Jazan University
Mechanical Engineering Department

CHAPTER 3

FORCE ANALYSIS OF MECHANISMS

EngM271 Theory of Machines

DR. Amr Assie

September 2018

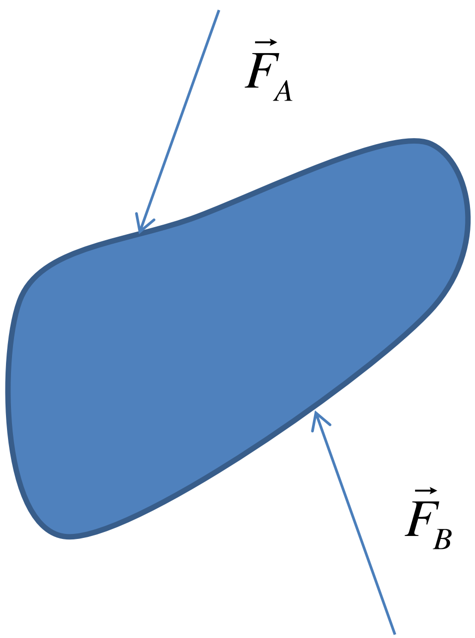
STATIC FORCE ANALYSIS OF MECHANISMS

Two force members

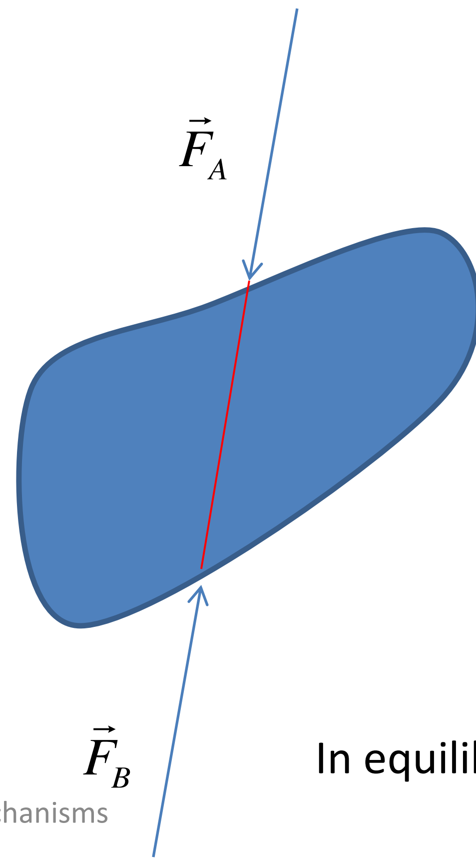
$$\sum \vec{F} = \vec{F}_A + \vec{F}_B = 0$$

$$\vec{F}_A = -\vec{F}_B$$

This requires that two force members should have forces equal in magnitudes and opposite in directions for equilibrium.



Not in equilibrium

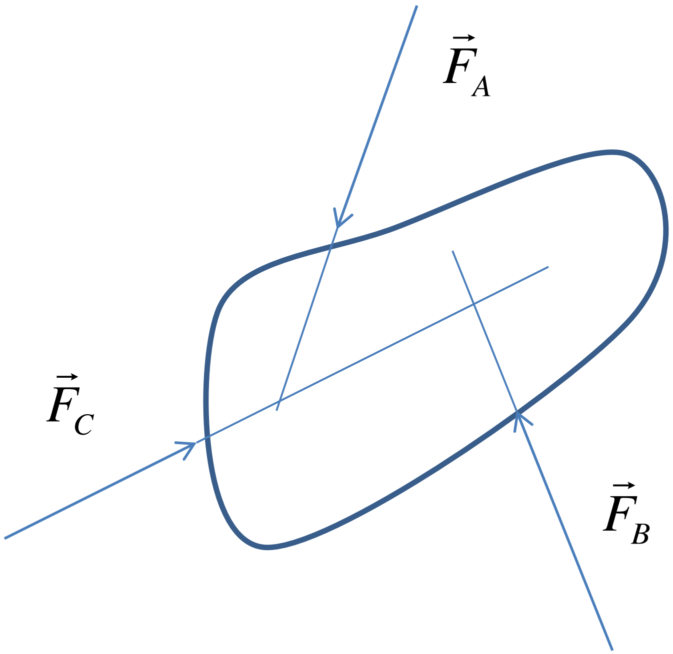


In equilibrium

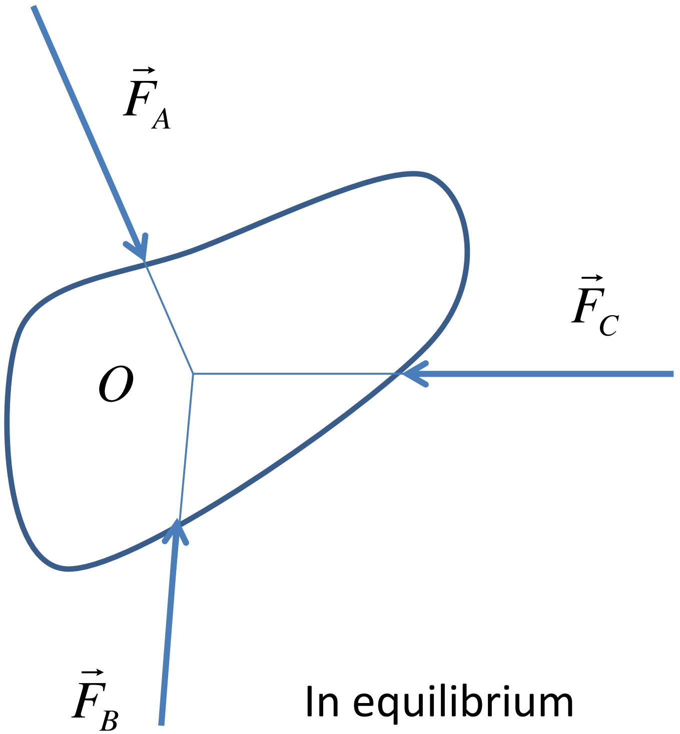
Three force members

$$\sum \vec{F} = \vec{F}_A + \vec{F}_B + \vec{F}_C = 0$$

If three forces are coplanar, if their line of actions intersects at a common point and if the vector sum of the forces is zero three forces are in equilibrium.

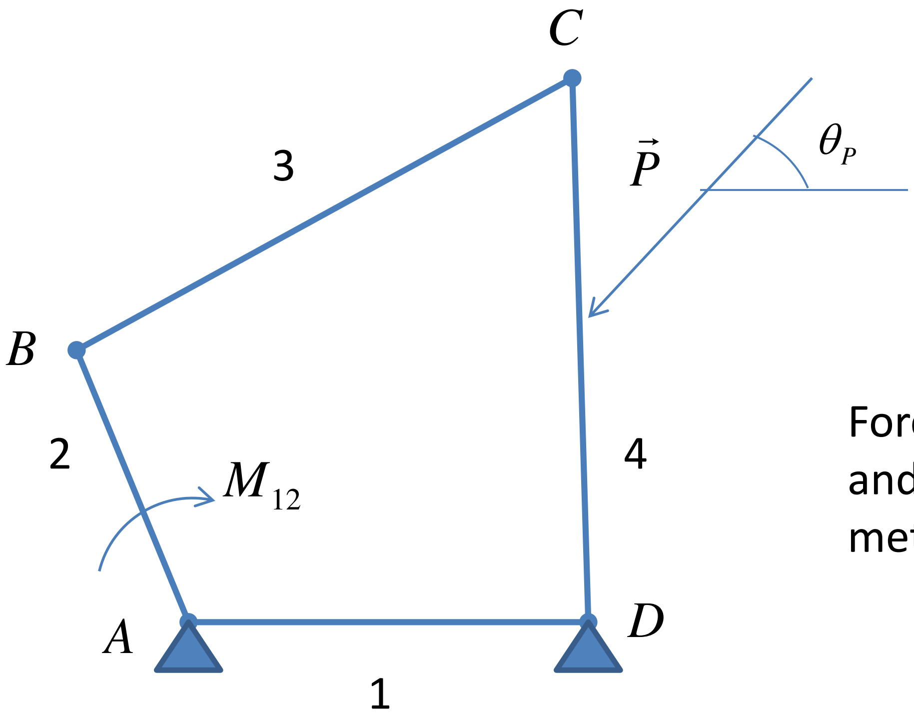


Not in equilibrium



In equilibrium

Four-bar mechanism
Graphical solution

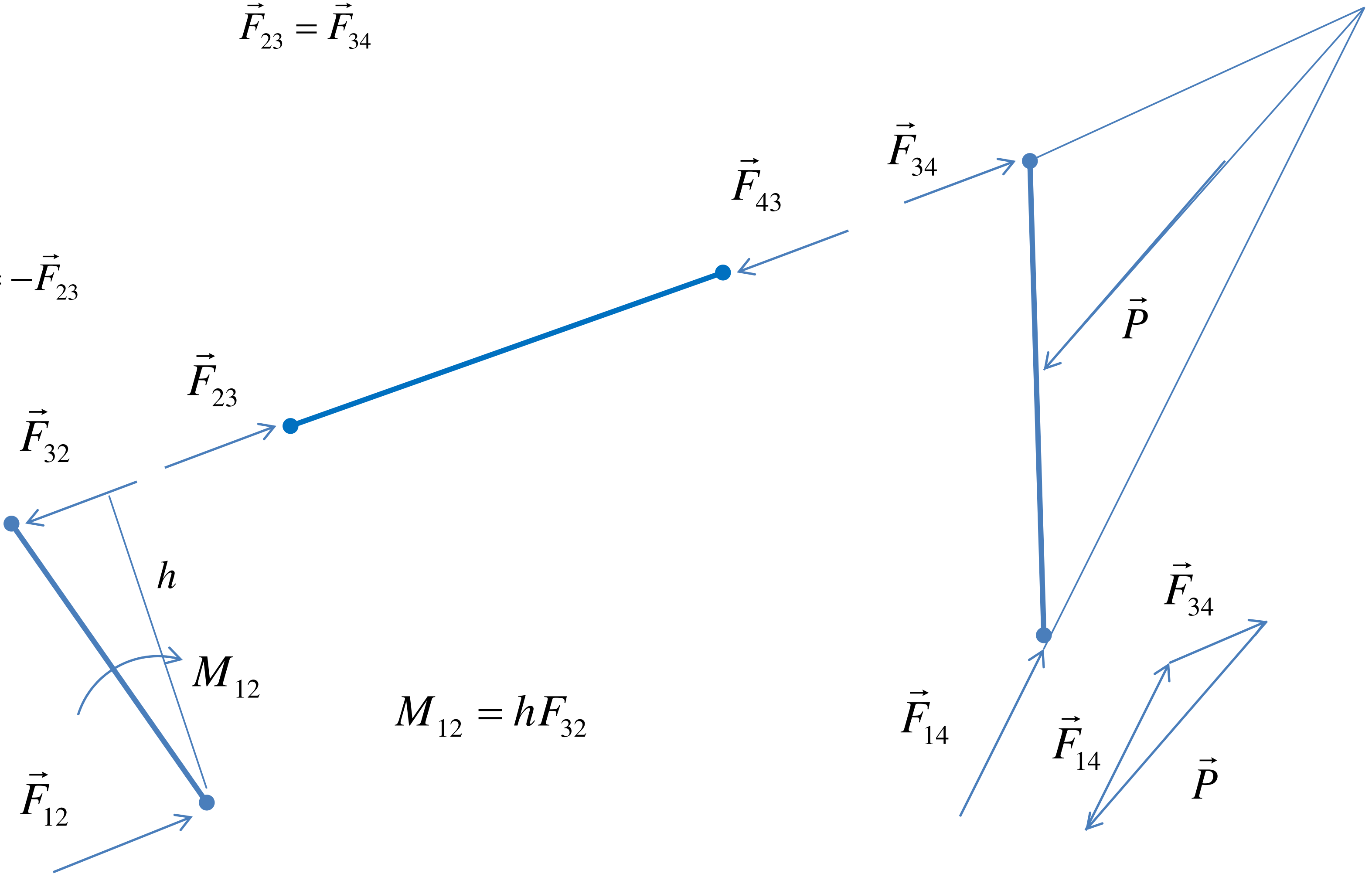


Force P is given, find moment M and reactions at joints by graphical method.

$$\vec{F}_{43} = -\vec{F}_{34}$$

$$\vec{F}_{23} = \vec{F}_{34}$$

$$\vec{F}_{32} = -\vec{F}_{23}$$

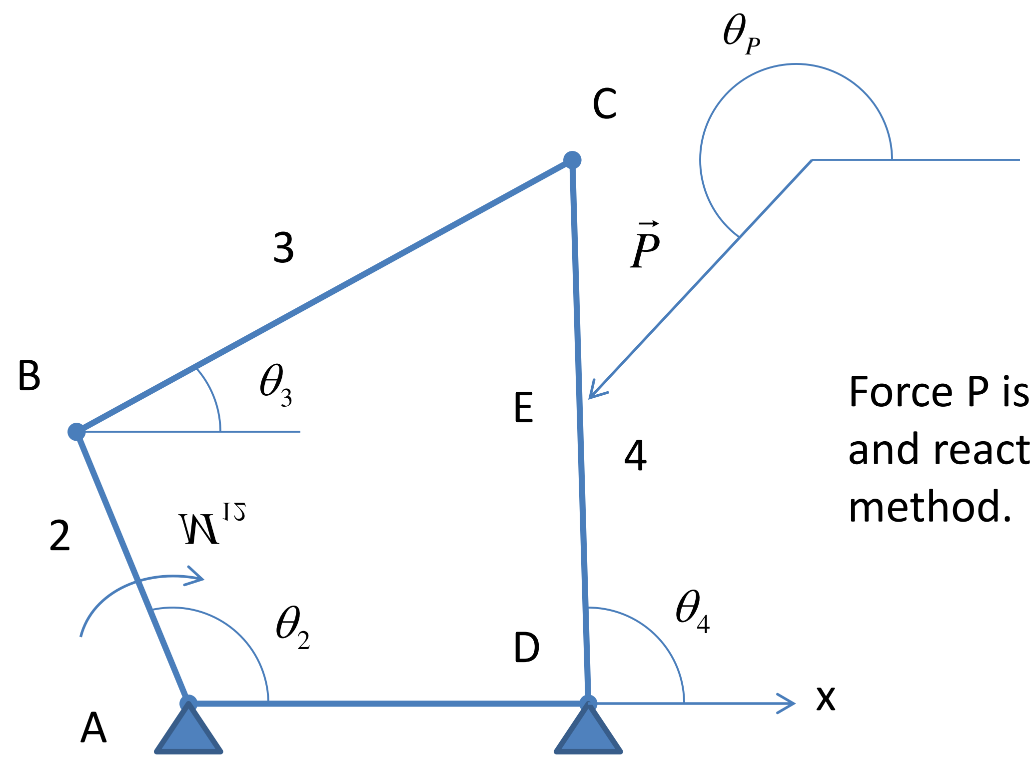


$$M_{12} = hF_{32}$$

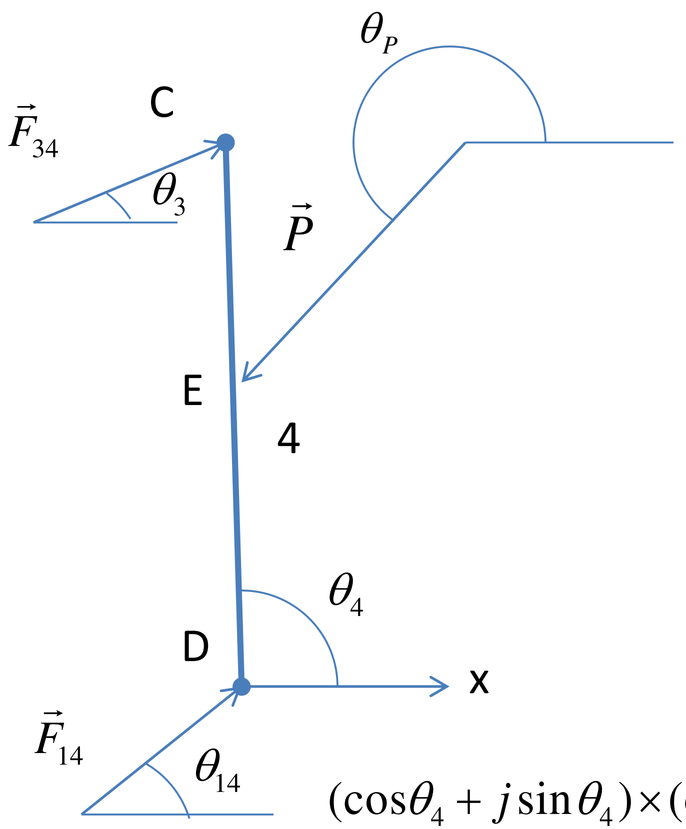
$$\vec{F}_{12} = -\vec{F}_{32}$$

$$\vec{F}_{34} + \vec{P} + \vec{F}_{14} = 0$$

Analytical solution



Force P is given, find moment M_{12} and reactions at joints by analytical method.



