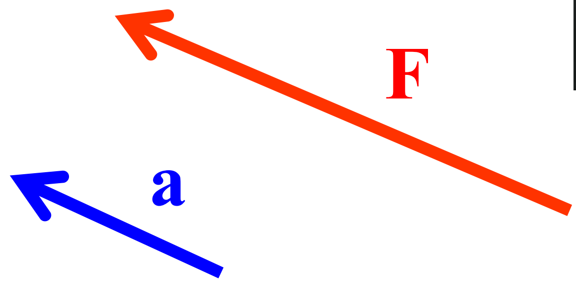
First Law: A particle originally at rest, or moving in a straight line at constant velocity, will remain in this state provided the particle is not subjected to an unbalanced force.



Second Law (Translational motion) : If a force is acting on rigid body through the **center of mass** in a given direction, the acceleration of the rigid body in the same direction is directly proportional to the force acting on it and is inversely proportional to the mass of the body. That is



$$\mathbf{F} \, \alpha \, \mathbf{a} \qquad \text{or} \qquad \mathbf{F} = m\,\mathbf{a}$$



$\mathbf{F}$: resultant unbalanced force acting on the body.

$\mathbf{a}$: acceleration of the body.

m : mass of the body.

$\mathbf{a}$	or	$\vec{a}$	: acceleration
$\mathbf{F}$	or	$\vec{F}$	: force

- Suppose that forces are acting on a body of mass m. If $\sum F$ is the sum of all forces acting on a mass m through the center of mass in a given direction, then

$$\sum \mathbf{F} = m\,\mathbf{a} \tag{3-7}$$

- a: is the resulting absolute acceleration in that direction.
- The line of action of the force acting on a body must pass through the center of mass of the body. Otherwise, rotational motion will also be involved.

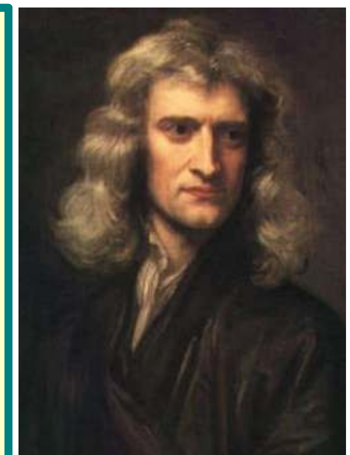
(Rotational Motion): For a rigid body in pure rotation about a fixed axis, Newton's second law states that

$$\sum T = J\,\alpha \tag{3-8}$$

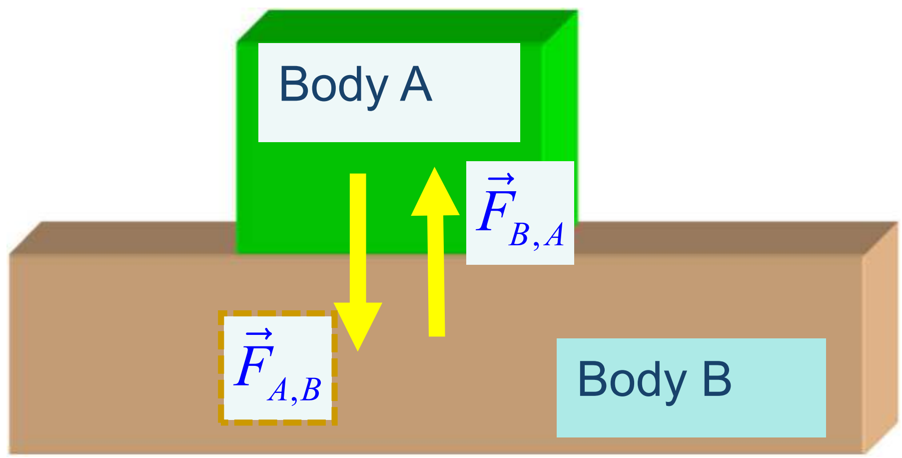
- $\sum T$: Sum of all torques acting about a given axis
- I : moment of inertia of a body about that axis, and
- α : the angular acceleration of the body

Third Law

- The mutual forces of action and reaction between two bodies are equal, opposite, and collinear.
- **Law of Action-Reaction**: For every action, there is an equal and opposite reaction.



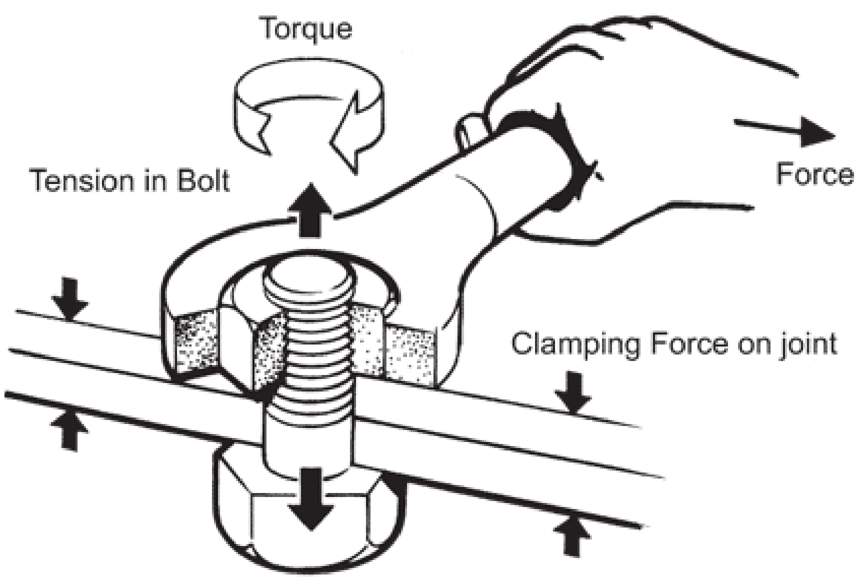
$$\vec{F}_{B,A} = -\vec{F}_{A,B}$$

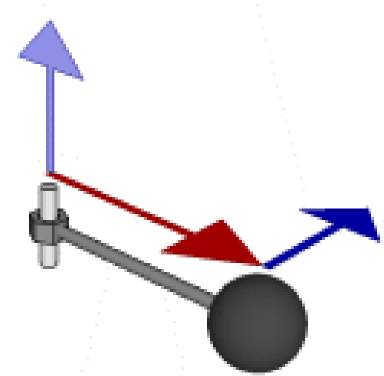


Torque or Moment of Force

- Torque is the product of a force and the perpendicular distance from a point of rotation to the line of action of the force.

$$\text{Torque} = \text{force} \times \text{distance} = [N\cdot m] \quad \text{in SI units}$$





$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$
$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

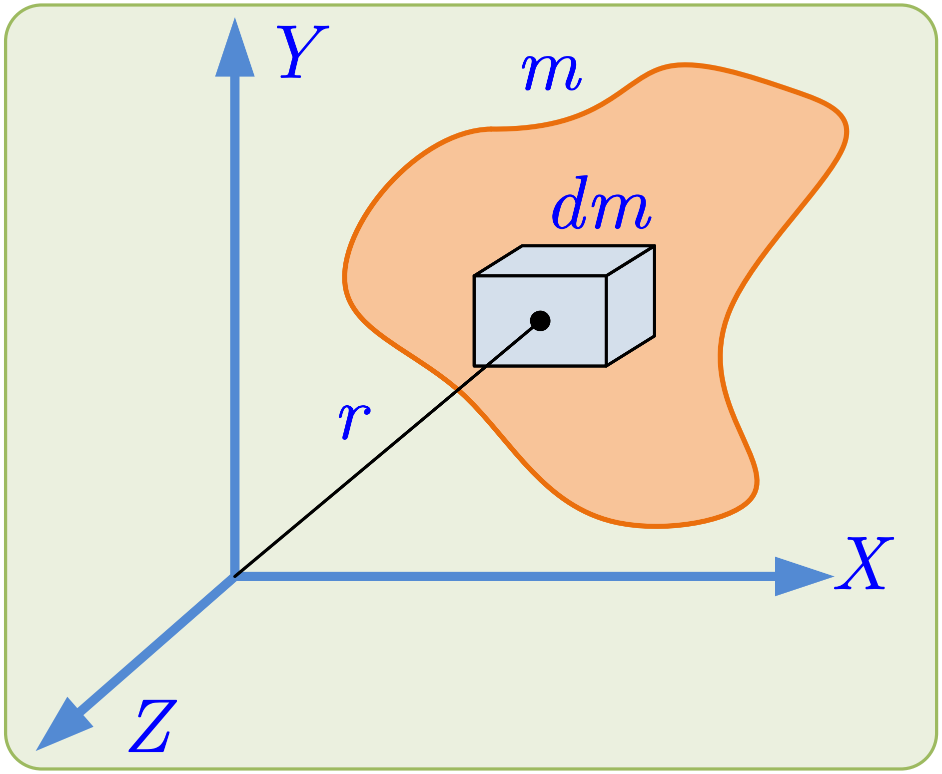
- The moment of inertia J of a rigid body about an axis is defined by

$$J = \int r^2 dm \tag{3-9}$$

dm: element of mass,

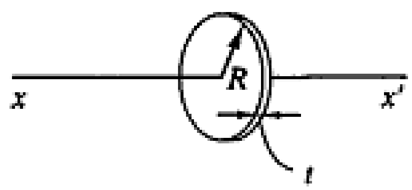
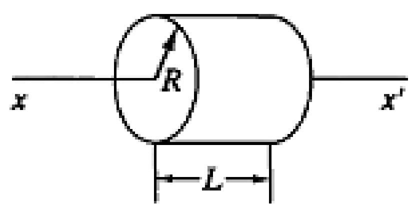
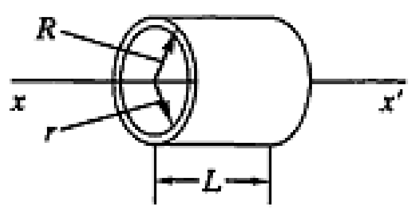
r: perpendicular distance from the axis to dm,

and integration is performed over the body.



- In considering moments of inertia, we assume that the rotating body is perfectly rigid.
- Physically, the moment of inertia of a body is a measure of the resistance of the body to angular acceleration.

Moment of Inertia of Some Common Shapes

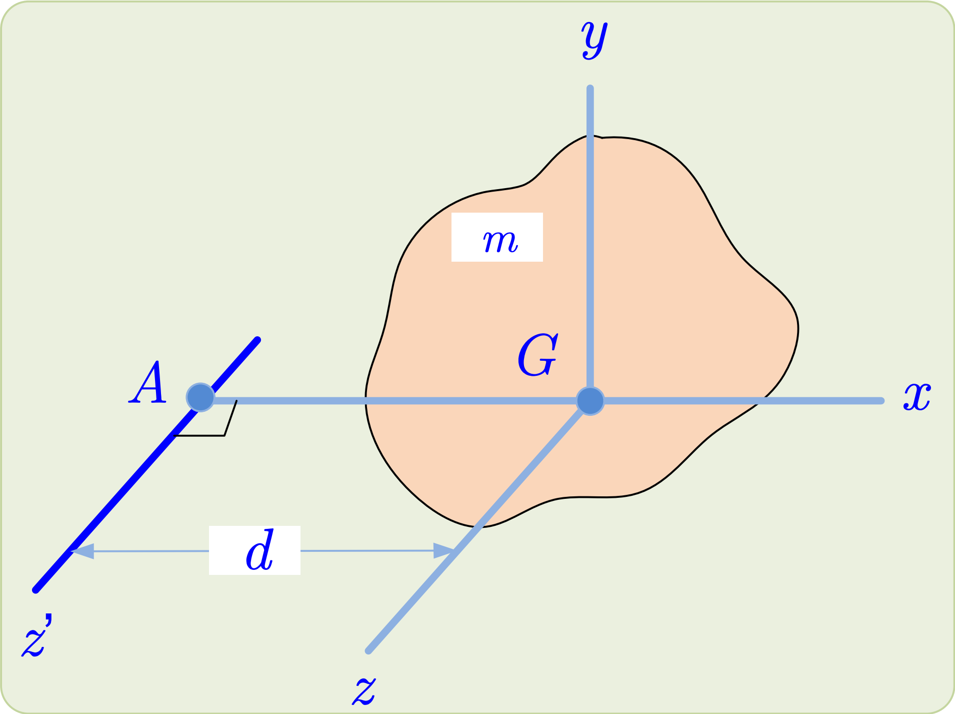
	$t \ll R$ m = mass of disk $J_x = \frac{1}{2} mR^2$
	m = mass of solid cylinder $J_x = \frac{1}{2} mR^2$
	m = mass of hollow cylinder $J_x = \frac{1}{2} m(R^2 + r^2)$

Parallel Axis Theorem

- The moment of inertia about an axis that is a distance d from the geometrical axis passing through the center of gravity of the body is the sum of the moment of inertia about the geometrical axis and the moment of inertia about the new axis when the mass of the body is considered concentrated at the center of gravity.

- For moment of inertia about the z' axis,

$$J_{z'} = J_G + md^2$$



J_G = MMI about the body's mass center

m = mass of the body

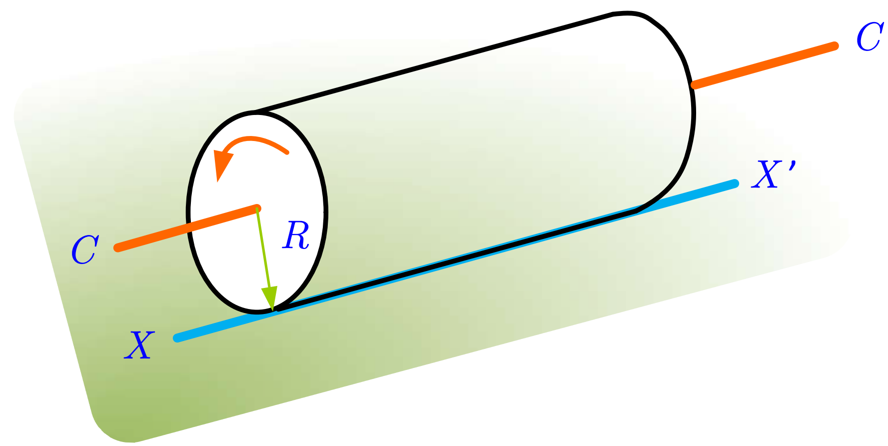
d = Perpendicular distance between the parallel axes

Example

- A homogeneous cylinder of mass m and radius R rolls on a flat surface. Find the moment of inertia, J_x, of the cylinder about its line of contact (axis XX') with the surface.

- Moment of inertia of the cylinder about axis CC' is

$$J_c = \frac{1}{2} m R^2$$

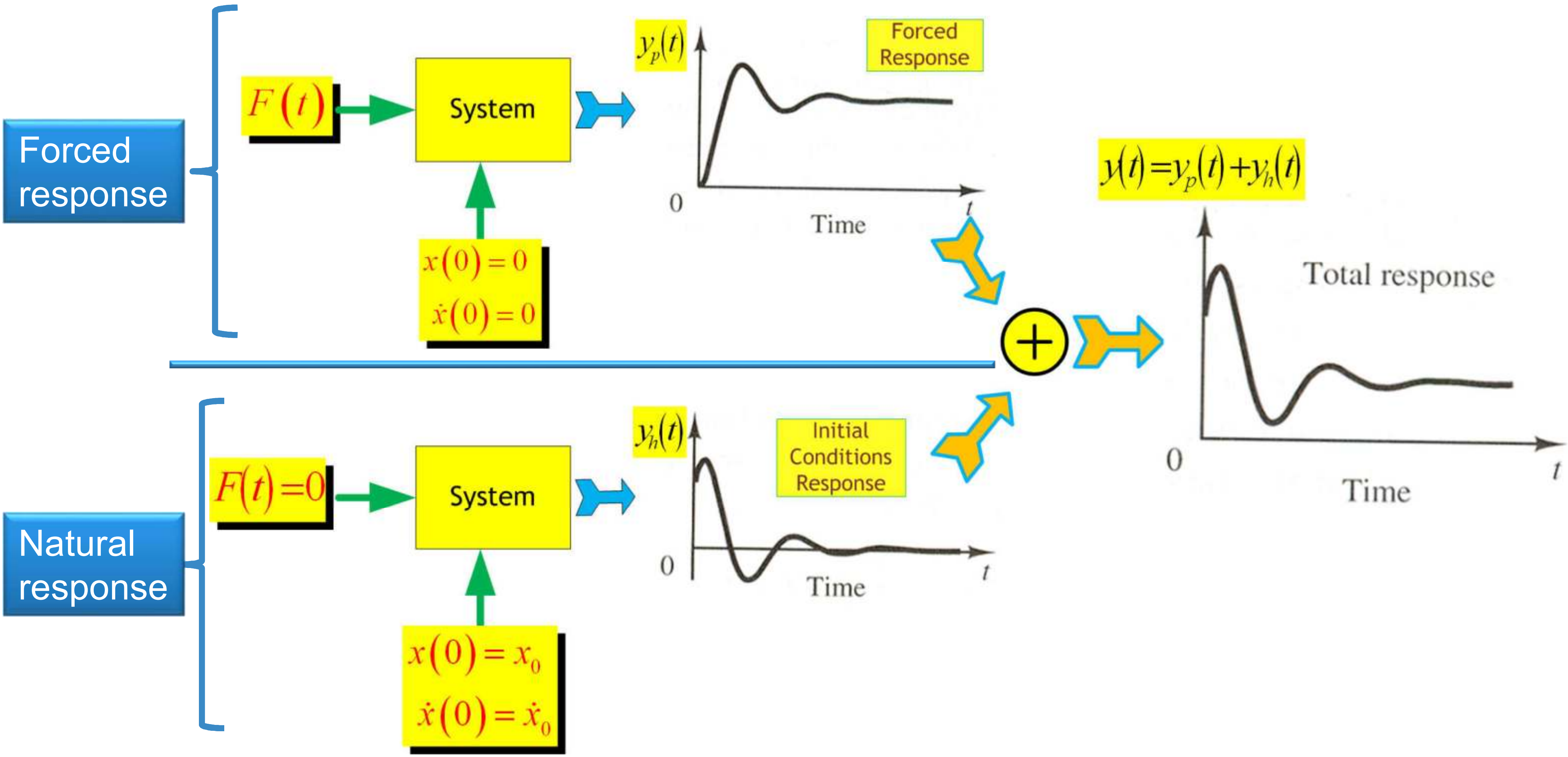


- Moment of inertia of the cylinder about axis XX' is

$$J_x = J_c + mR^2 = \frac{1}{2} m R^2 + mR^2 = \frac{3}{2} m R^2$$

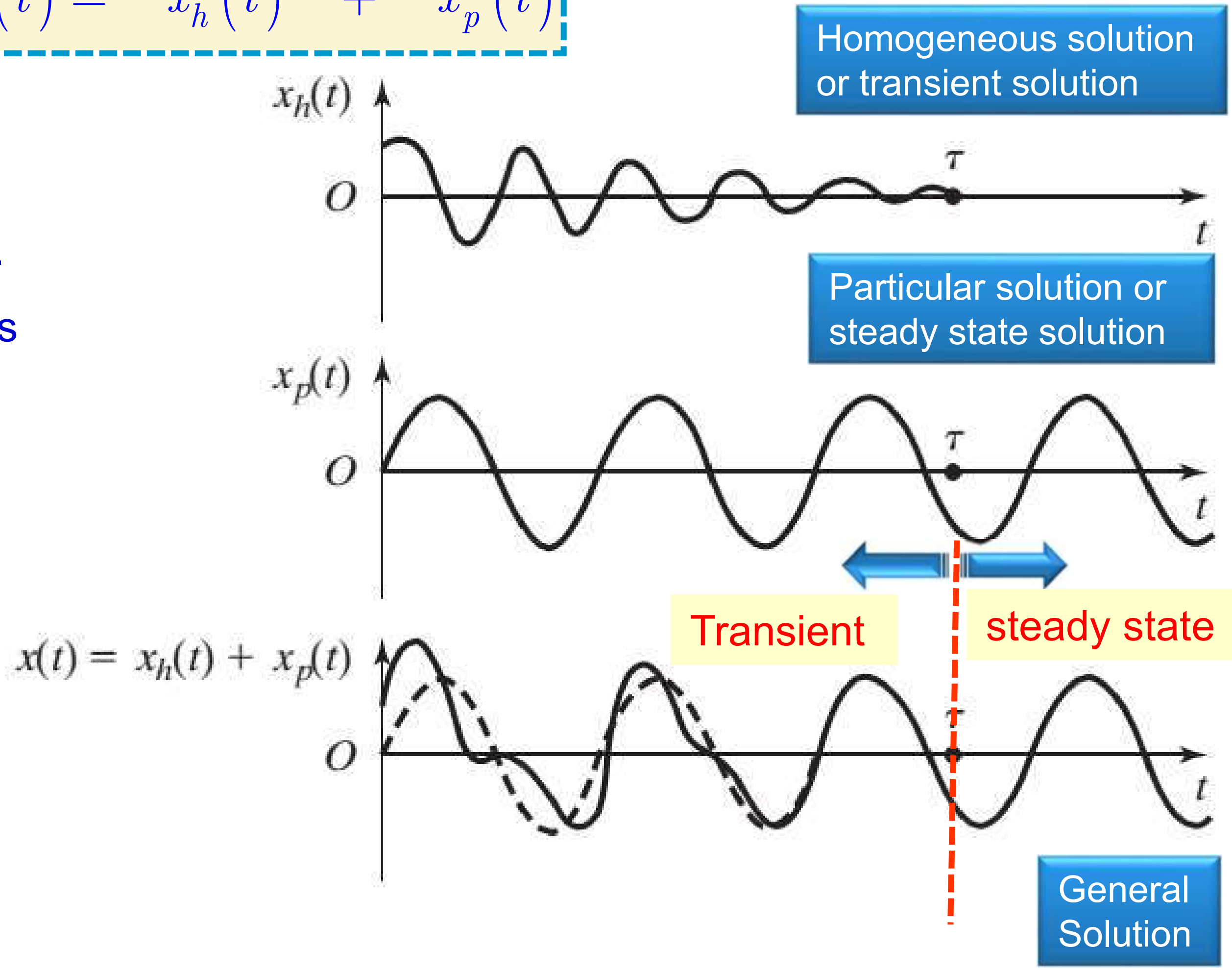
Forced Response and Natural response

- The behavior determined by a forcing function is called a forced response.
- Response due to initial conditions (initial energy storages) is called a natural (free) response.



$$x(t) = x_h(t) + x_p(t)$$

- The behavior determined by a **forcing function** is called a ***forced response***.
- The response due **initial conditions** is called ***natural response***.
- The period between initiation of a response and the ending is referred to as the ***transient period***.
- After the response has become negligibly small, conditions are said to have reached a ***steady state***.