$$\sum M_D = 0$$

$$\vec{r}_{DC} \times \vec{F}_{34} + \vec{r}_{DE} \times \vec{P} = 0$$

$$r_{DC} e^{j\theta_4} \times F_{34} e^{j\theta_3} + r_{DE} e^{j\theta_4} \times P e^{j\theta_p} = 0$$

$$r_{DC}(\cos\theta_4 + j\sin\theta_4) \times F_{34}(\cos\theta_3 + j\sin\theta_3)$$
$$+ r_{DE}(\cos\theta_4 + j\sin\theta_4) \times P(\cos\theta_p + j\sin\theta_p) = 0$$

$$(\cos\theta_4 + j\sin\theta_4) \times (\cos\theta_3 + j\sin\theta_3) = \cos\theta_4 \sin\theta_3 - \sin\theta_4 \cos\theta_3 = \sin(\theta_3 - \theta_4)$$

$$r_{DC} F_{34} \sin(\theta_3 - \theta_4) + r_{DE} P \sin(\theta_p - \theta_4) = 0$$

$$F_{34} = -\frac{r_{DE} \sin(\theta_p - \theta_4)}{r_{DC} \sin(\theta_3 - \theta_4)} P$$

$$\sum \vec{F} = 0 \qquad \vec{F}_{34} + \vec{P} + \vec{F}_{14} = 0$$

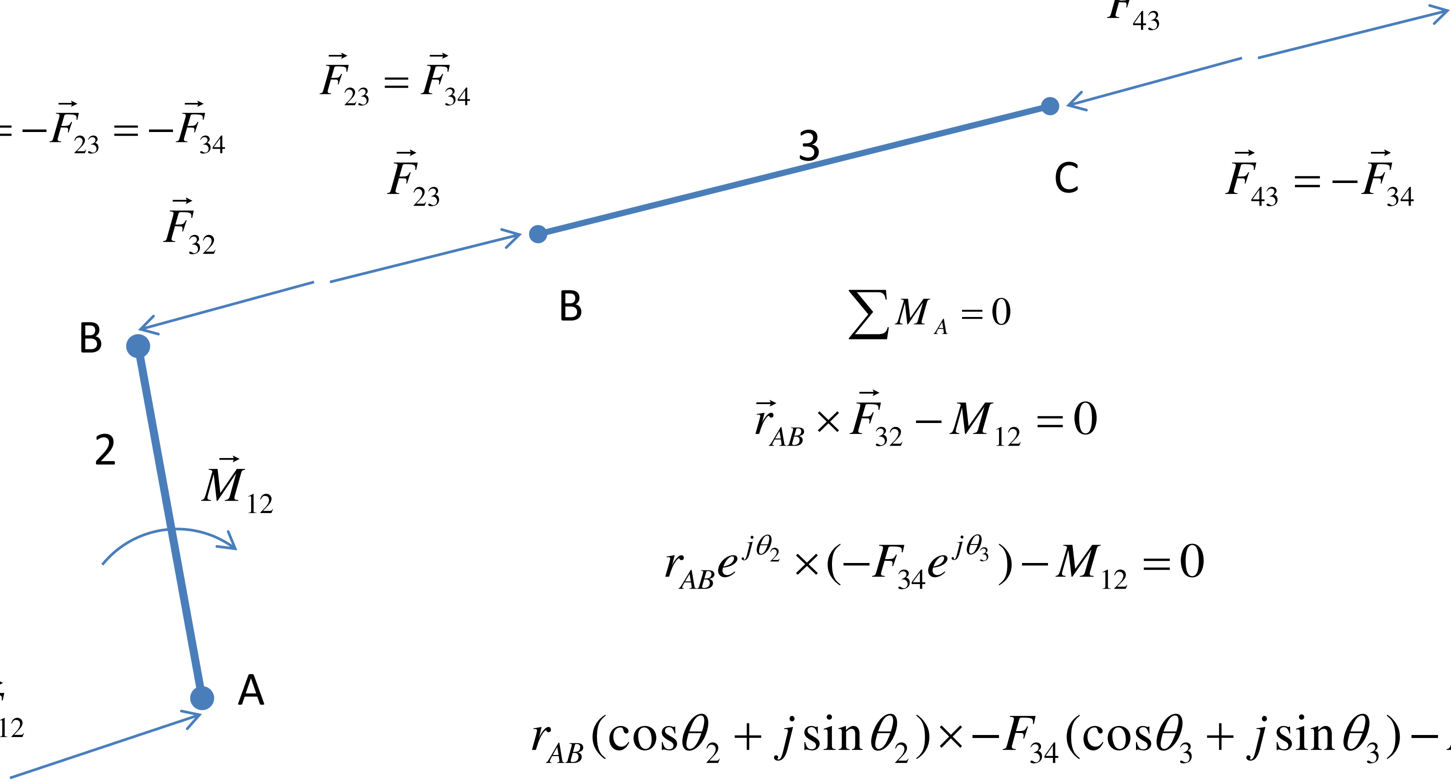
$$\vec{F}_{14} = -\vec{F}_{34} - \vec{P}$$

$$F_{14} e^{j\theta_{14}} = -F_{34}(\cos\theta_3 + j\sin\theta_3) - P(\cos\theta_p + j\sin\theta_p)$$

$$F_{14} e^{j\theta_{14}} = -(F_{34}\cos\theta_3 + P\cos\theta_p) - j(F_{34}\sin\theta_3 + jP\sin\theta_p)$$

$$F_{14} = \sqrt{(F_{34}\cos\theta_3 + P\cos\theta_p)^2 + (F_{34}\sin\theta_3 + P\sin\theta_p)^2}$$

$$\theta_{14} = \tan^{-1}\frac{F_{34}\sin\theta_3 + P\sin\theta_p}{F_{34}\cos\theta_3 + P\cos\theta_p}$$



$$\vec{F}_{34}$$

$$\vec{F}_{43}$$

$$\vec{F}_{23} = \vec{F}_{34}$$

$$\vec{F}_{32} = -\vec{F}_{23} = -\vec{F}_{34}$$

$$\vec{F}_{43} = -\vec{F}_{34}$$

$$\vec{F}_{23}$$

$$\vec{F}_{32}$$

$$\vec{M}_{12}$$

$$\vec{F}_{12}$$

$$\sum M_A = 0$$

$$\vec{r}_{AB} \times \vec{F}_{32} - M_{12} = 0$$

$$r_{AB} e^{j\theta_2} \times (-F_{34} e^{j\theta_3}) - M_{12} = 0$$

$$r_{AB}(\cos\theta_2 + j\sin\theta_2) \times -F_{34}(\cos\theta_3 + j\sin\theta_3) - M_{12} = 0$$

$$M_{12} = -r_{AB} F_{34} \sin(\theta_3 - \theta_2)$$

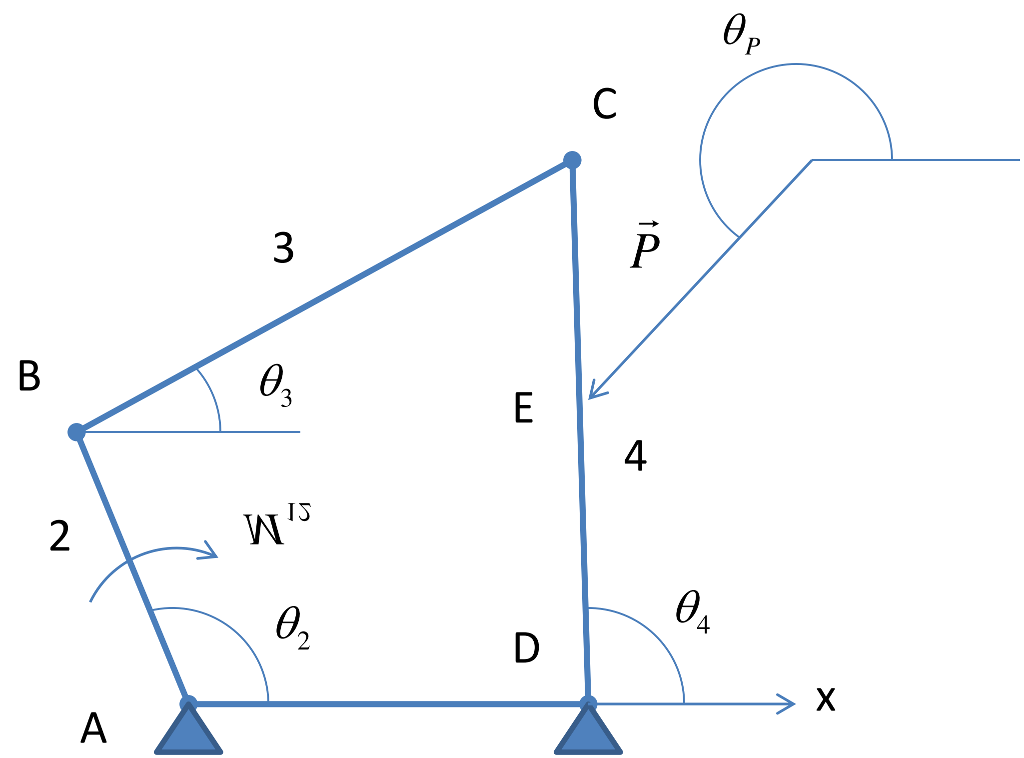
$$\vec{F}_{12} = -\vec{F}_{32} = \vec{F}_{34}$$

$$\vec{F}_{21} = -\vec{F}_{12}$$

$$\vec{F}_{41} = -\vec{F}_{14}$$

$\vec{F}_{21}$ and $\vec{F}_{41}$ are called shaking forces because they shake the body while the mechanism works.

example



$AB = 3cm \quad BC = 7cm \quad AD = 6cm \quad DC = 9cm$

$\theta_2 = 110° \quad \theta_3 = 54° \quad \theta_4 = 109° \quad \theta_p = 220° \quad P = 100N$

determine moment M_{12} and reaction forces on the links.

$$F_{34} = -\frac{r_{DE} P \sin(\theta_p - \theta_4)}{r_{DC} \sin(\theta_3 - \theta_4)} = -\frac{5(100)\sin(220-109)}{9\sin(54-109)} = 63.32N$$

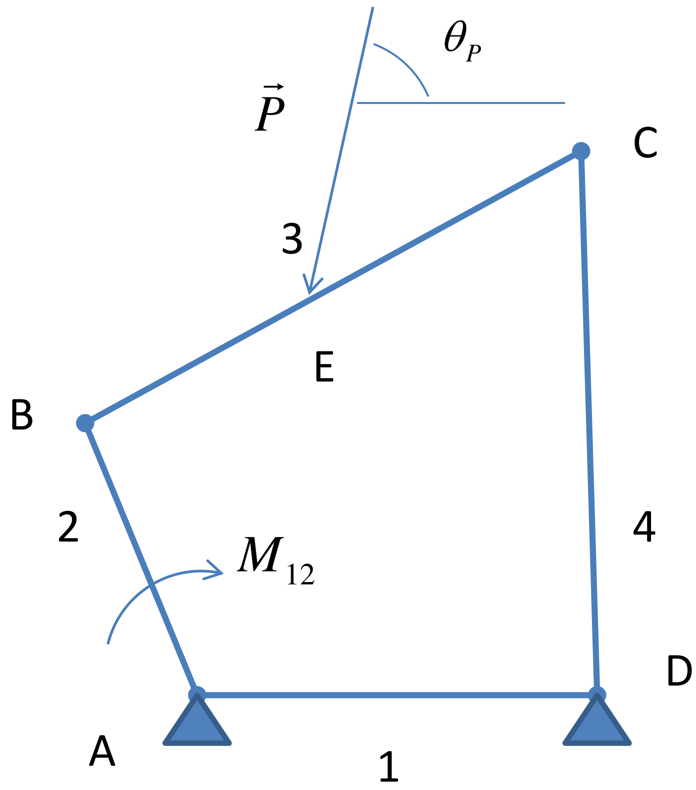
$$F_{14} = \sqrt{(F_{34}\cos\theta_3 + P\cos\theta_p)^2 + (F_{34}\sin\theta_3 + P\sin\theta_p)^2}$$
$$= \sqrt{(63.32\cos 54 + 100\cos 220)^2 + (63.32\sin 54 + 100\sin 220)^2}$$
$$= 41.49N$$