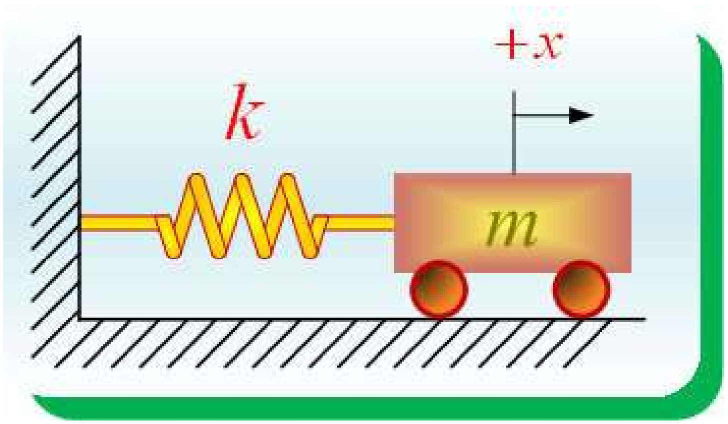
http://math.mit.edu/daimp/ForcedDampedVib.html

http://www.aw-bc.com/ide/idefiles/media/JavaTools/vibefdmp.html

Procedure (Translational Motion)

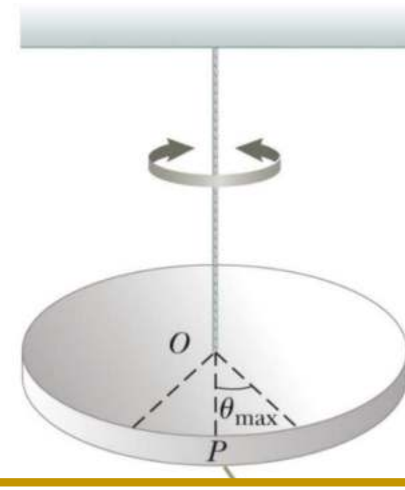
1. Select a suitable coordinate to describe the position of the mass in the system.
2. Determine the static equilibrium configuration of the system and measure the displacement of the mass from its static equilibrium.
3. Draw the Free Body Diagram (FBD) of the mass when a positive displacement and velocity are given to it. Indicate all the active and reactive forces acting on the mass.
4. Apply Newton's Second Law (NSL) of motion to the mass shown by the FBD.

Translational motion



- Linear Displacement, x
- Linear Velocity, x
- Linear Acceleration, x
- Linear Momentum, mx
- Forces

Rotational motion



- Angular Displacement, θ
- Angular Velocity, θ
- Angular Acceleration, θ
- Angular Momentum, $J\theta$
- Moments (torques)

- Find the free response of a rotor that is mounted in bearings as shown in figure 3-8. The moment of inertia of the rotor about the axis of rotation is J . The friction in bearings is viscous friction b. No external torque is applied to the rotor.

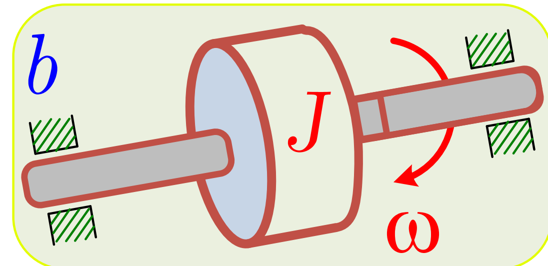


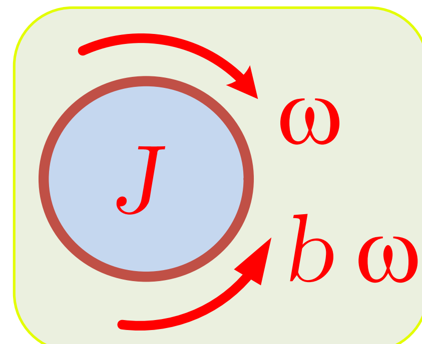
Fig. 3-8 Rotor mounted in bearings

- Apply Newton's second law for a system in rotation.

$$\sum M = J\ddot{\theta} = J\dot{\omega} \qquad \omega(0) = \omega_o$$

$$J\dot{\omega} + b\omega = 0 \quad \Rightarrow \quad \dot{\omega} + (b/J)\,\omega = 0$$

$$\dot{\omega} + \frac{1}{(J/b)}\omega = 0 \quad \Rightarrow \quad \dot{\omega} + \frac{1}{\tau}\omega = 0 \qquad \omega(0) = \omega_o$$



FBD

where $\tau = (J/b)$ is the time constant.

$$\dot{\omega} + \frac{1}{\tau}\omega = 0 \qquad \omega(0) = \omega_o$$

- The above Eq. represents the EOM as well as the mathematical model of the system shown. It represents a ***first order system***. To find the response $\omega(t)$, take LT of both sides of the previous equation

$$\underbrace{\{s\Omega(s) - \omega(0)\}}_{\mathcal{L}[\dot{\omega}]} + \frac{1}{\tau}\underbrace{\{\Omega(s)\}}_{\mathcal{L}[\omega]} = 0$$

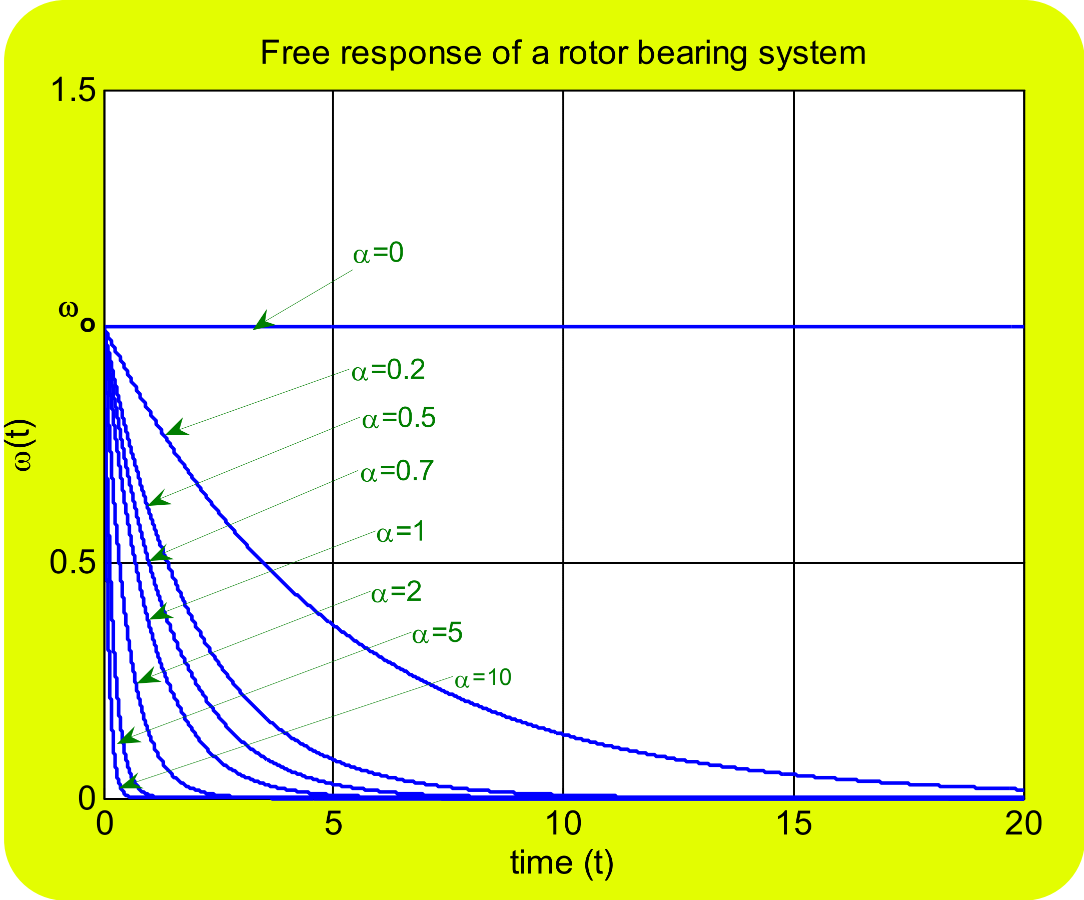
where $\Omega(s) = \mathcal{L}\{\omega(t)\}$

- Rearranging and solving for $\Omega(s)$ gives ➡ $$\Omega(s) = \frac{\omega_\circ}{s + (1/\tau)}$$
- The denominator $s + (1/\tau)$ is known as the characteristic polynomial.
- The equation $s + (1/\tau) = 0$ is known as the characteristic equation.

- Taking inverse LT of the previous Eq. gives the expression of $\omega(t)$

$$\omega(t) = \omega_\circ e^{-(b/J)t} = \omega_\circ e^{-(1/\tau)t} = \omega_\circ e^{-\alpha t}$$

- It is clear that the angular velocity decreases exponentially as shown.
- Since $\lim_{t\to\infty} e^{-(1/\tau)t} = 0$; then for such decaying system, it is convenient to depict the response in terms of a time constant.



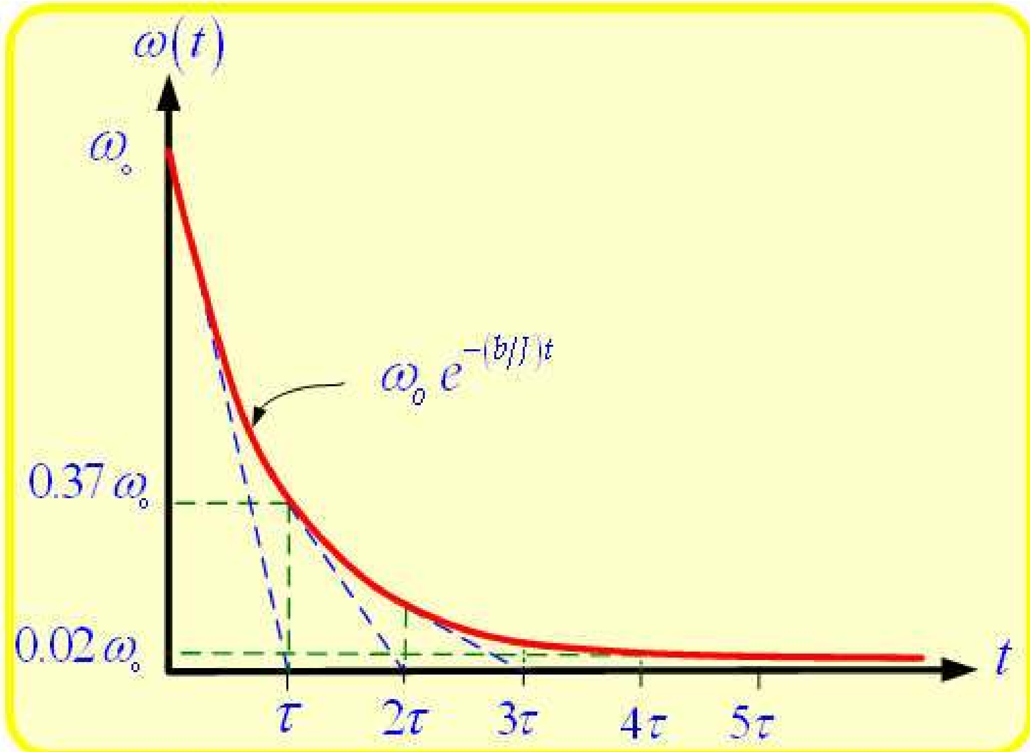
Time Constant

- A ***time constant*** is that value of time that makes the exponent equal to -1.
- For this system, the time constant $\tau = (J/b)$
- When $t = \tau$, the exponent factor is

$$e^{-(t/\tau)} = e^{-(\tau/\tau)} = e^{-1} = 0.368 = 36.8\ \%$$

- This means that when $t = \tau$, the time response is reduced to 36.8 % of its initial value.
- We also have

$$\tau = J\ /\ b \quad \Rightarrow \quad \text{time constant}$$
$$\omega(\tau) = 0.37\ \omega_\circ$$
$$\omega(4\tau) = 0.02\ \omega_\circ$$



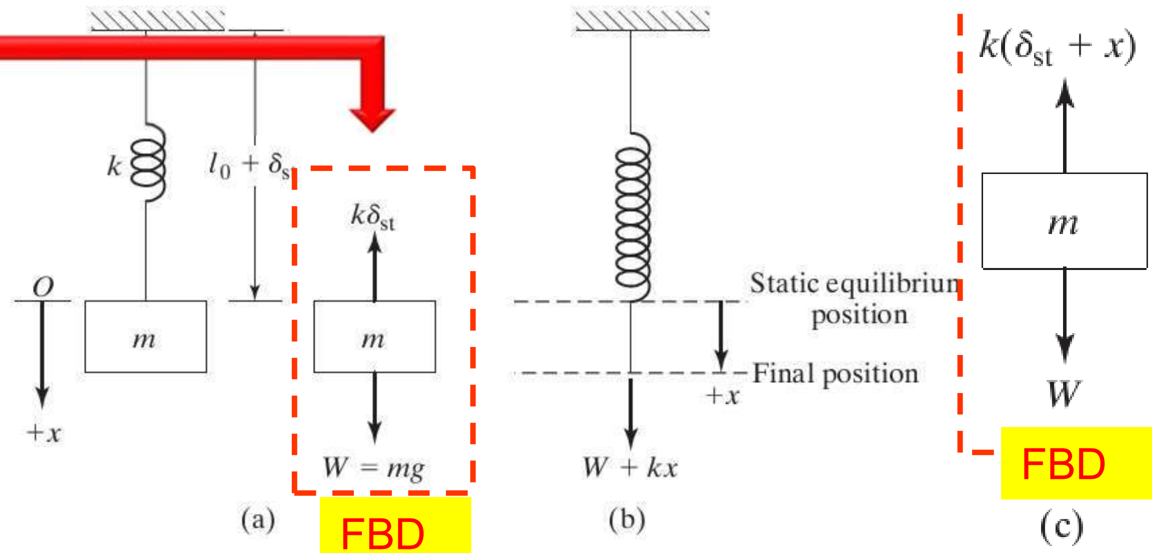
http://www.sciences.univ-nantes.fr/physique/perso/gtulloue/equadiff/equadiff.html

- Consider the configuration of the spring-mass system shown in the figure below.

For static equilibrium

$$W = mg = k\delta_{st}$$

where W = weight of mass m,
δ_{st} = static deflection
g = acceleration due to gravity



- Applying NSL of motion to mass m gives

$$m\ddot{x} = -k(x + \delta_{st}) + W$$

- Since $k\delta_{st} = W$, we obtain

$$m\ddot{x} = -kx \underbrace{-k\delta_{st} + W}$$

static equilibrium

$$\ddot{x} + \left(k/m\right)x = 0$$

- When a mass moves in a vertical direction, we can ignore its weight, provided we measure the position of m from its static equilibrium position.

http://phet.colorado.edu/sims/mass-spring-lab/mass-spring-lab_en.html

Free Vibration

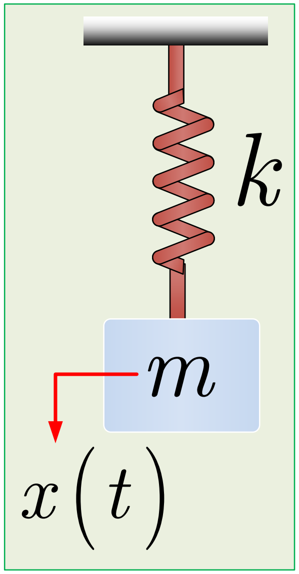
- The EOM of the mass- spring system shown in the Figure can be given by

$$\ddot{x} + \left(k/m\right)x = 0 \quad \text{or} \quad \ddot{x} + \omega_n^2\, x = 0$$

where

$$\omega_n = \sqrt{k/m}$$

is the natural frequency of the system, [rad/s] .



Mass-spring system

- Taking LT of both sides of the above Eq. where $x(0) = x_\circ$ and $x(0) = x_\circ$

$$\underbrace{s^2X(s) - sx(0) - \dot{x}(0)}_{\mathfrak{L}[\ddot{x}]} + \omega_n^2\, X(s) = 0$$

- Rearranging and solving for $X(s)$ gives

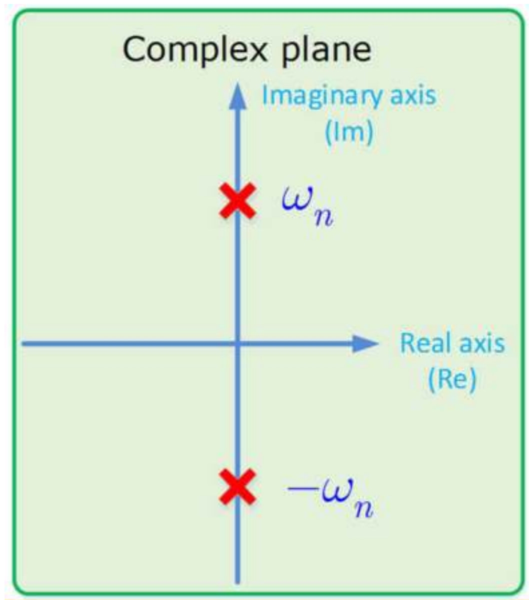
$$X(s) = \frac{sx_\circ + \dot{x}_\circ}{s^2 + \omega_n^2}$$

- Characteristic Eq.

$$s^2 + \omega_n^2 = 0$$

- Poles of the characteristic Eq.

$$s_{1,2} = \pm j\omega_n$$



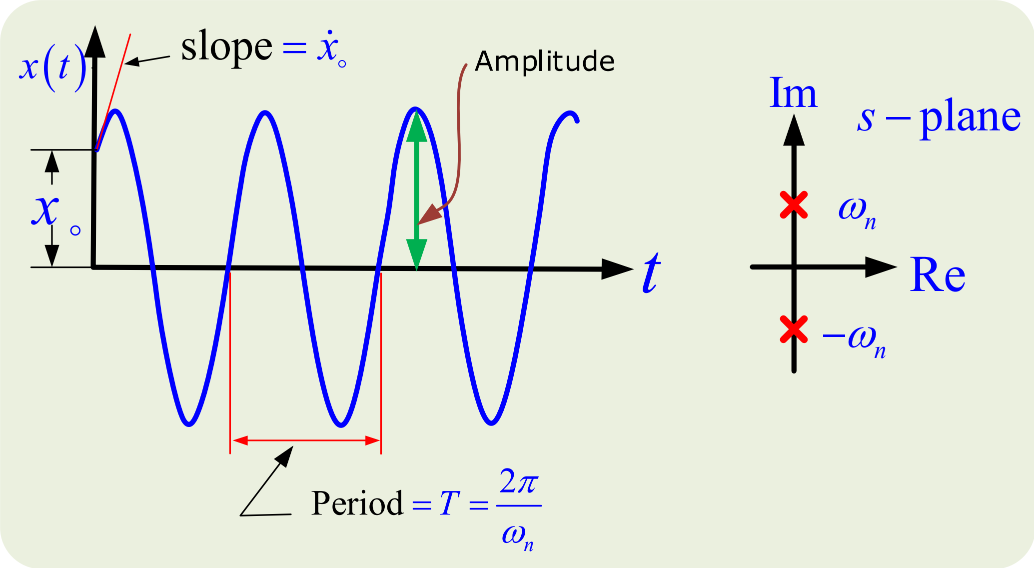
$$X(s) = \frac{sx_\circ + \dot{x}_\circ}{s^2 + \omega_n^2} = \frac{\dot{x}_\circ}{\omega_n}\frac{\omega_n}{s^2 + \omega_n^2} + x_\circ \frac{s}{s^2 + \omega_n^2}$$

- The response $x(t)$ is given by

$$x(t) = \mathfrak{L}^{-1}[X(s)] = \frac{\dot{x}_\circ}{\omega_n}\sin(\omega_n t) + x_\circ \cos(\omega_n t)$$

- The response $x(t)$ consists of a sine and cosine terms and depends on the values of the I. C's $x_\circ$ and $\dot{x}_\circ$.
- Periodic motion described by the above equation is called ***Simple Harmonic Motion (SHM)***.
- If $x(0) = x_\circ = 0$,, the previous equation reduces to

$$x(t) = x_\circ \cos(\omega_n t)$$



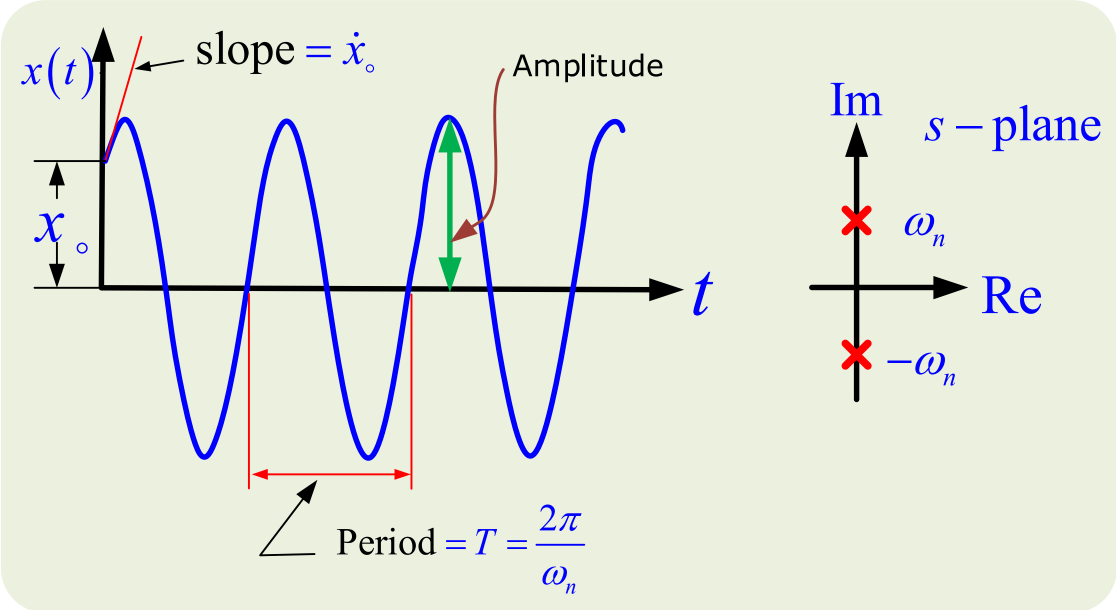
http://www.sciences.univ-nantes.fr/physique/perso/gtulloue/equadiff/equadiff.html

- **The period (T)**: is the time required for a periodic motion to repeat itself.

$$T = \frac{2\pi}{\omega_n}$$

- **The frequency (f)**: of a periodic motion is the number of cycles per second (cps), and the standard unit of frequency is the Hertz (Hz); that is 1 Hz = 1 cps.