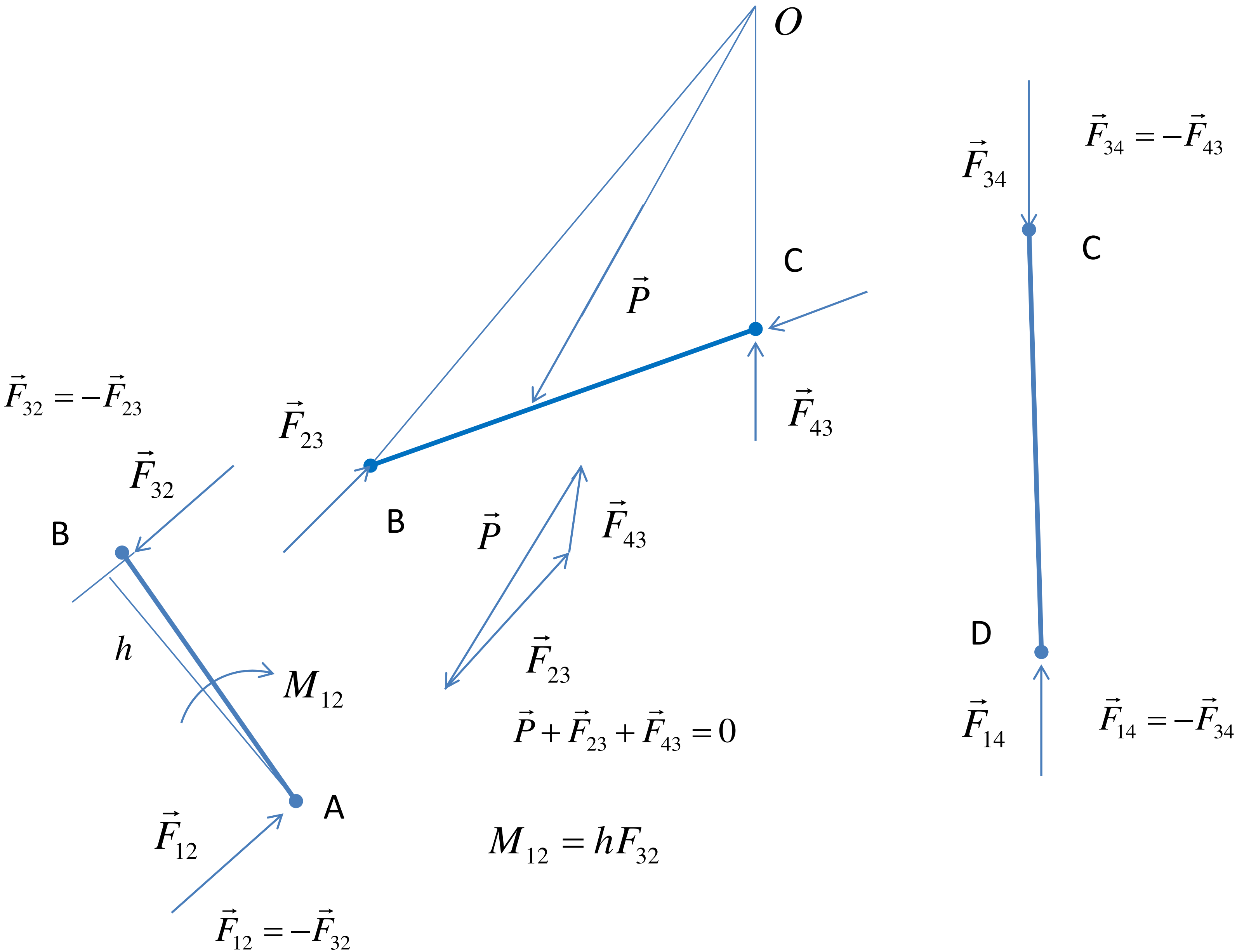
$$\theta_{14} = \tan^{-1}\frac{F_{34}\sin\theta_3 + P\sin\theta_p}{F_{34}\cos\theta_3 + P\cos\theta_p} = \tan^{-1}\frac{63.32\sin 54 + 100\sin 220}{63.32\cos 54 + 100\cos 220}$$
$$= 18.33°$$

$$\vec{F}_{23} = \vec{F}_{34} = 63.32N$$

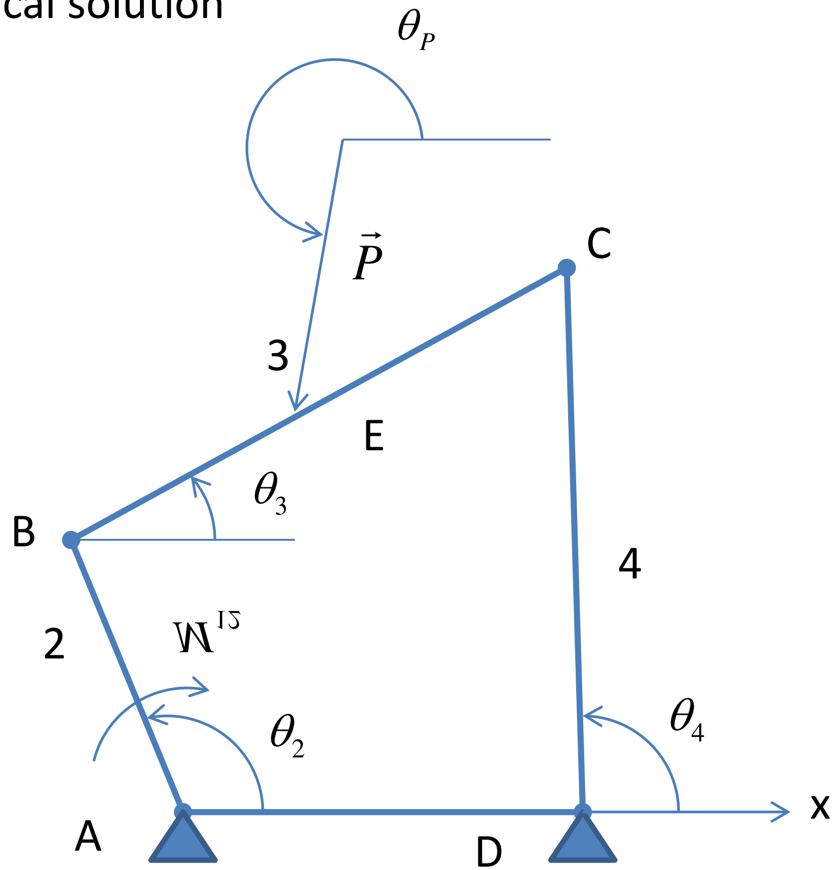
$$M_{12} = -r_{AB} F_{34} \sin(\theta_3 - \theta_2) = -3(63.32)\sin\left(54-110\right) = 157.48Ncm$$

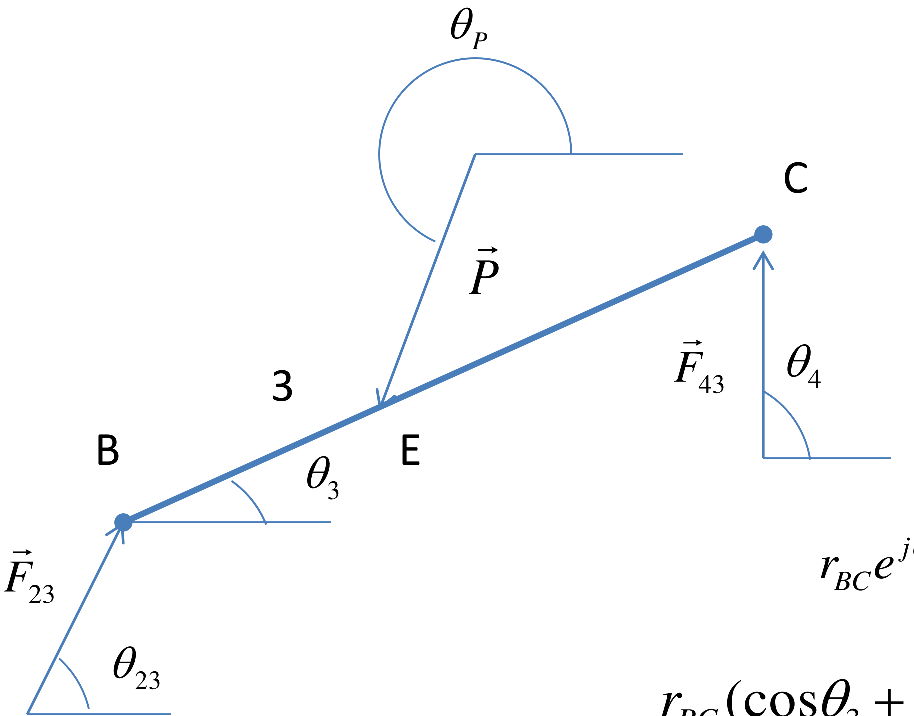
Four-bar mechanism
Graphical solution




O
C
$\vec{P}$
$\vec{F}_{43}$
$\vec{F}_{23}$
B
$\vec{F}_{32} = -\vec{F}_{23}$
$\vec{F}_{32}$
B
h
M_{12}
A
$\vec{F}_{12}$
$\vec{F}_{12} = -\vec{F}_{32}$
$\vec{P}$
$\vec{F}_{43}$
$\vec{F}_{23}$
$\vec{P} + \vec{F}_{23} + \vec{F}_{43} = 0$
$M_{12} = hF_{32}$
$\vec{F}_{34} = -\vec{F}_{43}$
$\vec{F}_{34}$
C
D
$\vec{F}_{14}$
$\vec{F}_{14} = -\vec{F}_{34}$

Analytical solution





$$\sum M_B = 0$$

$$\vec{r}_{BC} \times \vec{F}_{43} + \vec{r}_{BE} \times \vec{P} = 0$$

$$r_{BC} e^{j\theta_3} \times F_{43} e^{j\theta_4} + r_{BE} e^{j\theta_3} \times P e^{j\theta_p} = 0$$

$$r_{BC}(\cos\theta_3 + j\sin\theta_3) \times F_{43}(\cos\theta_4 + j\sin\theta_4)$$
$$+ r_{BE}(\cos\theta_3 + j\sin\theta_3) \times P(\cos\theta_p + j\sin\theta_p) = 0$$

$$r_{BC} F_{43} \sin(\theta_4 - \theta_3) + r_{BE} P \sin(\theta_p - \theta_3) = 0$$

$$F_{43} = -\frac{r_{BE}\sin(\theta_p - \theta_3)}{r_{BC}\sin(\theta_4 - \theta_3)} P$$

$$\sum \vec{F} = 0 \qquad \vec{F}_{23} + \vec{P} + \vec{F}_{43} = 0$$

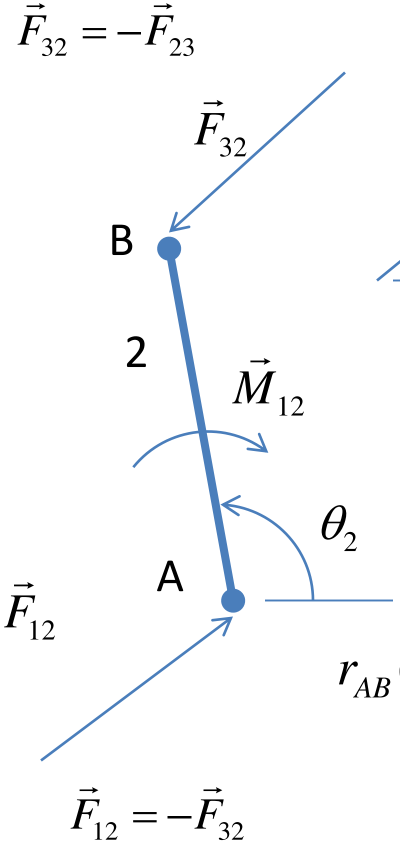
$$\vec{F}_{23} = -\vec{F}_{43} - \vec{P}$$

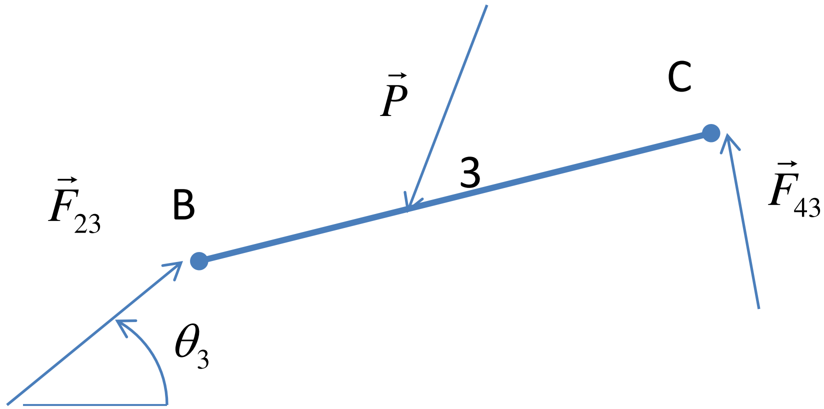
$$F_{23}e^{j\theta_{23}} = -F_{43}(\cos\theta_4 + j\sin\theta_4) - P(\cos\theta_p + j\sin\theta_p)$$