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

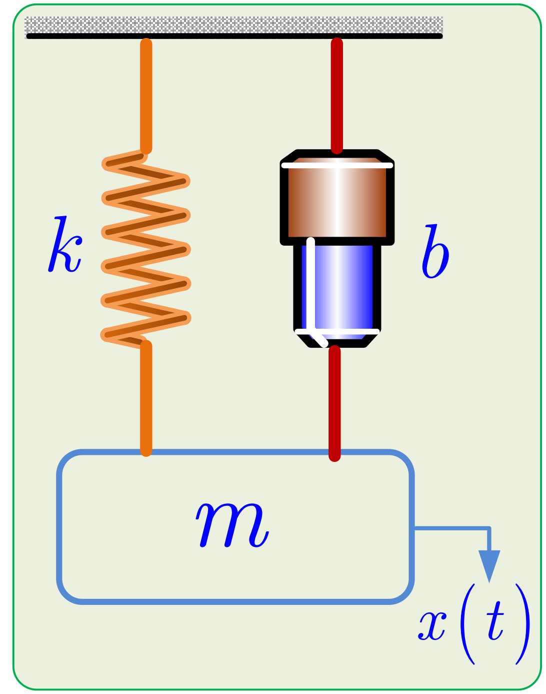


- **Underdamped frequency (ω_n)**: is the frequency in the free vibration of a system having no damping. If the natural frequency is measured in Hz or in cps, it is denoted by f_n. If it is measured in rad/sec, it is denoted by

$$\omega_n = 2\pi f_n = \sqrt{k/m}$$

Spring Mass Damper System

- Obtain the mathematical model of the mechanical system shown involving viscous damping.



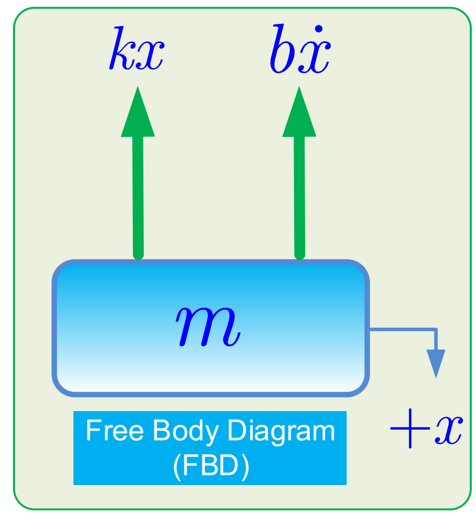
Solution

- The FBD is shown in the figure.
- Applying NSL of motion to a system in translation gives

$$\sum F = m\ddot{x} \quad \Rightarrow \quad -b\dot{x} - kx = m\ddot{x}$$

- Rearranging gives $$m\ddot{x} + b\dot{x} + kx = 0$$

- The above equation represents the mathematical model as well as the free vibration motion for a second order system.



- If $m = 0.1$ kg, $b = 0.4$ N/m.s and $k = 4$ N/m, the above Eq. becomes

$$0.1\ddot{x} + 0.4\dot{x} + 4x = 0 \quad \gg \quad \ddot{x} + 4\dot{x} + 40\,x = 0$$

- To obtain the free response $x(t)$, assume $x(0) = x_\circ$ and $x(0) = 0$.

- Taking LT of both sides of the previous Eq. gives

$$\underbrace{\left[s^2X(s) - s\,x(0) - \dot{x}(0)\right]}_{\mathfrak{L}[\ddot{x}(t)]} + 4\underbrace{\left[sX(s) - x(0)\right]}_{\mathfrak{L}[\dot{x}(t)]} + 40\underbrace{\left[X(s)\right]}_{\mathfrak{L}[x(t)]} = 0$$

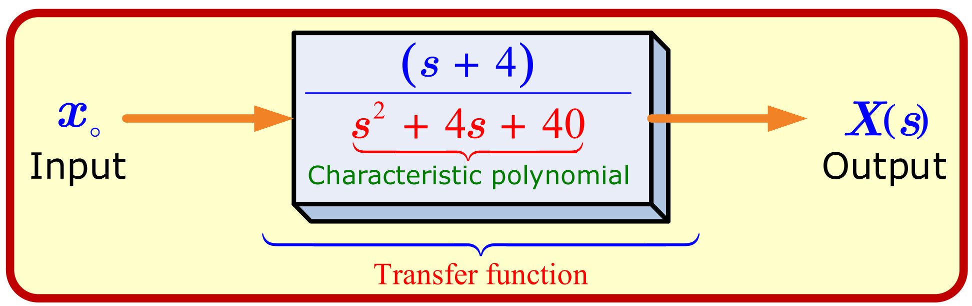
- Substituting $x(0) = x_\circ$ & $x(0) = 0$, rearranging and solving for $X(s)$ gives

$$X(s) = \frac{(sx_\circ + 4x_\circ)}{s^2 + 4s + 40} = \frac{(s+4)}{\underbrace{s^2 + 4s + 40}_{\text{Characteristic polynomial}}}x_\circ$$

- The above Eq. can be written as

$$G(s) = \frac{X(s)}{x_\circ} = \frac{(s+4)}{\underbrace{s^2 + 4s + 40}_{\text{Characteristic polynomial}}}$$

- where $G(s)$ is referred to as the ***transfer function*** of the system. It gives the relationship between the output $X(s)$ and the input $x_\circ$.



- The characteristic equation of the system is $s^2+4s+40=0$
- The roots of the characteristic Eq. can be found by completing the square ,i.e.,

$$s^2+4s+40=\underbrace{s^2+4s+4}_{(s+2)^2}+36=(s+2)^2+6^2=0 \quad \gg \quad s_{1,2}=-2\pm j6$$

- The expression of $X(s)$ can be written

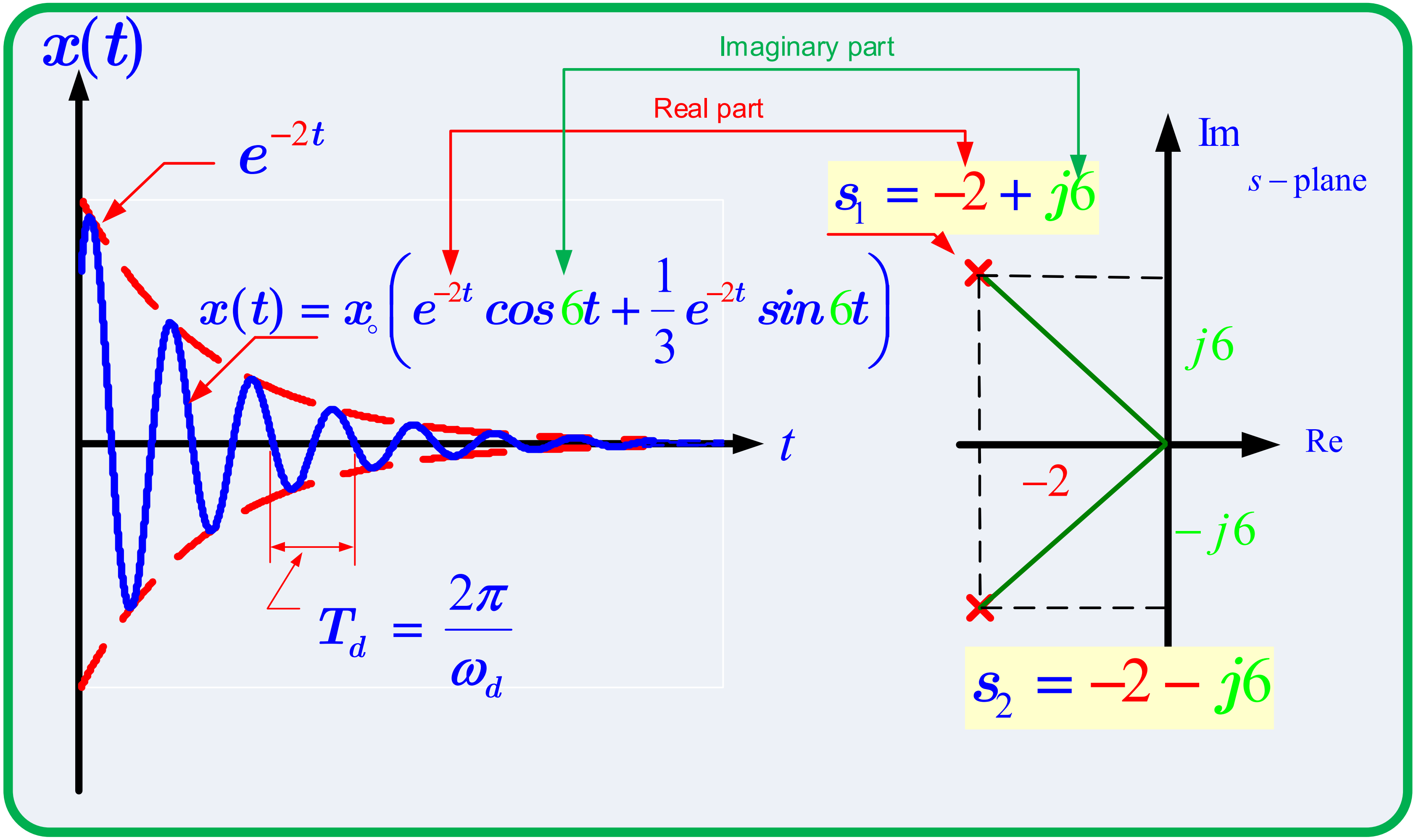
$$X(s)=\frac{(s+4)}{s^2+4s+40}x_\circ=\frac{(s+2+2)}{(s+2)^2+6^2}x_\circ=\frac{(s+2)}{(s+2)^2+6^2}x_\circ+\frac{2}{(s+2)^2+6^2}x_\circ$$
$$=\frac{(s+2)}{(s+2)^2+6^2}x_\circ+\frac{1}{3}\frac{6}{(s+2)^2+6^2}x_\circ$$

- Solving for $x(t)$ yields

$$x(t)=\mathcal{L}^{-1}\left[X(s)\right]=x_\circ\left(e^{-2t}\cos 6t+\frac{1}{3}e^{-2t}\sin 6t\right)$$

or

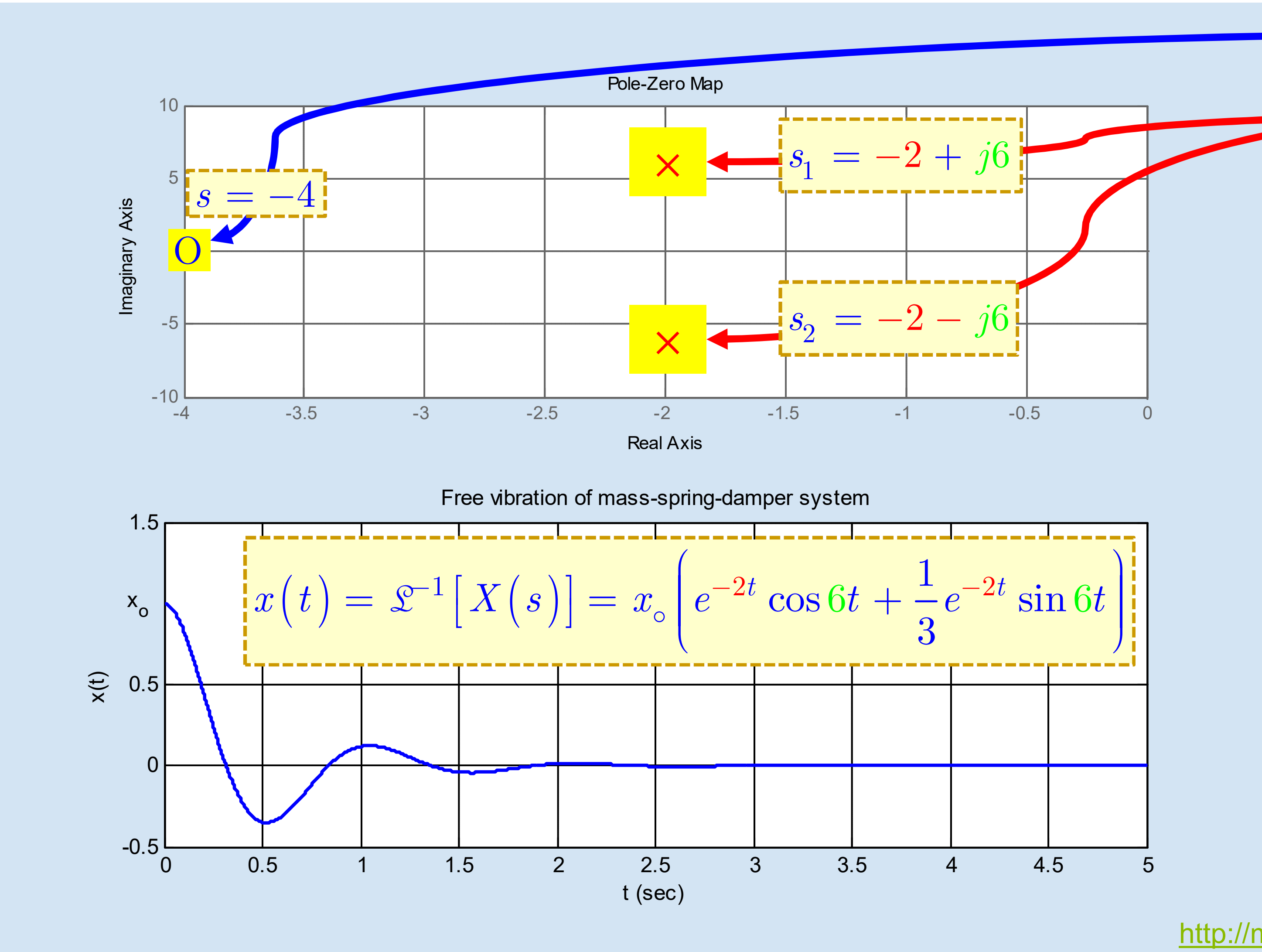
$$x(t)=\frac{\sqrt{10}}{3}x_\circ e^{-2t}\left(\sin 6t+71.56^\circ\right)$$



Free Vibration of the mass-spring-damper system described by

$x + 4x + 40\,x = 0$ with initial conditions $x(0) = x_{\circ}$ & $x(0) = 0$.

Pole-Zero Map



$$G(s) = \frac{(s+4)}{\underbrace{s^2 + 4s + 40}_{\text{Characteristic polynomial}}}$$

Zeros of $G(s)$

$s + 4 = 0$

$s = -4$

Symbol in PZ-map: O

Poles of $G(s)$

$s^2 + 4s + 40 = 0$

$s_{1,2} = -2 \pm j6$

Symbol in PZ-map: ×

http://mathlets.org/mathlets/damped-vibrations/

Free Vibration of the mass-spring-damper system described by $x + 4x + 40\,x = 0$ with I.C's $x(0) = x_\circ$ & $x(0) = 0$.

Work »

The ***work*** done in a mechanical system is the product of a force and a distance (or a torque and the angular displacement) through which the force is exerted with both force and distance measured in the same direction.

- Units of work in SI units:

$$[\text{work}] = [\text{force} \times \text{distance}] = [\text{N}\cdot\text{m}] = [\text{Joule}] = [\text{J}]$$

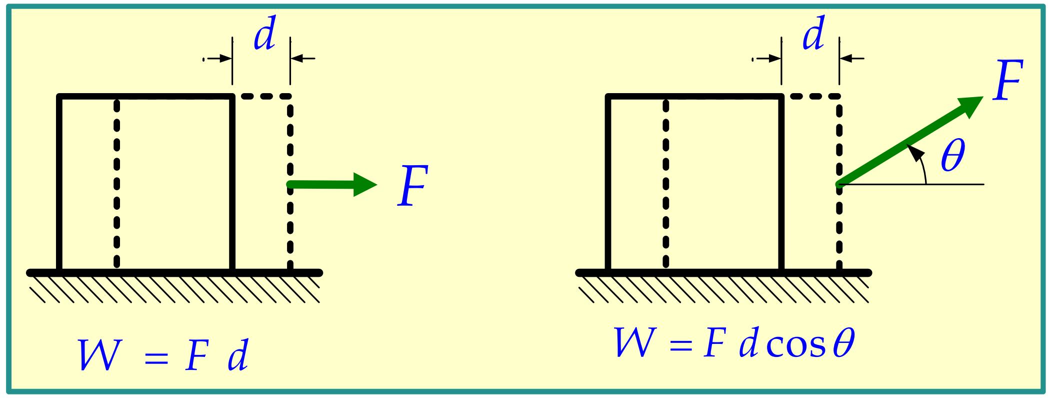
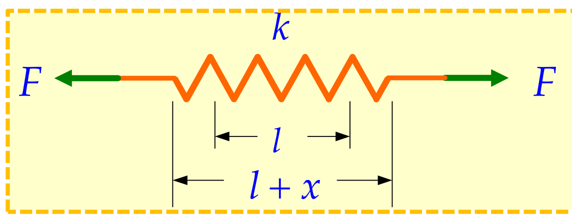
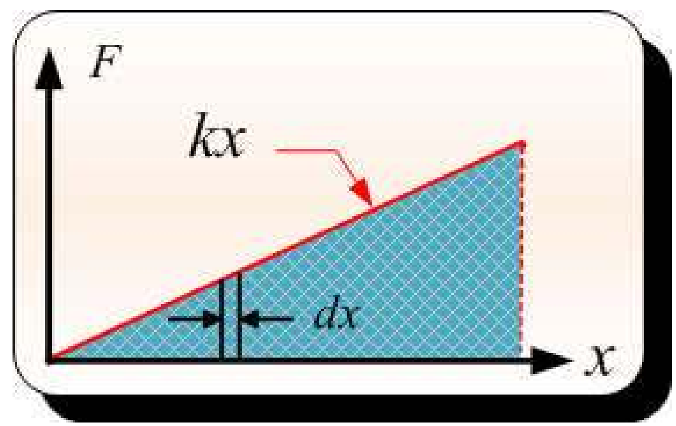


Figure 3-19 Work done by a force

- Work done by a linear spring

$$W = \int_0^x \underbrace{k\,x}_{F}\,dx = \frac{1}{2}k\,x^2$$





- Work done by a torsional spring

$$W = \int_0^\theta k_t\,\theta\,d\theta = \frac{1}{2}k_t\theta^2$$

Energy

- **Energy** can be defined as **ability to do work**.
- **Energy** can be found in **many different forms** and **can be converted from one form to another**.
- For instance, an **electric motor** converts electrical energy into mechanical energy, a **battery** converts chemical energy into electrical energy, and so forth.

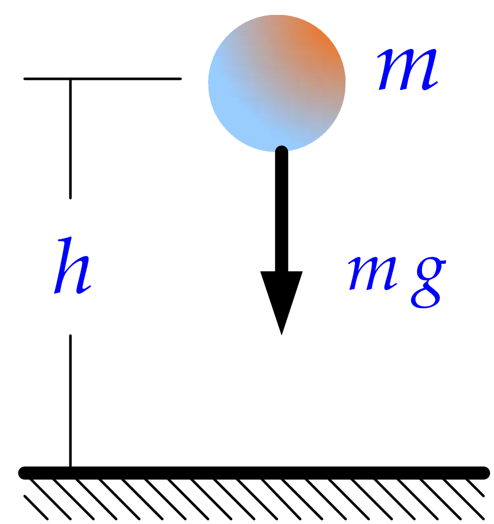
- According to the law of **conservation of energy**, **energy can be neither created nor destroyed**.
- This means that the increase in the total energy within a system is equal to the **net energy input** to the system. So if there is **no energy input**, there is **no change in the total energy** of the system.

Potential Energy

- The **Energy** that a body possesses because of its position is called ***potential energy.***
- In mechanical systems, only mass and spring can store potential energy.
- The change in the potential energy stored in a system equals the work required to change the system's configuration.
- **Potential energy** is always measured with reference to some chosen level and is relative to that level.

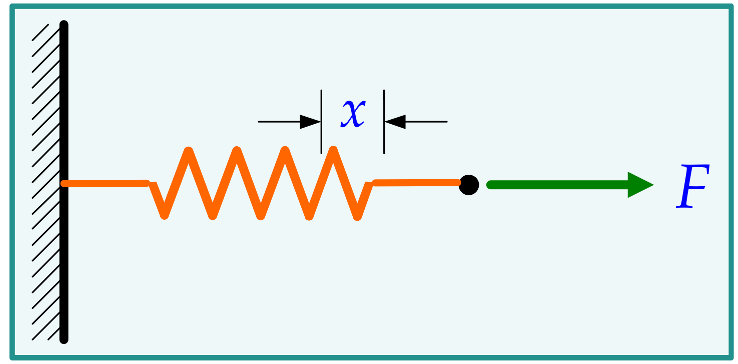
- Refer to Figure, the potential energy U of the mass m is

$$U = \int_0^h mg\,dx = mgh$$



- For a translational spring, the **potential energy** U , (sometimes called **strain energy** which is **potential energy** that is due to **elastic deformations**) is:

$$U = \int_0^x F \, dx = \int_0^x k\,x \, dx = \frac{1}{2} k\,x^2$$



- If the initial and final values of x are x_1 and x_1, respectively, then

Change in potential energy = $$\Delta U = \int_{x_1}^{x_2} F \, dx = \int_{x_1}^{x_2} k\,x \, dx = \frac{1}{2} k\,x_2^2 - \frac{1}{2} k\,x_1^2$$

- Similarly, for a torsional spring, we obtain

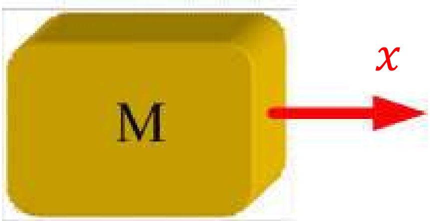
Change in potential energy = $$\Delta U = \int_{\theta_1}^{\theta_2} T \, d\theta = \int_{\theta_1}^{\theta_2} k_T\,\theta \, d\theta = \frac{1}{2} k_T\,\theta_2^2 - \frac{1}{2} k_T\,\theta_1^2$$

Kinetic Energy

- Only inertia elements can store **kinetic energy** in mechanical systems.