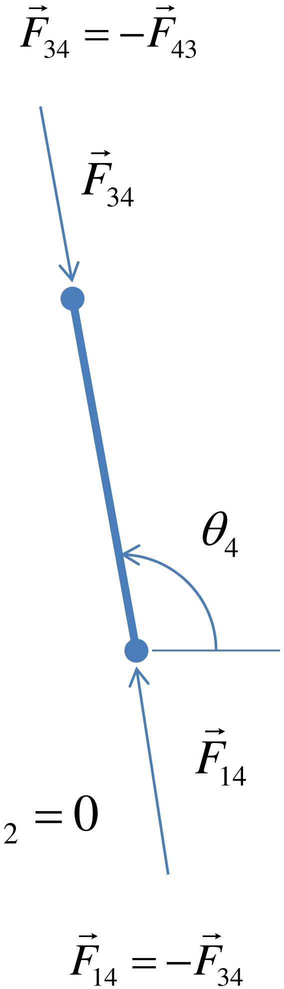
$$F_{23}e^{j\theta_{23}} = -(F_{43}\cos\theta_4 + P\cos\theta_p) - j(F_{43}\sin\theta_4 + jP\sin\theta_p)$$

$$F_{23} = \sqrt{(F_{43}\cos\theta_4 + P\cos\theta_p)^2 + (F_{43}\sin\theta_4 + P\sin\theta_p)^2}$$

$$\theta_{23} = \tan^{-1}\frac{F_{43}\sin\theta_4 + P\sin\theta_p}{F_{43}\cos\theta_4 + P\cos\theta_p}$$







$$\sum M_A = 0$$

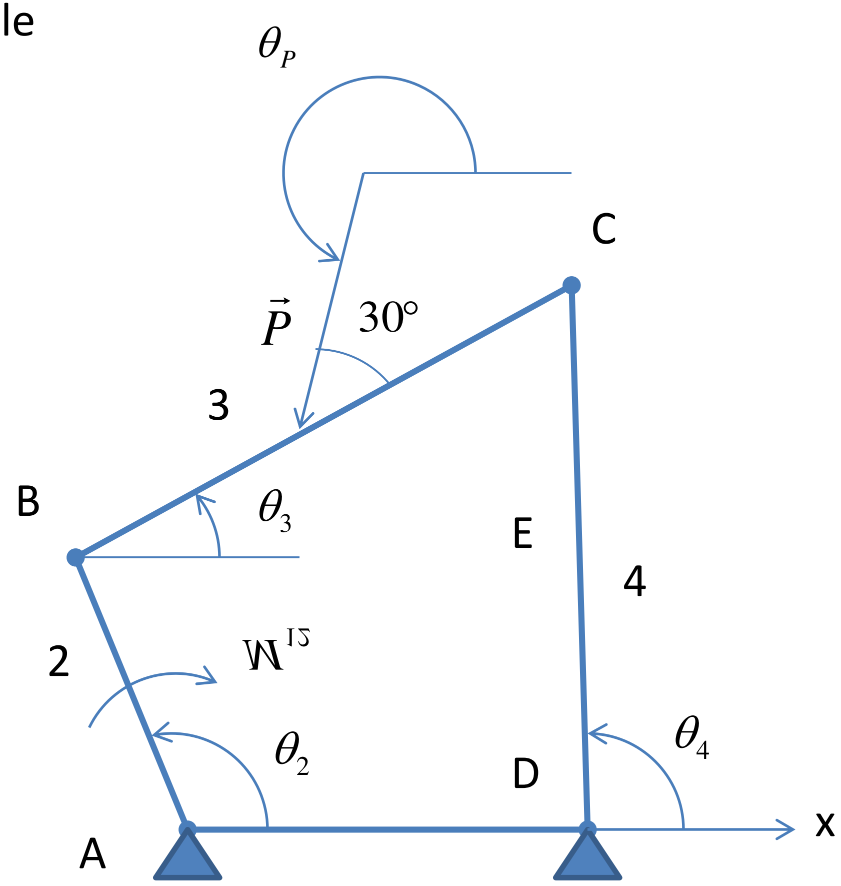
$$\vec{r}_{AB} \times \vec{F}_{32} - M_{12} = 0$$

$$r_{AB} e^{j\theta_2} \times (-F_{23} e^{j\theta_{23}}) - M_{12} = 0$$

$$r_{AB}(\cos\theta_2 + j\sin\theta_2) \times -F_{23}(\cos\theta_{23} + j\sin\theta_{23}) - M_{12} = 0$$

$$M_{12} = -r_{AB} F_{23} \sin(\theta_{23} - \theta_2)$$

example



$$AB = 3cm \quad BC = 7cm \quad AD = 6cm \quad DC = 9cm$$

$$\theta_2 = 110° \quad \theta_3 = 54° \quad \theta_4 = 109° \quad \theta_p = 180° + 54° + 30° = 264° \quad P = 100N$$

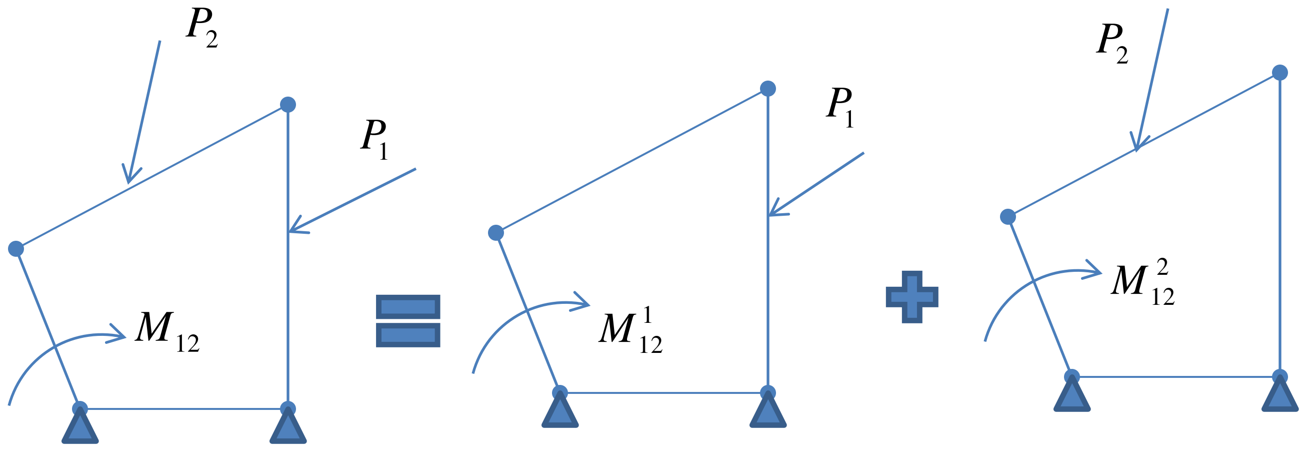
Determine moment M_{12} and reaction forces on the links.

$$F_{43} = -\frac{r_{BE} P \sin(\theta_p - \theta_3)}{r_{BC} \sin(\theta_4 - \theta_3)} = -\frac{4(100)\sin(264-54)}{7\sin(109-54)} = 34.88N$$

$$F_{23} = \sqrt{(F_{43}\cos\theta_4 + P\cos\theta_p)^2 + (F_{43}\sin\theta_4 + P\sin\theta_p)^2}$$
$$= \sqrt{(34.88\cos 109 + 100\cos 264)^2 + (34.88\sin 109 + 100\sin 264)^2}$$
$$= 69.96N$$

$$\theta_{23} = \tan^{-1}\frac{F_{43}\sin\theta_4 + P\sin\theta_p}{F_{43}\cos\theta_4 + P\cos\theta_p} = \tan^{-1}\frac{34.88\sin 109 + 100\sin 264}{34.88\cos 109 + 100\cos 264}$$
$$= 71.83°$$

$$M_{12} = -r_{AB} F_{23} \sin(\theta_{23} - \theta_2) = -3(69.96)\sin(71.83-110) = 129.7N$$

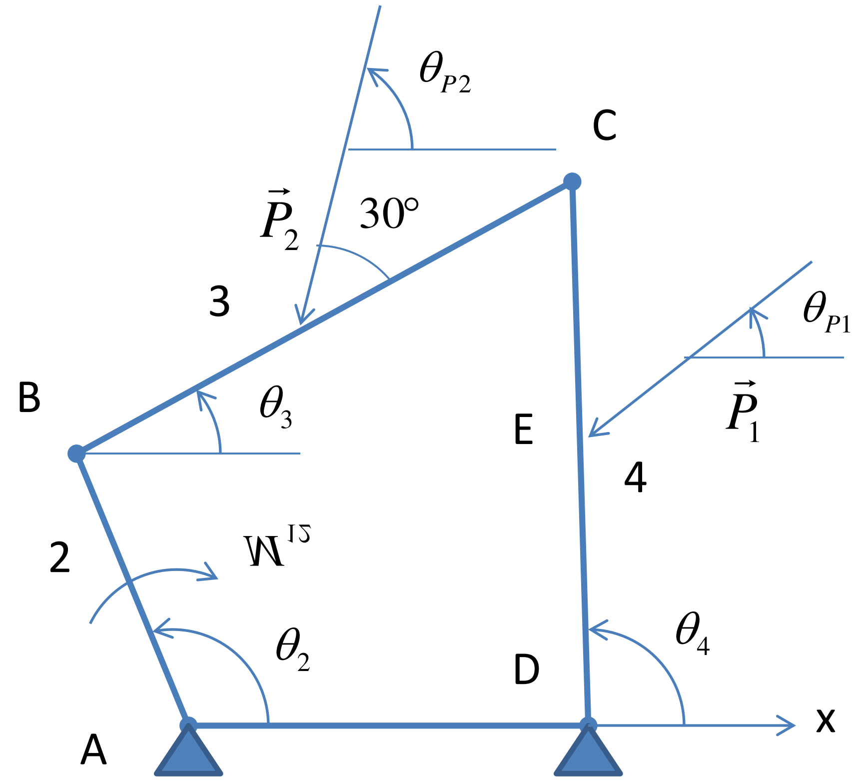


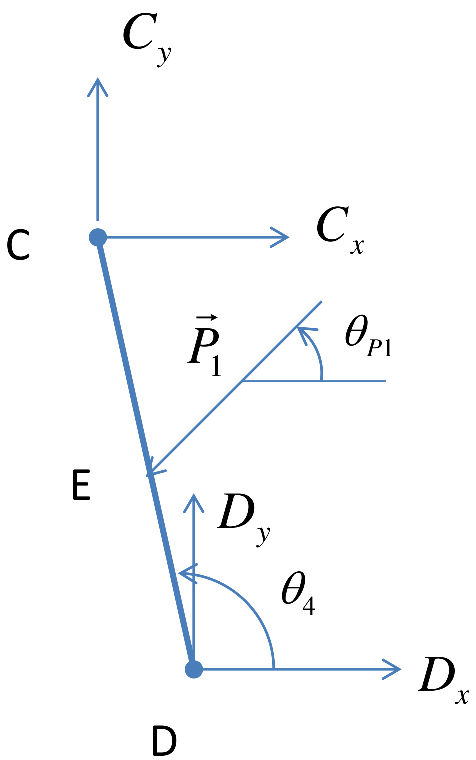
$$M_{12} = M_{12}^1 + M_{12}^2$$

If each link has a force we should solve the problem for one force for the mechanism and superpose the solutions to find total reaction moment and reaction forces

Analytical solution

Second method





$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum M_D = 0$$

$$D_x + C_x - P_1 \cos\theta_{p1} = 0$$

$$D_x + C_x = P_1 \cos\theta_{p1} \quad \#1$$

$$D_y + C_y - P_1 \sin\theta_{p1} = 0$$

$$D_y + C_y = P_1 \sin\theta_{p1} \quad \#2$$

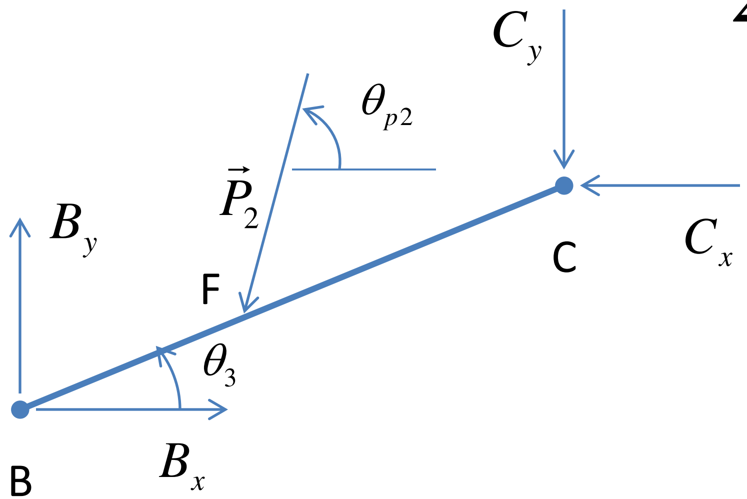
$$\vec{r}_{DE} \times \vec{P}_1 + \vec{r}_{DC} \times \vec{C} = 0$$

$$r_{DE} e^{j\theta_4} \times P_1 e^{j\theta_{p1}} + r_{DC} e^{j\theta_4} \times (C_x + jC_y) = 0$$

$$r_{DE}(\cos\theta_4 + j\sin\theta_4) \times P_1(\cos\theta_{p1} + j\sin\theta_{p1}) + r_{DC}(\cos\theta_4 + j\sin\theta_4) \times (C_x + jC_y) = 0$$

$$r_{DE} P_1 \sin(\theta_{p1} - \theta_4) + r_{DC} \cos\theta_4 C_y - r_{DC} \sin\theta_4 C_x = 0$$

$$r_{DC} \cos\theta_4 C_y - r_{DC} \sin\theta_4 C_x = -r_{DE} P_1 \sin(\theta_{p1} - \theta_4) \quad \#3$$



$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum M_B = 0$$

$$B_x - C_x - P_2 \cos\theta_{p2} = 0$$

$$B_x - C_x = P_2 \cos\theta_{p2} \quad \#4$$

$$B_y - C_y - P_2 \sin\theta_{p2} = 0$$

$$B_y - C_y = P_2 \sin\theta_{p2} \quad \#5$$

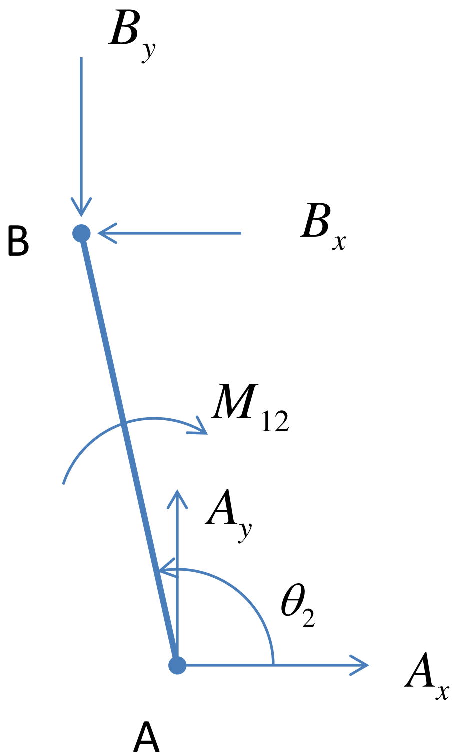
$$\vec{r}_{BF} \times \vec{P}_2 + \vec{r}_{BC} \times \vec{C} = 0$$

$$r_{BE} e^{j\theta_3} \times P_2 e^{j\theta_{p2}} + r_{BC} e^{j\theta_3} \times (-C_x - jC_y) = 0$$

$$r_{BF}(\cos\theta_3 + j\sin\theta_3) \times P_2(\cos\theta_{p2} + j\sin\theta_{p2}) + r_{BC}(\cos\theta_3 + j\sin\theta_3) \times (-C_x - jC_y) = 0$$

$$r_{BF} P_2 \sin(\theta_{p2} - \theta_3) - r_{BC}\cos\theta_3 C_y + r_{BC}\sin\theta_3 C_x = 0$$

$$-r_{BC}\cos\theta_3 C_y + r_{BC}\sin\theta_3 C_x = -r_{BF} P_2 \sin(\theta_{p2} - \theta_3) \quad \#6$$



$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum M_A = 0$$

$$A_x - B_x = 0 \quad \#7$$

$$A_y - B_y = 0 \quad \#8$$

$$\vec{r}_{AB} \times \vec{B} - M_{12} = 0$$

$$r_{AB} e^{j\theta_2} \times (-B_x - jB_y) - M_{12} = 0$$

$$r_{AB}(\cos\theta_2 + j\sin\theta_2) \times (-B_x - jB_y) - M_{12} = 0$$

$$-r_{AB}\cos\theta_2 B_y + r_{AB}\sin\theta_2 B_x - M_{12} = 0 \quad \#9$$

unkonwns are $A_x, A_y, B_x, B_y, C_x, C_y, D_x, D_y, M_{12}$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -r_{DC}\sin\theta_4 & r_{DC}\cos\theta_4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r_{BC}\sin\theta_3 & -r_{BC}\cos\theta_3 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r_{AB}\sin\theta_2 & -r_{AB}\sin\theta_2 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ B_x \\ B_y \\ C_x \\ C_y \\ D_x \\ D_y \\ M_{12} \end{bmatrix} = \begin{bmatrix} P_1\cos\theta_{p1} \\ P_1\sin\theta_{p1} \\ -r_{DE}P_1\sin(\theta_{p1}-\theta_4) \\ P_2\cos\theta_{p2} \\ P_2\sin\theta_{p2} \\ -r_{BF}P_2\sin(\theta_{p2}-\theta_3) \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{Ax} = \mathbf{B}$$

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{B}$$

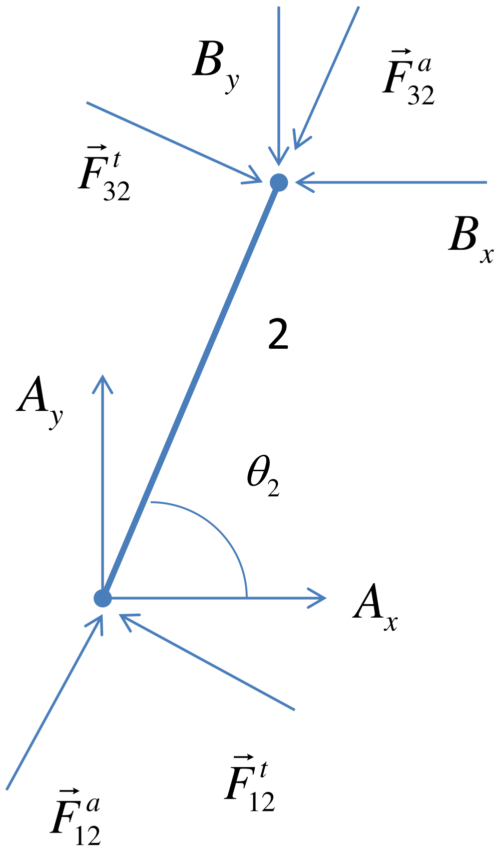
Forces at joints can be calculated from

$$A = \sqrt{A_x^2 + A_y^2}$$

$$B = \sqrt{B_x^2 + B_y^2}$$

$$C = \sqrt{C_x^2 + C_y^2}$$

$$D = \sqrt{D_x^2 + D_y^2}$$



Axial and transversal components of the forces are

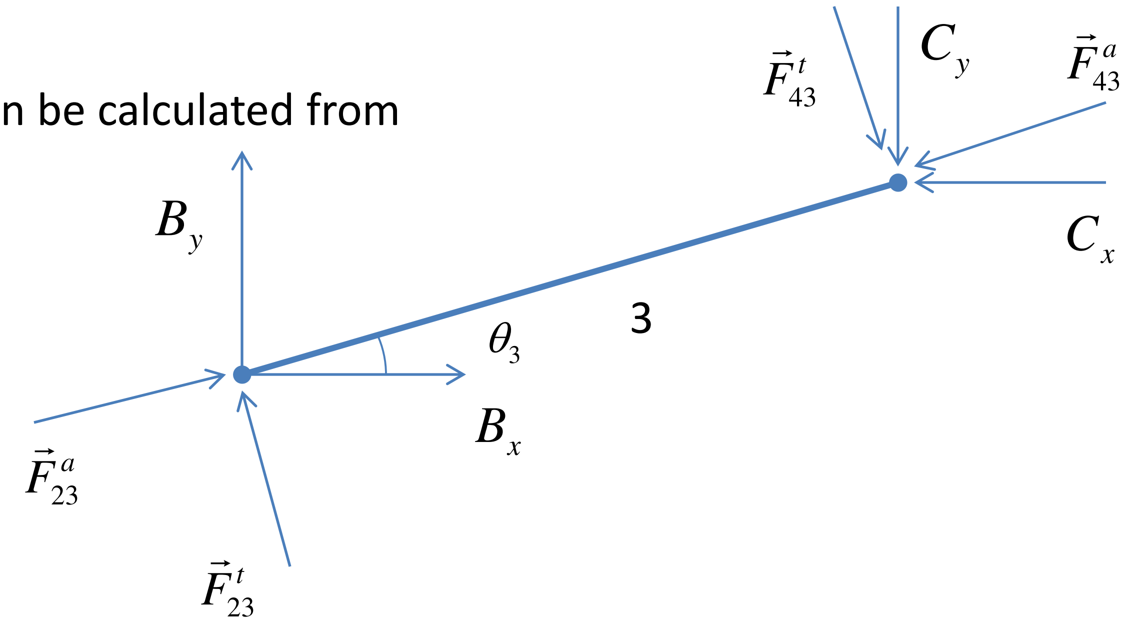
$$F_{12}^a = A_x \cos\theta_2 + A_y \sin\theta_2$$

$$F_{12}^t = -A_x \sin\theta_2 + A_y \cos\theta_2$$

$$F_{32}^a = -B_x \cos\theta_2 - B_y \sin\theta_2$$

$$F_{32}^t = B_x \sin\theta_2 - B_y \cos\theta_2$$

Forces at joints can be calculated from



Axial and transversal components of the forces are

$$F_{23}^a = B_x \cos\theta_3 + B_y \sin\theta_3$$

$$F_{23}^t = -B_x \sin\theta_3 + B_y \cos\theta_3$$

$$F_{43}^a = -C_x \cos\theta_3 - C_y \sin\theta_3$$

$$F_{43}^t = C_x \sin\theta_3 - C_y \cos\theta_3$$

Forces at joints can be calculated from

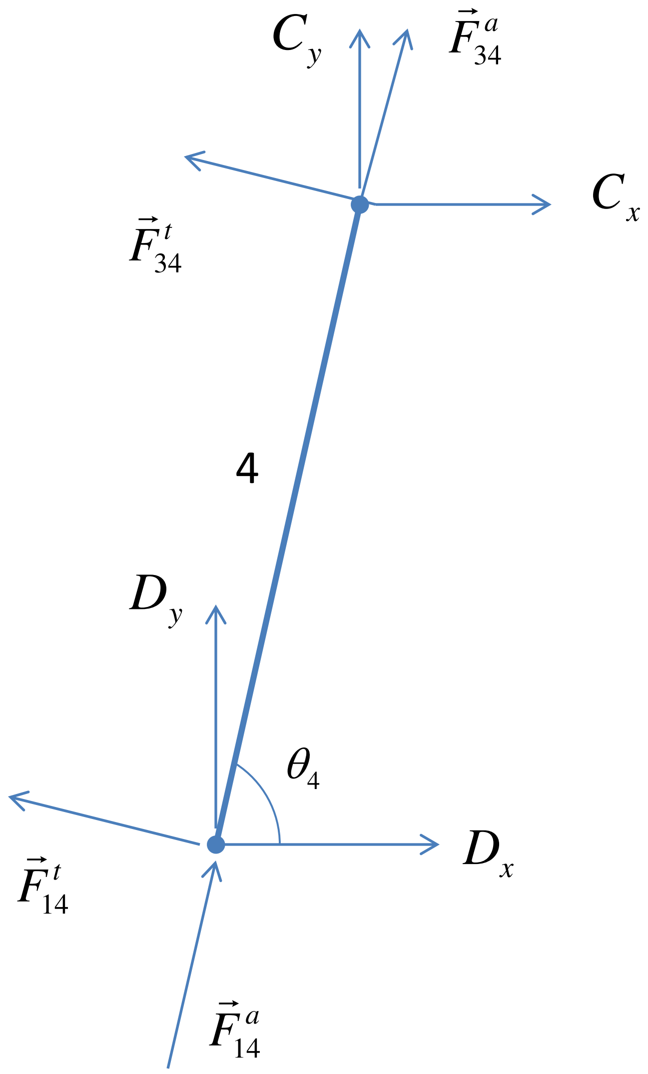
Axial and transversal components of the forces are

$$F_{14}^{a} = D_x \cos\theta_4 + D_y \sin\theta_4$$

$$F_{14}^{t} = -D_x \sin\theta_4 + D_y \cos\theta_4$$

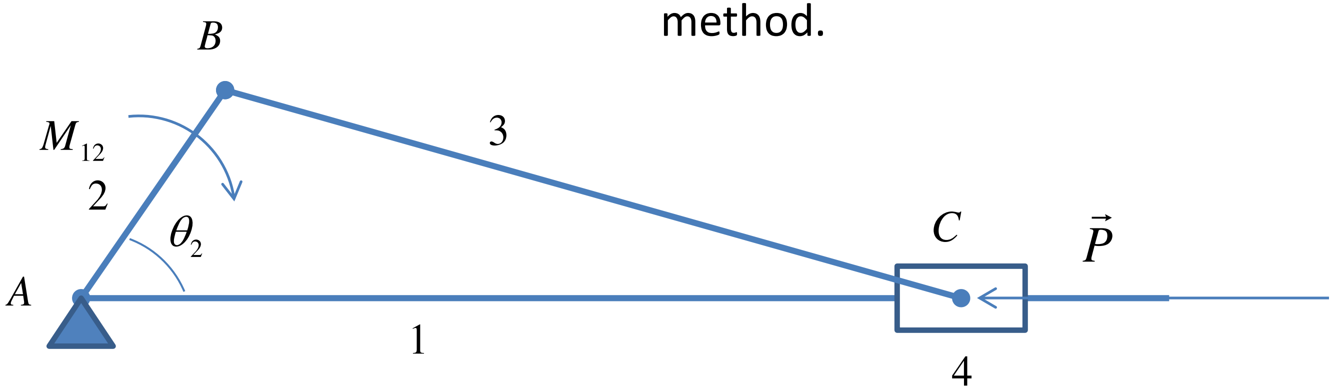
$$F_{34}^{a} = C_x \cos\theta_4 + C_y \sin\theta_4$$

$$F_{34}^{t} = -C_x \sin\theta_4 + C_y \cos\theta_4$$

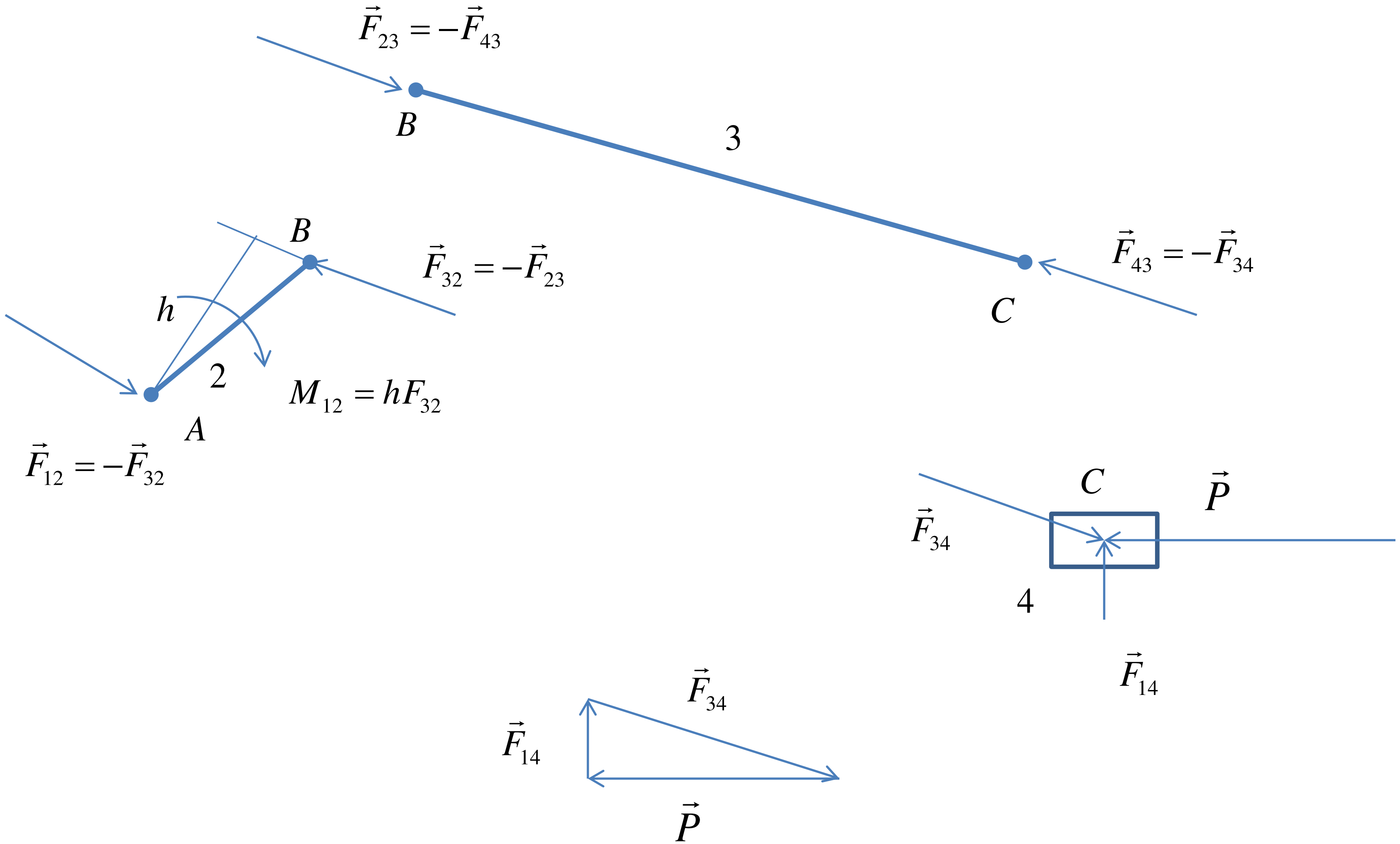


Slider-crank mechanism
Graphical solution

Force P is given, find moment M_{12} and reactions at joints by graphical method.