- **Kinetic energy** of a body in **translation** having a mass m and a linear velocity x

Translation

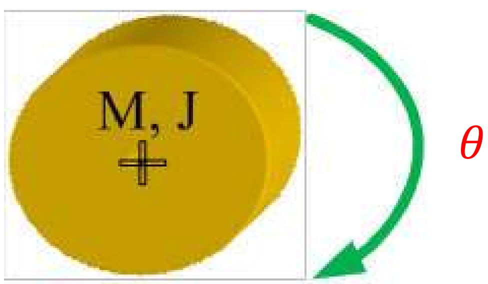


$$T = \frac{1}{2} m \, \dot{x}^2$$

- **Change in Kinetic energy:**

$$\Delta T = \Delta W = \frac{1}{2} m \, \dot{x}_2^2 - \frac{1}{2} m \, \dot{x}_1^2$$

- **Kinetic energy** of a body in **rotation** having a mass moment of inertia J and an angular velocity θ



Rotation

$$T = \frac{1}{2} J \, \dot{\theta}^2$$

- **Change in Kinetic energy:**

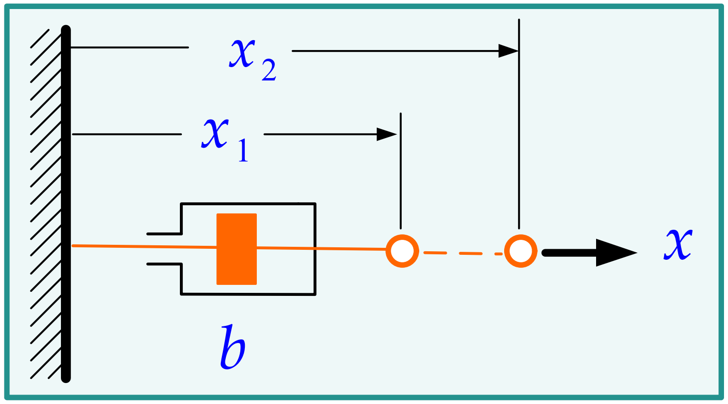
$$\Delta T = \Delta W = \frac{1}{2} J \, \dot{\theta}_2^2 - \frac{1}{2} J \, \dot{\theta}_1^2$$

Dissipated Energy

- Consider the damper shown in figure in which one end is fixed and the other end is moved from x_1 to x_2 .

- **The dissipated energy** of a body in **translation** having a mass W in the damper is equal to.

$$\Delta W = \int_{x_1}^{x_2} F \, dx = \int_{x_1}^{x_2} \underbrace{b \, \dot{x}}_{F} \, dx = b \int_{t_1}^{t_2} \dot{x} \frac{dx}{dt} \, dt = b \int_{t_1}^{t_2} \dot{x}^2 \, dt$$



Power »

- Power is the time rate of doing work. That is,

$$\text{Power} = P = \frac{dW}{dt}$$

dW denotes work done during time interval dt

- The unit of power is

$$[\text{Power}] = \left[\frac{\text{N}\cdot\text{m}}{\text{s}}\right] = \left[\frac{\text{Joule}}{\text{s}}\right] = [\text{Watt}] = \text{W}$$

Passive Elements »

- Non-energy producing element.
- They can only store energy, not generate it such as springs and masses.

Active Elements »

- Energy producing elements such as external forces and torques.

Energy Method for Deriving Equations of Motion (EOM)

- Equations of motion (EOM) are derived from the fact that the total energy of a system remains the same if no energy enters or leaves the system.

Conservative Systems

- Systems that do not involve friction (damping) are called *conservative systems*.

$$\underbrace{\Delta\left(T+U\right)}_{\text{Change in the total energy}} = \underbrace{\Delta W}_{\text{Net work done on the system by external forces}}$$

- If no external energy enters the system, ($\Delta W = 0$), no work done by external forces, then

$$\Delta\left(T+U\right) = 0$$

or

$$T + U = \text{Constant}$$

Conservation of energy only for conservative systems (No friction or damping)

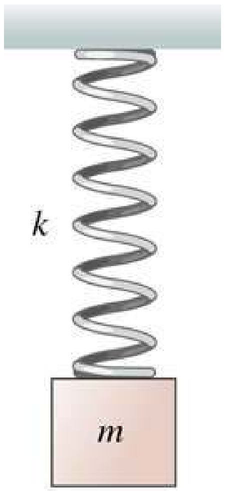
Example

A system is said to be conservative if no energy is lost due to friction or energy dissipating.

- If no work is done on the conservative system by external forces, the total energy of the system remains constant. Thus the principle of conservation of energy can be expressed as:

$$T + U = \text{constant} \quad \gg \quad \frac{d}{dt}(T + U) = 0 \qquad \text{(E-1)}$$

- Kinetic energy $\gg$ $T = \frac{1}{2} m x^2$ (E-2)
- Strain energy $\gg$ $U = \frac{1}{2} k x^2$ (E-3)



- Substitution of Eqs. (E-2) & (E-3) into Eq. (E-1) yields the desired equation

$$T + U = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2$$

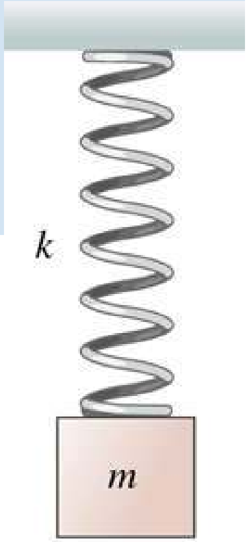
$$\frac{d}{dt}(T + U) = \frac{d}{dt}\left(\frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2\right) = 0 \quad \gg \quad \frac{1}{2} 2m\dot{x}\ddot{x} + \frac{1}{2} 2kx\dot{x} = 0$$

$\dot{x}\left(m\ddot{x} + kx\right) = 0$ $\gg$ Since $x \neq 0$, then $\gg$ $\ddot{x} + \left(k/m\right)x = 0$ (E-4)

Example

- The natural frequency of a conservative system can be obtained from a consideration of the kinetic energy and the potential energy of the system.
- Assuming that the datum line for the potential energy at the equilibrium state is zero.
- For such a conservative system, the max. kinetic energy equals the max. potential energy ,

$$T_{max} = U_{max}$$



Example

- The spring force is a conservative force, so in a frictionless system the energy is constant.
- The displacement of the mass-spring system is given by

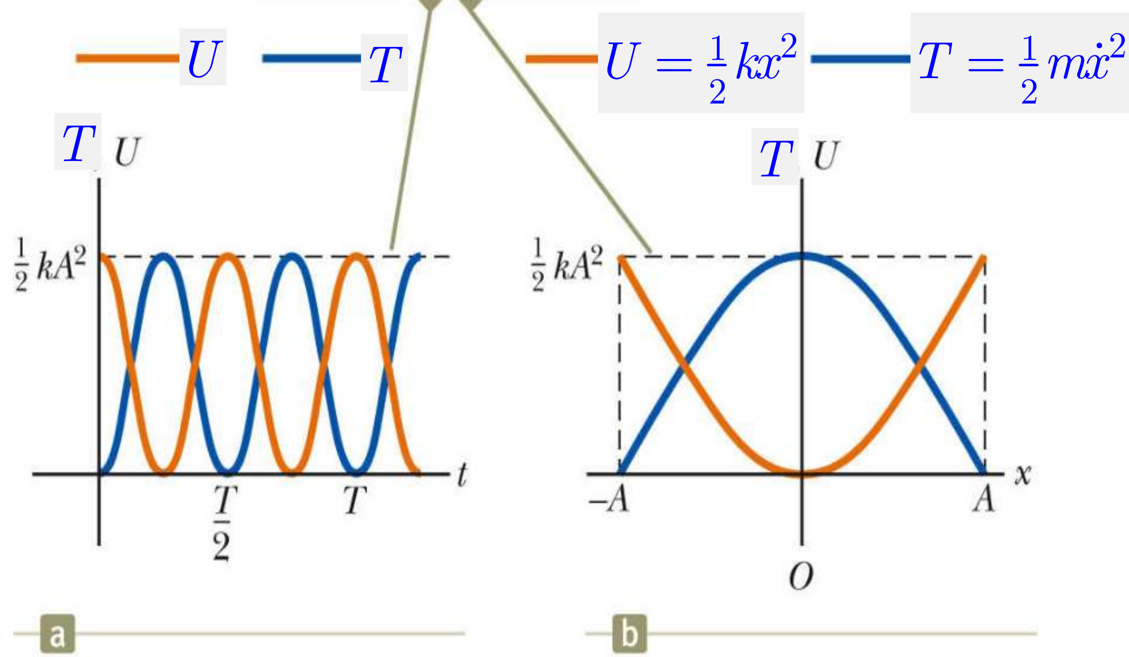
$$x = A\sin\left(\omega t + \phi\right)$$

- **Kinetic energy** >> $T = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}\left(\frac{k}{\omega^2}\right)\omega^2 A^2\cos^2\left(\omega t + \phi\right) = \frac{1}{2}kA^2\cos^2\left(\omega t + \phi\right)$
- **Strain energy** >> $U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\sin^2\left(\omega t + \phi\right)$
- **Total energy** >> $E = T + U = \frac{1}{2}kA^2\left[\cos^2\left(\omega t + \phi\right) + \sin^2\left(\omega t + \phi\right)\right] = \frac{1}{2}kA^2$

- **Total energy** »» $E = T + U = \frac{1}{2}kA^2$

In either plot, notice that $T + U = \text{constant}$

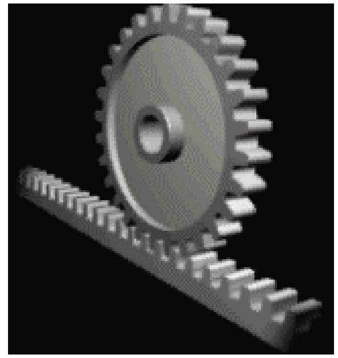
- The total energy is $E = \frac{1}{2}kA^2$ constant at all times, and is (proportional to the square of the amplitude).
- Energy is continuously being transferred between potential energy stored in the spring, and the kinetic energy of the block.

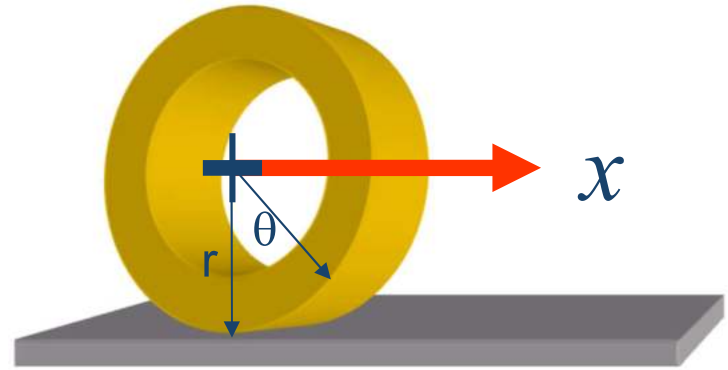


- The disc shown in the figure rolls without slipping find the equivalent mass (inertia).