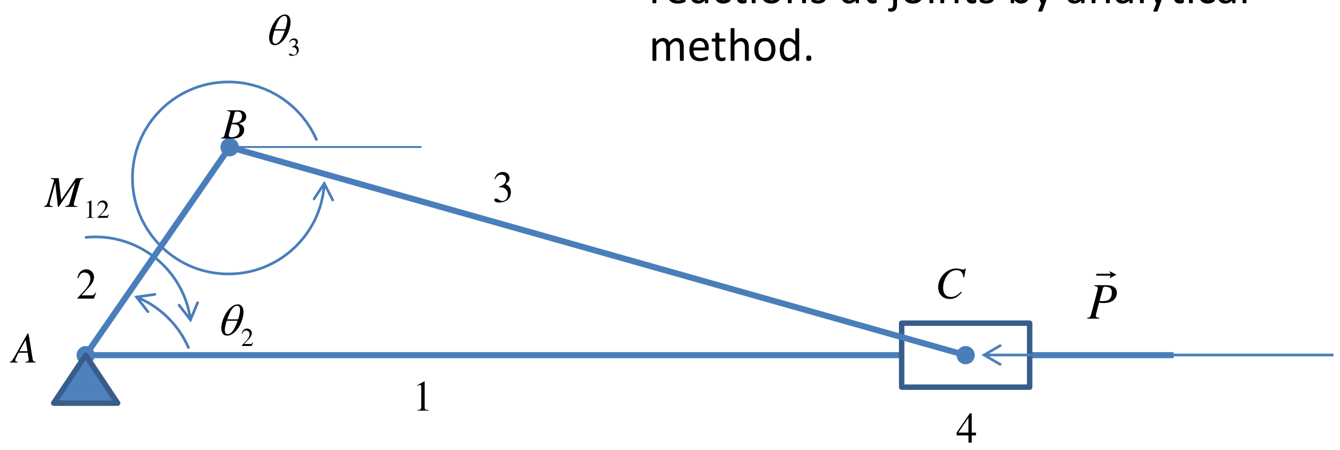


Slider-crank mechanism
Graphical solution



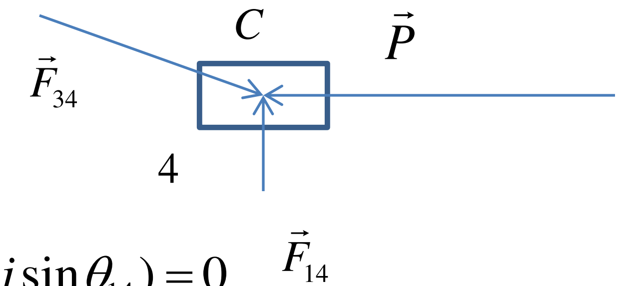
Slider-crank mechanism
analytical solution

Force P is given, find moment M_{12} and reactions at joints by analytical method.



$$\sum \vec{F} = 0 \qquad \vec{F}_{34} + \vec{P} + \vec{F}_{14} = 0$$

$$F_{34} e^{j\theta_3} + P e^{j\theta_p} + F_{14} e^{j\theta_{14}} = 0$$



$$F_{34}(\cos\theta_3 + j\sin\theta_3) + P(\cos\theta_p + j\sin\theta_p) + F_{14}(\cos\theta_{14} + j\sin\theta_{14}) = 0$$

$$F_{34}\cos\theta_3 + P\cos\theta_p + F_{14}\cos\theta_{14} = 0$$

$$F_{34}\sin\theta_3 + P\sin\theta_p + F_{14}\sin\theta_{14} = 0$$

$$F_{34}\cos\theta_3 + P\cos 180 + F_{14}\cos 90 = 0$$

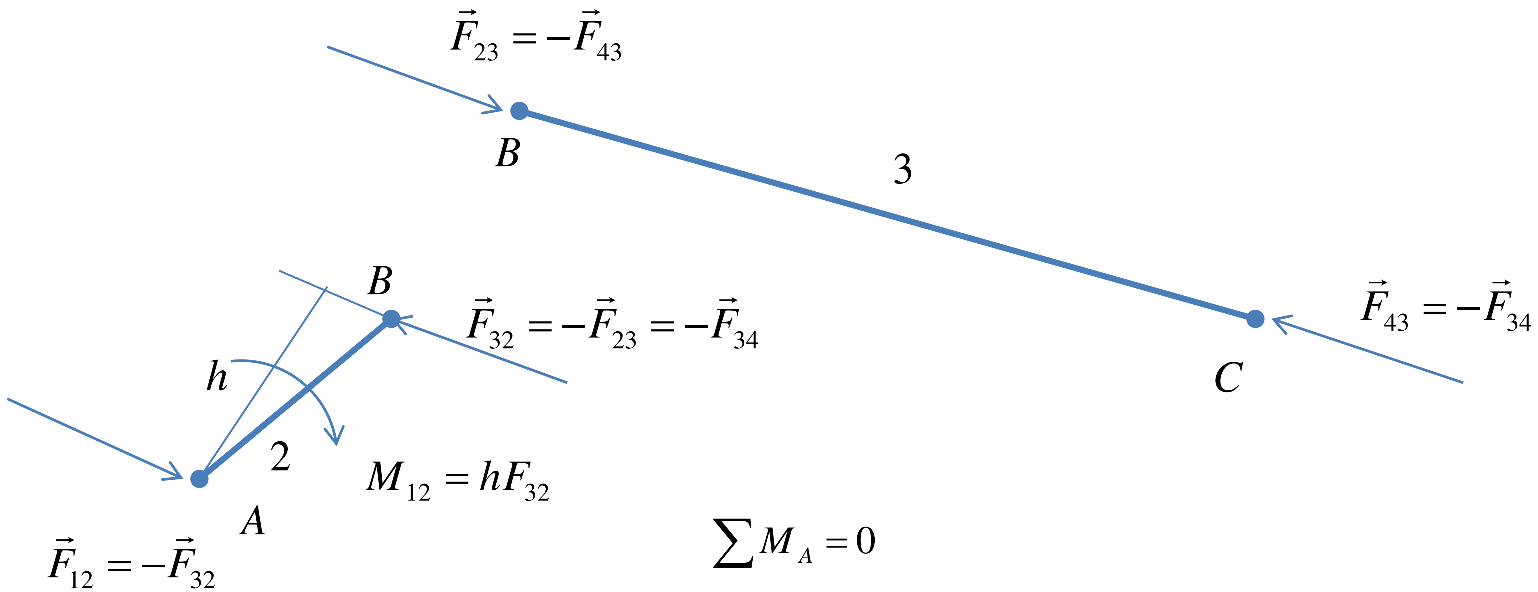
$$F_{34}\cos\theta_3 - P = 0$$

$$F_{34} = \frac{P}{\cos\theta_3}$$

$$F_{34}\sin\theta_3 + P\sin 180 + F_{14}\sin 90 = 0$$

$$F_{34}\sin\theta_3 + F_{14} = 0$$

$$F_{14} = -F_{34}\sin\theta_3 = -\frac{P}{\cos\theta_3}\sin\theta_3 = -P\tan\theta_3$$



$$\sum M_A = 0$$

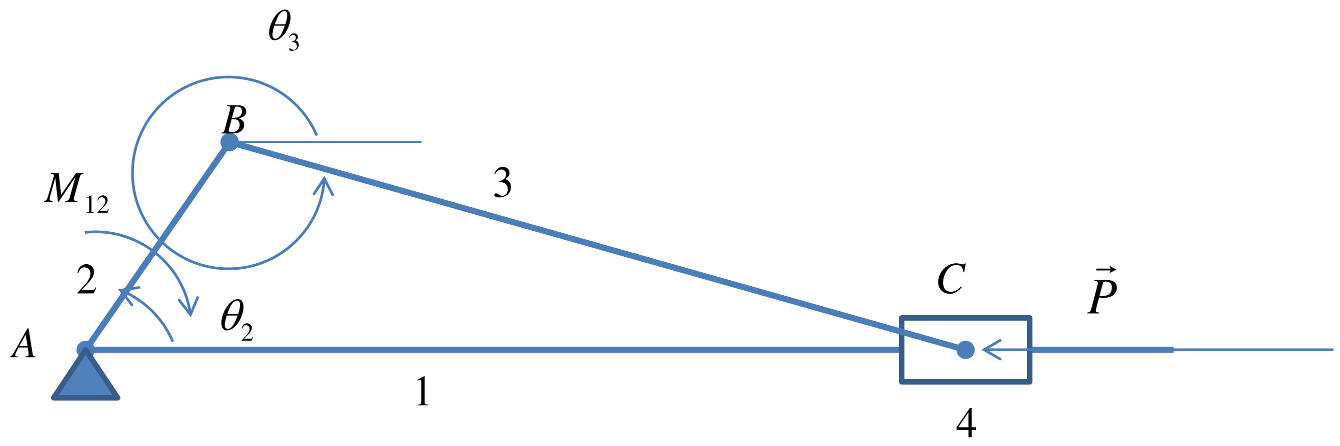
$$\vec{r}_{AB} \times \vec{F}_{32} - M_{12} = 0$$

$$r_{AB} e^{j\theta_2} \times (-F_{34} e^{j\theta_3}) - M_{12} = 0$$

$$r_{AB} (\cos\theta_2 + j \sin\theta_2) \times -F_{34} (\cos\theta_3 + j \sin\theta_3) - M_{12} = 0$$

$$M_{12} = -r_{AB} F_{34} \sin(\theta_3 - \theta_2) = -r_{AB} \frac{P}{\cos\theta_3} \sin(\theta_3 - \theta_2) = -r_{AB} \frac{\sin(\theta_3 - \theta_2)}{\cos\theta_3} P$$

example



$AB = 2cm \quad BC = 6cm$

$\theta_2 = 61° \quad \theta_3 = 343° \quad P = 100N$

Determine moment M_{12} and reaction forces on the links.

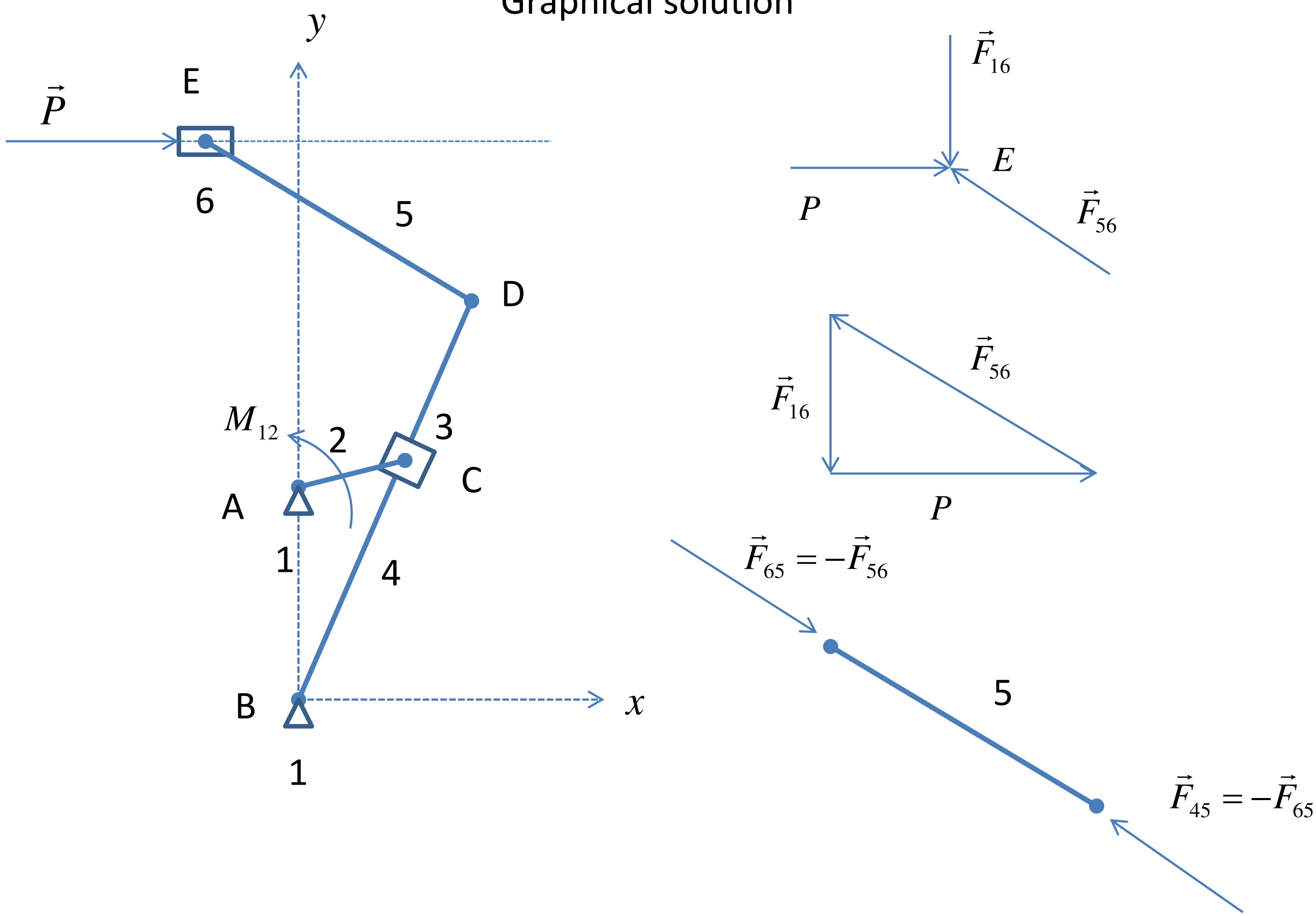
$$F_{34} = \frac{P}{\cos\theta_3} = \frac{100}{\cos 343} = 104.57N$$

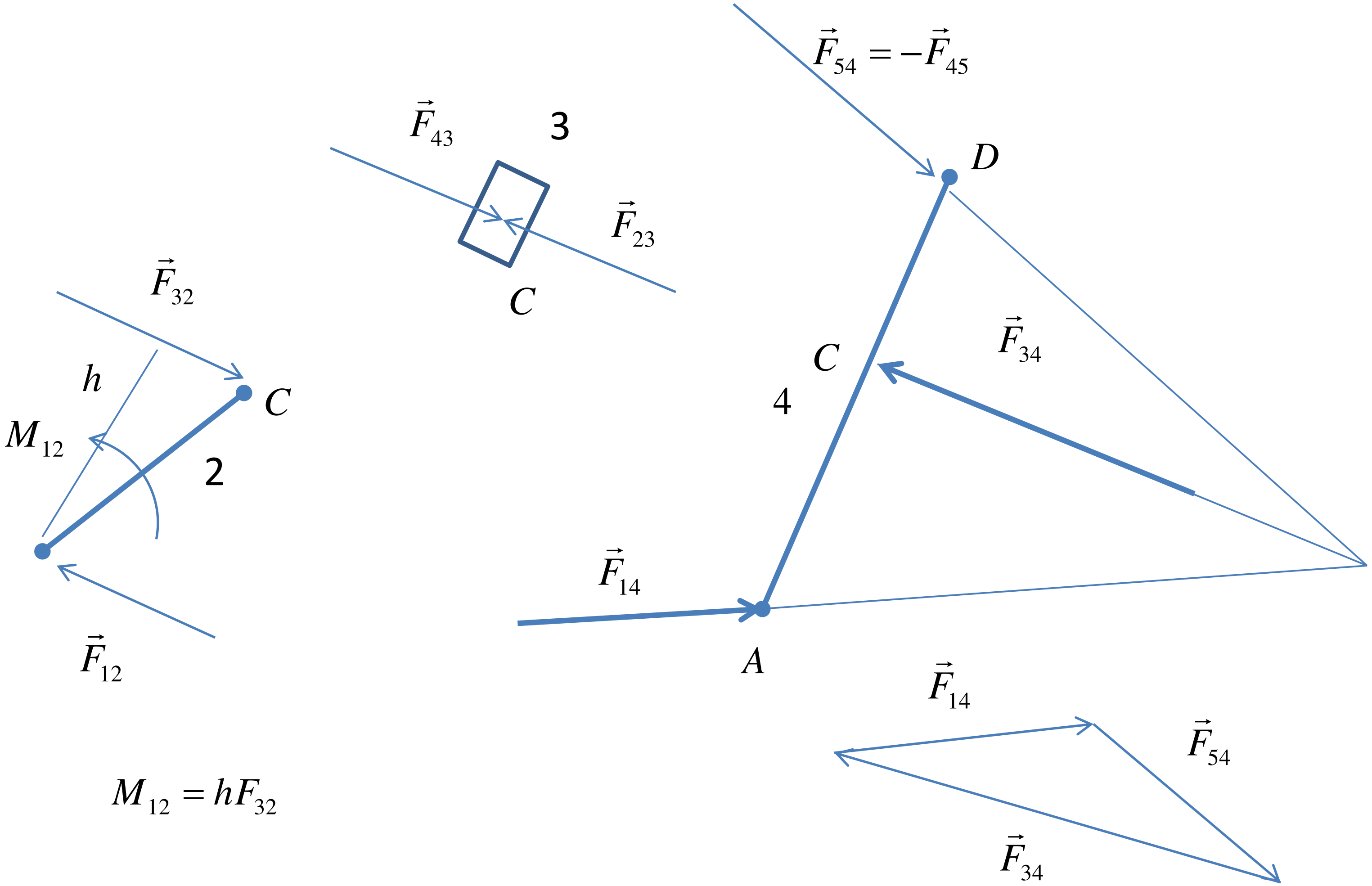
$$F_{14} = -P\tan\theta_3 = -100\tan 343 = 30.57N$$

$$M_{12} = -r_{AB}F_{34}\sin(\theta_3 - \theta_2) = -2(104.57)\sin(343-61) = 204.57Ncm$$

Quick return mechanism

Graphical solution



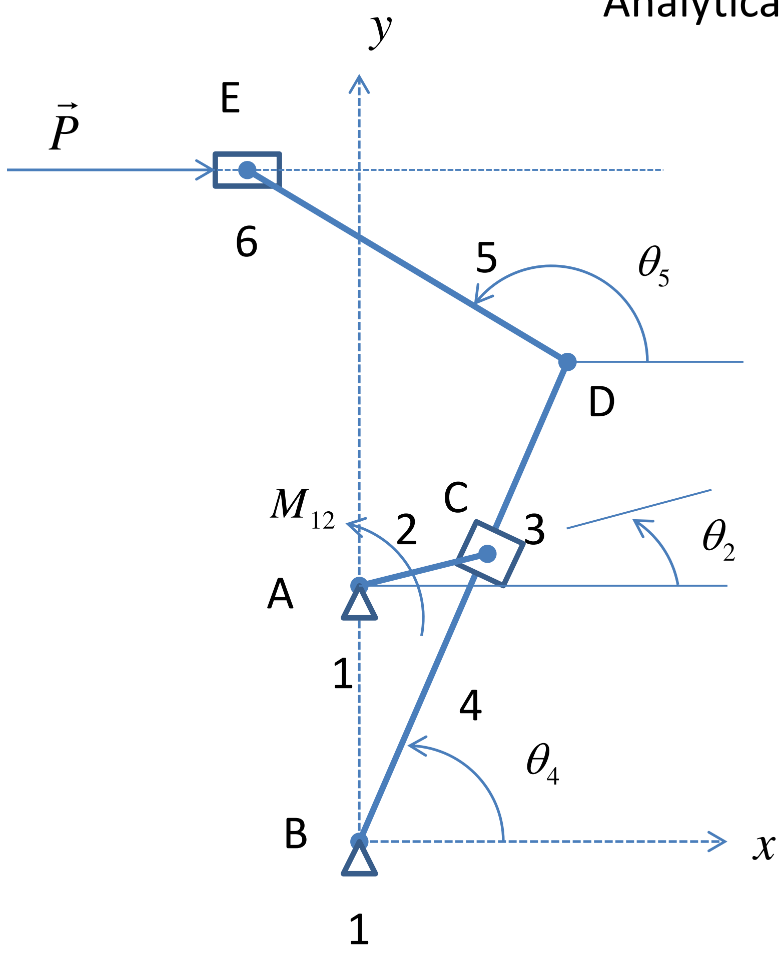


$\vec{F}_{43}$
3
$\vec{F}_{23}$
C
$\vec{F}_{54} = -\vec{F}_{45}$
D
$\vec{F}_{32}$
h
C
M_{12}
2
$\vec{F}_{12}$
C
4
$\vec{F}_{34}$
$\vec{F}_{14}$
A
$\vec{F}_{14}$
$\vec{F}_{54}$
$\vec{F}_{34}$

$$M_{12} = hF_{32}$$

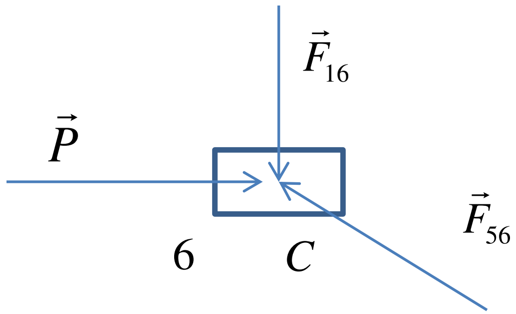
Quick return mechanism

Analytical solution



$$\sum \vec{F} = 0 \qquad \vec{F}_{56} + \vec{P} + \vec{F}_{16} = 0$$

$$F_{56}e^{j\theta_5} + Pe^{j\theta_p} + F_{16}e^{j\theta_{16}} = 0$$



$$F_{56}(\cos\theta_5 + j\sin\theta_5) + P(\cos\theta_p + j\sin\theta_p) + F_{16}(\cos\theta_{16} + j\sin\theta_{16}) = 0$$

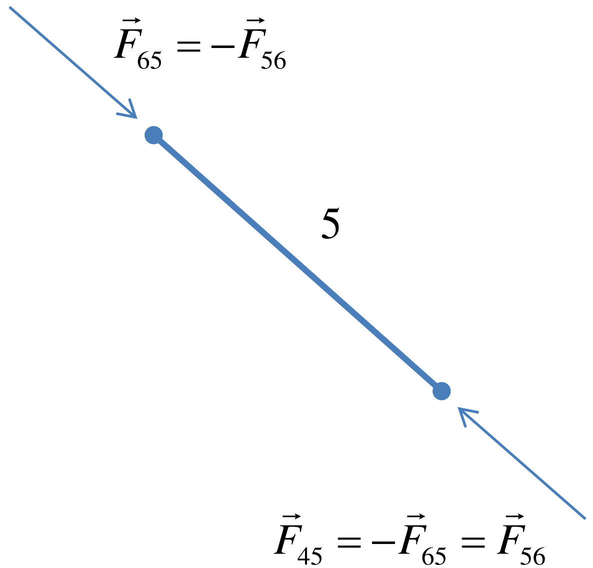
$$F_{56}\cos\theta_5 + P\cos\theta_p + F_{16}\cos\theta_{16} = 0$$

$$F_{56}\sin\theta_5 + P\sin\theta_p + F_{16}\sin\theta_{16} = 0$$

$$F_{56}\cos\theta_5 + P\cos 0 + F_{16}\cos 270 = 0$$

$$F_{56}\cos\theta_5 + P = 0$$

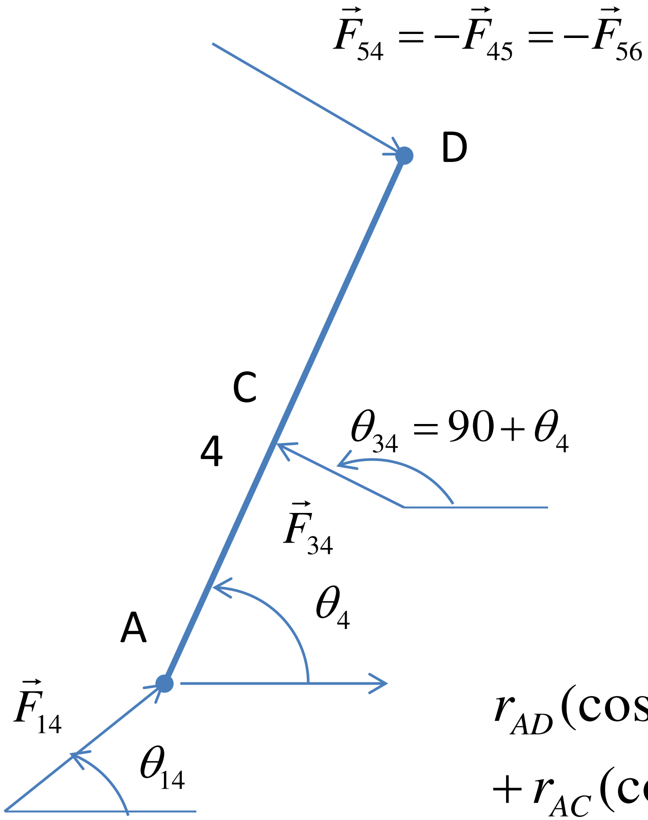
$$F_{56} = -\frac{P}{\cos\theta_5}$$



$$F_{56}\sin\theta_5 + P\sin 0 + F_{16}\sin 270 = 0$$

$$F_{56}\sin\theta_5 - F_{16} = 0$$

$$F_{16} = F_{56}\sin\theta_5 = -\frac{P}{\cos\theta_5}\sin\theta_{35} = -P\tan\theta_5$$



$$\sum M_A = 0$$

$$\vec{r}_{AD} \times \vec{F}_{54} + \vec{r}_{AC} \times \vec{F}_{34} = 0$$

$$r_{AD} e^{j\theta_4} \times -F_{56} e^{j\theta_5} + r_{AC} e^{j\theta_4} \times F_{34} e^{j\theta_{34}} = 0$$

$$r_{AD}(\cos\theta_4 + j\sin\theta_4) \times -F_{56}(\cos\theta_5 + j\sin\theta_5)$$
$$+ r_{AC}(\cos\theta_4 + j\sin\theta_4) \times F_{34}(\cos(90+\theta_4) + j\sin(90+\theta_4)) = 0$$

$$-r_{AD} F_{56} \sin(\theta_5 - \theta_4) + r_{AC} F_{34} \sin(90 + \theta_4 - \theta_4) = 0$$

$$-r_{AD} F_{56} \sin(\theta_5 - \theta_4) + r_{AC} F_{34} = 0$$

$$F_{34} = \frac{r_{AD} F_{56} \sin(\theta_5 - \theta_4)}{r_{AC}} = -\frac{r_{AD}}{r_{AC}} \frac{\sin(\theta_5 - \theta_4)}{\cos\theta_5} P$$

$$\sum \vec{F} = 0 \qquad \vec{F}_{14} + \vec{F}_{34} + \vec{F}_{54} = 0$$

$$\vec{F}_{14} + \vec{F}_{34} - \vec{F}_{56} = 0$$

$$\vec{F}_{14} = -\vec{F}_{34} + \vec{F}_{56}$$

$$\vec{F}_{14} = -F_{34}e^{j\theta_{34}} + F_{56}e^{j\theta_5} = -F_{34}(\cos\theta_{34} + j\sin\theta_{34}) + F_{56}(\cos\theta_5 + j\sin\theta_5)$$

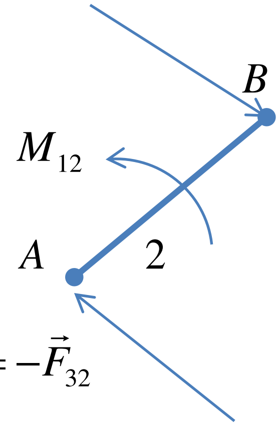
$$\vec{F}_{14} = -F_{34}(\cos(90+\theta_4) + j\sin(90+\theta_4)) + F_{56}(\cos\theta_5 + j\sin\theta_5)$$

$$\vec{F}_{14} = (F_{34}\sin\theta_4 + F_{56}\cos\theta_5) + j(F_{34}\cos\theta_4 + F_{56}\sin\theta_5)$$

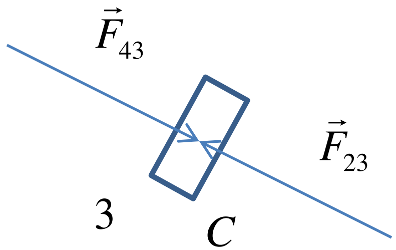
$$F_{14} = \sqrt{(F_{34}\sin\theta_4 + F_{56}\cos\theta_5)^2 + j(F_{34}\cos\theta_4 + F_{56}\sin\theta_5)^2}$$

$$\theta_{14} = \tan^{-1}\frac{F_{34}\cos\theta_4 + F_{56}\sin\theta_5}{F_{34}\sin\theta_4 + F_{56}\cos\theta_5}$$

$$\vec{F}_{32} = -\vec{F}_{23} = \vec{F}_{43} = -\vec{F}_{34}$$



$$\vec{F}_{12} = -\vec{F}_{32}$$



$$\sum M_A = 0$$

$$\vec{r}_{AB} \times \vec{F}_{32} + M_{12} = 0$$

$$r_{AB} e^{j\theta_2} \times (-F_{34} e^{j(90+\theta_4)}) + M_{12} = 0$$

$$r_{AB}(\cos\theta_2 + j\sin\theta_2) \times -F_{34}(\cos(90+\theta_4) + j\sin(90+\theta_4)) + M_{12} = 0$$

$$-r_{AB} F_{34} \sin(90+\theta_4 - \theta_2) + M_{12} = 0$$

$$M_{12} = r_{AB} F_{34} \sin(90+\theta_4 - \theta_2) = r_2 F_{34} \cos(\theta_4 - \theta_2)$$

DYNAMIC FORCE ANALYSIS

$$\sum \vec{F} = m\vec{a}_G$$

$$\sum \vec{M}_G = I_G \vec{\alpha}$$

These equations show that when an unbalanced system of forces acts upon a rigid body, the body experiences a linear acceleration $\vec{a}_G$ of its mass center in the same direction as the resultant force $\sum \vec{F}$. The body also experience an angular acceleration $\vec{\alpha}$ due to the moment of the forces and torques about the center of mass, in the same direction as the resultant moment $\sum \vec{M}_G$. If the forces and moments are known, these equations can be used to determine the resultant accelerations $\vec{a}_G$ and $\vec{\alpha}$.

In engineering design the motion of the machine members is usually specified in advance by other machine requirements. The problem then is; given the motion of the machine elements, what forces are required to produce this motion.

The problem therefore requires;

1. Kinematic analysis in order to determine the linear and angular acceleration of the various members.

2. A definition of the actual shape, dimension, and material of the member; otherwise the masses and moment of inertias could not be determined. (subject of machine design).