Solution

$$T = \underbrace{\frac{1}{2} m\dot{x}^2}_{Translation} + \underbrace{\frac{1}{2} J\dot{\theta}^2}_{Rotation}$$





Non slip condition »

$$x = r\theta$$

- **Equivalent translational mass**

$$T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} J \frac{\dot{x}^2}{r^2} = \frac{1}{2}\left(m + \frac{J}{r^2}\right)\dot{x}^2 = \frac{1}{2} m_{eq}\dot{x}^2$$

»

$$m_{eq} = \left(m + \frac{J}{r^2}\right)$$

- **Equivalent rotational inertia**

$$T = \frac{1}{2} mr^2\dot{\theta}^2 + \frac{1}{2} J\dot{\theta}^2 = \frac{1}{2}\left(mr^2 + J\right)\dot{\theta}^2 = \frac{1}{2} J_{eq}\dot{\theta}^2$$

»

$$J_{eq} = \left(mr^2 + J\right)$$

Example:

- Use the energy method to calculate the EOM and natural frequency of an airplane's steering mechanism for the nose wheel of its landing gear. The mechanism is modeled as the SDOF system illustrated in the figure below. The steering wheel and tire assembly are modeled as being fixed at ground for this calculation. The steering rod gear system is modeled as a linear spring and mass system (m, k_2) oscillating in the x -direction. The shaft-gear mechanism is modeled as the disk of inertia J and torsional stiffness k_1. The gear J turns through the angle θ such that the disk does not slip on the mass. Obtain an equation in the linear motion x.

Solution

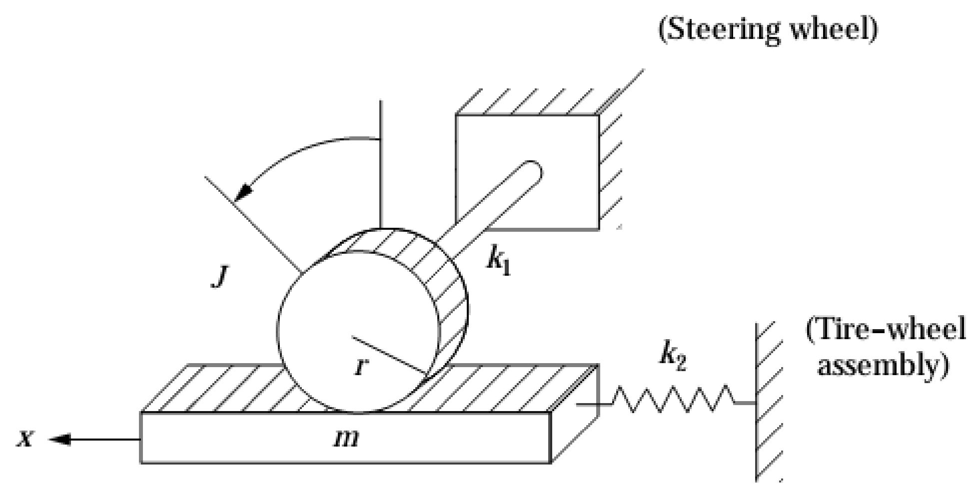
- Kinematic relation $x = r\theta$ and $x = r\theta$

- Kinetic Energy: »

$$T = \underbrace{\frac{1}{2}m\dot{x}^2}_{\text{Translation}} + \underbrace{\frac{1}{2}J\dot{\theta}^2}_{\text{Rotation}}$$

- Potential Energy: »

$$U = \underbrace{\frac{1}{2}k_2 x^2}_{\text{Translation}} + \underbrace{\frac{1}{2}k_1\theta^2}_{\text{Rotation}}$$



- Total Energy:

$$T + U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}k_2 x^2 + \frac{1}{2}k_1\theta^2$$

- Substitute $\theta = x/r$ and $\theta = x/r$ and rearranging gives

$$T + U = \frac{1}{2}\underbrace{\left(m + \frac{J}{r^2}\right)}_{M_{eq}}\dot{x}^2 + \frac{1}{2}\underbrace{\left(k_2 + \frac{k_1}{r^2}\right)}_{K_{eq}}x^2$$

Equivalent Mass:

$$M_{eq} = m + J/r^2$$

Equivalent Stiffness:

$$J_{eq} = k_2 + k_1/r^2$$

- Differentiating the above expression wrt time gives

$$\frac{\mathrm{d}}{\mathrm{d}t}(T + U) = M_{eq}\dot{x}\ddot{x} + K_{eq}x\dot{x} = 0$$

- Since $x \neq 0$, the EOM simplifies to

$$M_{eq}\ddot{x} + K_{eq}x = 0$$ or $$\ddot{x} + \omega_n^2 x = 0$$

- The natural frequency of the system is given by

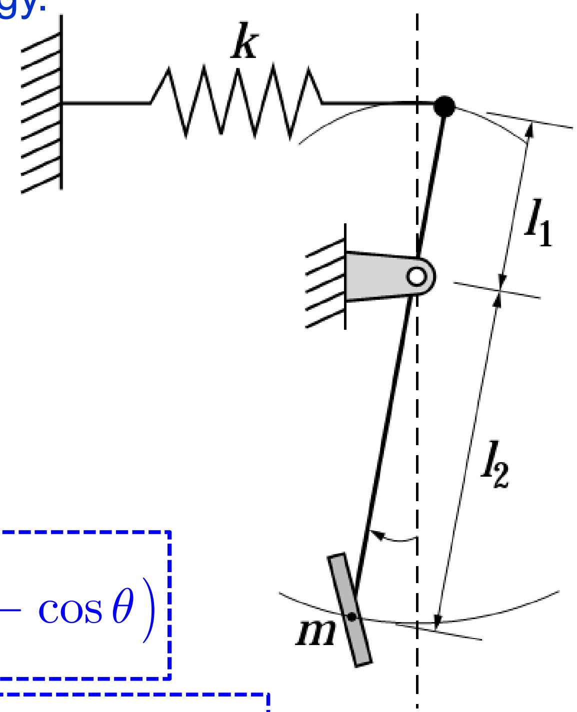
$$\omega_n = \sqrt{\frac{K_{eq}}{M_{eq}}} = \sqrt{\frac{k_2 + \frac{k_1}{r^2}}{m + \frac{J}{r^2}}}$$

Example:

- A control pedal of an aircraft can be modeled as the single-degree-of-freedom system as shown in the figure below. Consider the lever as a massless shaft and the pedal as a lumped mass at the end of the shaft. Use the energy method to determine the equation of motion in θ and calculate the natural frequency of the system. Assume the spring to be unstretched at $\theta = 0$.

Solution

- Let the mass at $\theta = 0$ be the lowest point for potential energy.
- Then, the height of the mass m is $(1 \quad \cos\theta) l_2$.



- Kinematic relation $x = l_1 \theta$.
- Kinetic Energy: »

$$T = \frac{1}{2} m \dot{x}^2 = \frac{1}{2} m\, l_2^2\, \dot{\theta}^2$$

- Potential Energy: »

$$U = \frac{1}{2} k \left(l_1 \theta\right)^2 + m g l_2 \left(1 - \cos\theta\right)$$

- Total Energy: »

$$T + \mathrm{U} = \frac{1}{2} m\, l_2^2\, \dot{\theta}^2 + \frac{1}{2} k \left(l_1 \theta\right)^2 + m g l_2 \left(1 - \cos\theta\right)$$

- Derivative of Total Energy:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(T + \mathrm{U}\right) = m\, l_2^2\, \dot{\theta}\ddot{\theta} + k\left(l_1^2 \theta\right)\dot{\theta} + m g l_2 \left(\sin\theta\right)\dot{\theta} = 0$$

- For small angles measured in radians $\sin\theta \approx \theta$.
- Dividing by $\theta \neq 0$, and rearranging gives :

$$m\,l_2^2\,\ddot{\theta} + \left(kl_1^2 + mgl_2\right)\theta = 0$$

Equation of motion of the system.

$$I_{eq}\,\ddot{\theta} + K_{eq}\theta = 0$$

Equivalent Inertia: $I_{eq} = ml_2^2$

Equivalent Stiffness: $K_{eq} = kl_1^2 + mgl_2$

- The above Eq. is a harmonic Eq. that can be written in a standard form as

$$\ddot{\theta} + \frac{K_{eq}}{I_{eq}}\theta = 0$$

or

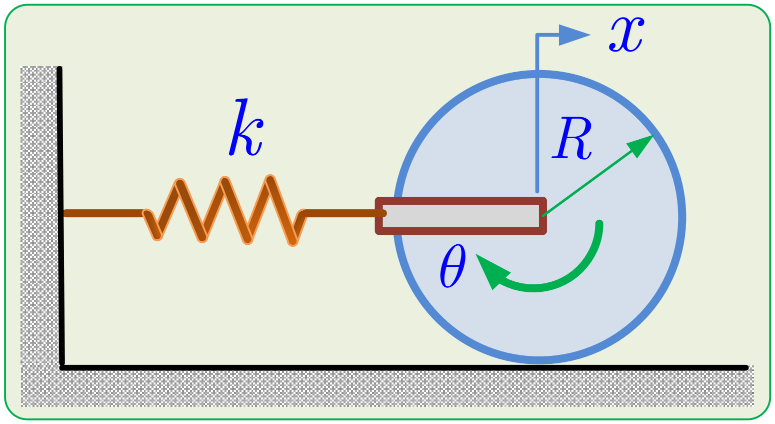
$$\ddot{\theta} + \omega_n^2\theta = 0$$

$$\omega_n = \sqrt{\frac{K_{eq}}{I_{eq}}} = \sqrt{\frac{kl_1^2 + mgl_2}{m\,l_2^2}}$$

- Figure 3-17 shows a homogeneous cylinder of radius R and mass m that is free to rotate about its axis of rotation and that is connected to the wall through a spring.
- Assuming that the cylinder rolls on a rough surface without sliding, obtain the kinetic energy and potential energy of the system.
- Derive the EOM from the fact that the total energy is constant. Assume that x and θ are measured from respective equilibrium positions.

Solution

- Kinematic relation $x = R\theta$.



- Kinetic Energy:

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J\dot{\theta}^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J\left(\frac{\dot{x}}{R}\right)^2 = \frac{1}{2}\underbrace{\left(m + \frac{J}{R^2}\right)}_{m_{eq}}\dot{x}^2$$

- Potential Energy: $$U = \frac{1}{2}kx^2$$

- Since the total energy $(T + U) = \text{const}$ in this conservative system, it follows that

$$T + U = \frac{1}{2}\left(m + \frac{J}{R^2}\right)\dot{x}^2 + \frac{1}{2}kx^2 = \text{const.}$$

- Since $J = \frac{1}{2}mR^2$, the previous Eq. can be written as

$$T + U = \frac{3}{4}m\dot{x}^2 + \frac{1}{2}kx^2 = \text{const.}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(T + \mathrm{U}\right) = \frac{3}{2}m\dot{x}\ddot{x} + kx\dot{x} = 0$$

or

$$\frac{3}{2}m\ddot{x} + kx = 0$$

or

$$\ddot{x} + \frac{2k}{3m}x = 0 \qquad \ddot{x} + \omega_n^2 x = 0$$

- The natural frequency of the system is $\omega_n = \sqrt{\frac{2k}{3m}}$

http://www.mwit.ac.th/~physicslab/applet_04/fun@learning/JAVA/dho/dho2.html
http://www.geneseo.edu/~heap/applets/Spring/spring.html
http://www.cabrillo.edu/~jmccullough/Applets/oscillations.html
http://phet.colorado.edu/en/simulation/mass-spring-lab

QUESTIONS … …?