D' Alembert's Principle;

The vector sum of all the external forces and the inertia forces acting upon a rigid body is zero. The vector sum of all the external moments and the inertia torques acting upon a rigid body is also separately zero.

$$\sum \vec{F} - m\vec{a}_G = 0$$

$$\sum \vec{M}_G - I_G\vec{\alpha} = 0$$

$-m\vec{a}_G$ is called inertial force

$-I_G\vec{\alpha}$ is called inertial moment

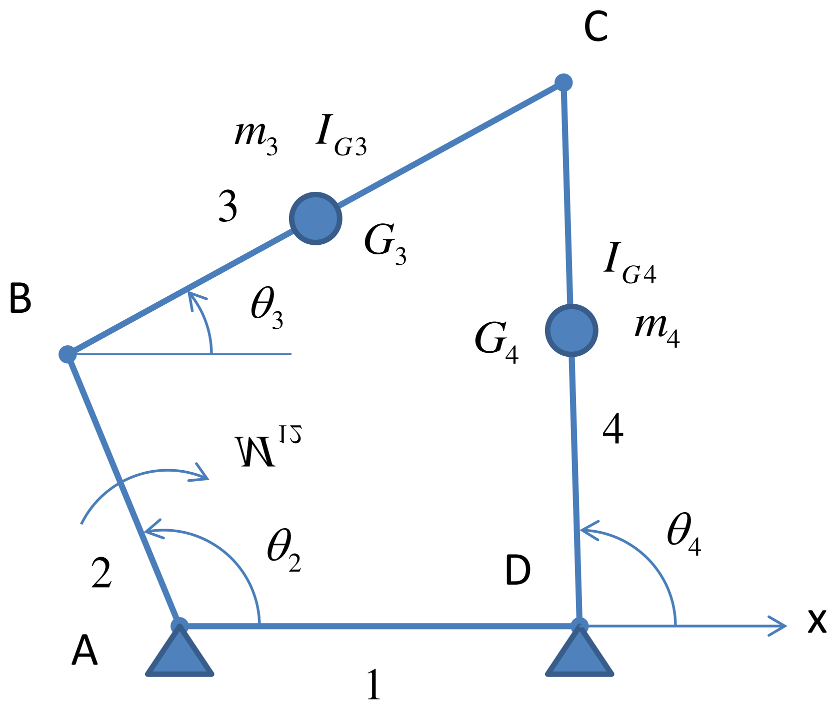
The principle of superposition:

Linear systems are those in which effect is proportional to cause. This means that the response or output of the system depends directly upon the drive or input to the system.

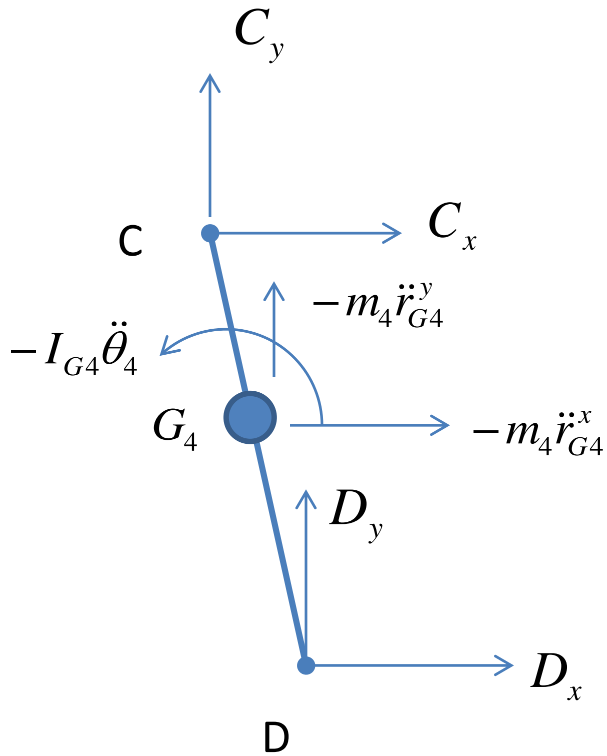
$F=kx$ output force F is proportional to the input distance x

The principle of superposition is used to solve problems by considering separately each of the drives or inputs to a system. If the system is linear the responses to each of these inputs can be summed or superposed on each other to determine the total response of the system. Thus the principle of superposition states that for linear systems the individual responses to several disturbances or driving functions can be superposed on each other to obtain the total response.

Dynamic forces for a four-bar mechanism



Determine reactions at links and reaction moment M_{12} at crank under dynamic loads.



$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum M_D = 0$$

$$D_x + C_x - m_4 \ddot{r}_{G4}^x = 0$$

$$D_x + C_x = m_4 \ddot{r}_{G4}^x \quad \#1$$

$$D_y + C_y - m_4 \ddot{r}_{G4}^y = 0$$

$$D_y + C_y = m_4 \ddot{r}_{G4}^y \quad \#2$$

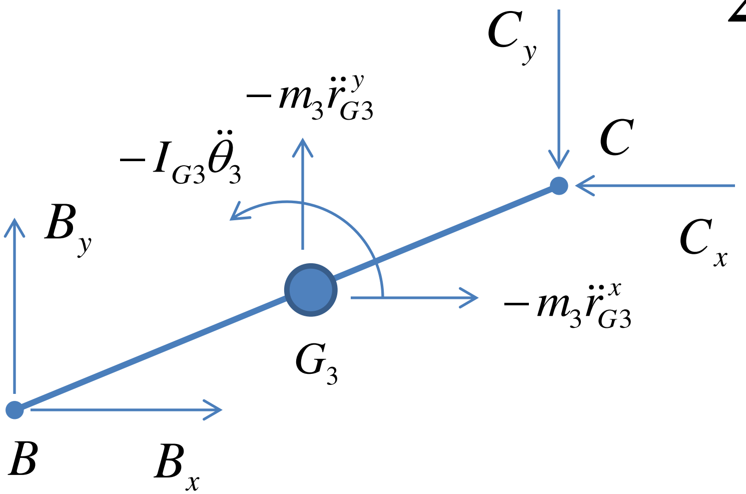
$$\vec{r}_{DG4} \times (-m_4 \ddot{r}_{G4}^x - j m_4 \ddot{r}_{G4}^y) + \vec{r}_4 \times (C_x + jC_y) - I_{G4} \ddot{\theta}_4 = 0$$

$$r_{DG4} e^{j\theta_4} \times (-m_4 \ddot{r}_{G4}^x - j m_4 \ddot{r}_{G4}^y) + r_4 e^{j\theta_4} \times (C_x + jC_y) - I_{G4} \ddot{\theta}_4 = 0$$

$$r_{DG4} (\cos\theta_4 + j\sin\theta_4) \times (-m_4 \ddot{r}_{G4}^x - j m_4 \ddot{r}_{G4}^y) + r_4 (\cos\theta_4 + j\sin\theta_4) \times (C_x + jC_y) - I_{G4} \ddot{\theta}_4 = 0$$

$$r_{DG4} m_4 (-\ddot{r}_{G4}^y \cos\theta_4 + \ddot{r}_{G4}^x \sin\theta_4) + r_4 \cos\theta_4 C_y - r_4 \sin\theta_4 C_x - I_{G4} \ddot{\theta}_4 = 0$$

$$r_4 \cos\theta_4 C_y - r_4 \sin\theta_4 C_x = -r_{DG4} m_4 (-\ddot{r}_{G4}^y \cos\theta_4 + \ddot{r}_{G4}^x \sin\theta_4) + I_{G4} \ddot{\theta}_4 \quad \#3$$



$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum M_B = 0$$

$$B_x - C_x - m_3\ddot{r}_{G3}^x = 0$$

$$B_x - C_x = m_3\ddot{r}_{G3}^x \quad \#4$$

$$B_y - C_y - m_3\ddot{r}_{G3}^y = 0$$

$$B_y - C_y = m_3\ddot{r}_{G3}^y \quad \#5$$

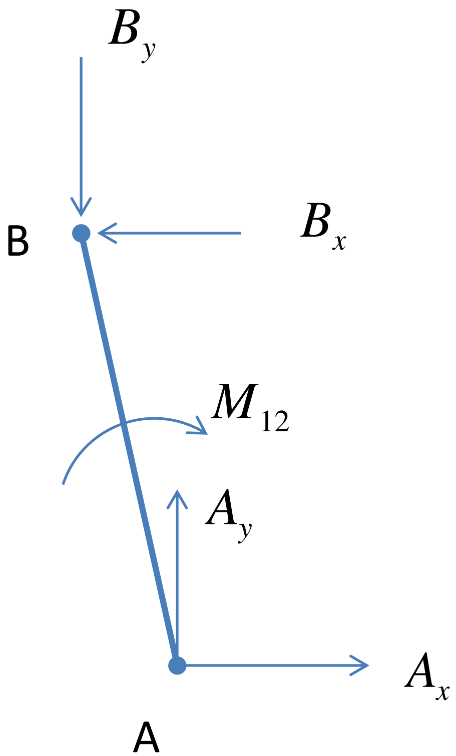
$$\vec{r}_{BG3} \times (-m_3\ddot{r}_{G3}^x - jm_3\ddot{r}_{G3}^y) + \vec{r}_3 \times (-C_x - jC_y) - I_{G3}\ddot{\theta}_3 = 0$$

$$r_{BG3}e^{j\theta_3} \times (-m_3\ddot{r}_{G3}^x - jm_3\ddot{r}_{G3}^y) + r_3 e^{j\theta_3} \times (-C_x - jC_y) - I_{G3}\ddot{\theta}_3 = 0$$

$$r_{BG3}(\cos\theta_3 + j\sin\theta_3) \times (-m_3\ddot{r}_{G3}^x - jm_3\ddot{r}_{G3}^y) + r_3(\cos\theta_3 + j\sin\theta_3) \times (-C_x - jC_y) - I_{G3}\ddot{\theta}_3 = 0$$

$$r_{BG3}m_3(-\ddot{r}_{G3}^y\cos\theta_3 + \ddot{r}_{G3}^x\sin\theta_3) - r_3\cos\theta_3 C_y + r_3\sin\theta_3 C_x - I_{G3}\ddot{\theta}_3 = 0$$

$$-r_3\cos\theta_3 C_y + r_3\sin\theta_3 C_x = -r_{BG3}m_3(-\ddot{r}_{G3}^y\cos\theta_3 + \ddot{r}_{G3}^x\sin\theta_3) + I_{G3}\ddot{\theta}_3 \quad \#6$$



$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum M_A = 0$$

$$A_x - B_x = 0 \quad \#7$$

$$A_y - B_y = 0 \quad \#8$$

$$\vec{r}_2 \times \vec{B} - M_{12} = 0$$

$$r_2 e^{j\theta_2} \times (-B_x - jB_y) - M_{12} = 0$$

$$r_2(\cos\theta_2 + j\sin\theta_2) \times (-B_x - jB_y) - M_{12} = 0$$

$$-r_2\cos\theta_2 B_y + r_2\sin\theta_2 B_x - M_{12} = 0 \quad \#9$$

unkonwns are $A_x, A_y, B_x, B_y, C_x, C_y, D_x, D_y, M_{12}$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -r_4\sin\theta_4 & r_4\cos\theta_4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r_3\sin\theta_3 & -r_3\cos\theta_3 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r_2\sin\theta_2 & -r_2\sin\theta_2 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ B_x \\ B_y \\ C_x \\ C_y \\ D_x \\ D_y \\ M_{12} \end{bmatrix} = \begin{bmatrix} m_4\ddot{r}_{G4}^x \\ m_4\ddot{r}_{G4}^y \\ -r_{DG4}m_4(-\ddot{r}_{G4}^y\cos\theta_4 + \ddot{r}_{G4}^x\sin\theta_4) + I_{G4}\ddot{\theta}_4 \\ m_3\ddot{r}_{G3}^x \\ m_3\ddot{r}_{G3}^y \\ -r_{BG3}m_3(-\ddot{r}_{G3}^y\cos\theta_3 + \ddot{r}_{G3}^x\sin\theta_3) + I_{G3}\ddot{\theta}_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{Ax} = \mathbf{B}$$

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{B}$$

To find the inertial forces

$$\vec{r}_{G4} = \vec{r}_1 + \vec{r}_{DG4} = r_1 e^{j\theta_1} + r_{DG4} e^{j\theta_4}$$

$$\dot{\vec{r}}_{G4} = r_1 j\dot{\theta}_1 e^{j\theta_1} + r_{DG4} j\dot{\theta}_4 e^{j\theta_4}$$

$$\ddot{\vec{r}}_{G4} = r_1 j\ddot{\theta}_1 e^{j\theta_1} + r_1 j^2 \dot{\theta}_1^2 e^{j\theta_1} + r_{DG4} j\ddot{\theta}_4 e^{j\theta_4} + r_{DG4} j^2 \dot{\theta}_4^2 e^{j\theta_4}$$

$$\ddot{\vec{r}}_{G4} = r_1 j\ddot{\theta}_1 e^{j\theta_1} - r_1 \dot{\theta}_1^2 e^{j\theta_1} + r_{DG4} j\ddot{\theta}_4 e^{j\theta_4} - r_{DG4} \dot{\theta}_4^2 e^{j\theta_4} \qquad (j^2 = -1)$$

$$\ddot{\vec{r}}_{G4} = (r_1 j\ddot{\theta}_1 - r_1 \dot{\theta}_1^2) e^{j\theta_1} + (r_{DG4} j\ddot{\theta}_4 - r_{DG4} \dot{\theta}_4^2) e^{j\theta_4}$$

$$\ddot{\vec{r}}_{G4} = (r_1 j\ddot{\theta}_1 - r_1 \dot{\theta}_1^2)(\cos\theta_1 + j\sin\theta_1) + (r_{DG4} j\ddot{\theta}_4 - r_{DG4} \dot{\theta}_4^2)(\cos\theta_4 + j\sin\theta_4)$$

$$\begin{aligned}\ddot{\vec{r}}_{G4} = {} & jr_1\ddot{\theta}_1 \cos\theta_1 - r_1\ddot{\theta}_1 \sin\theta_1 - r_1\dot{\theta}_1^2 \cos\theta_1 - jr_1\dot{\theta}_1^2 \sin\theta_1 \\ & + jr_{DG4}\ddot{\theta}_4 \cos\theta_4 - r_{DG4}\ddot{\theta}_4 \sin\theta_4 - r_{DG4}\dot{\theta}_4^2 \cos\theta_4 - jr_{DG4}\dot{\theta}_4^2 \sin\theta_4\end{aligned}$$

$$\begin{aligned}\ddot{\vec{r}}_{G4} = {} & -r_1\ddot{\theta}_1 \sin\theta_1 - r_1\dot{\theta}_1^2 \cos\theta_1 - r_{DG4}\ddot{\theta}_4 \sin\theta_4 - r_{DG4}\dot{\theta}_4^2 \cos\theta_4 \\ & + j(r_1\ddot{\theta}_1 \cos\theta_1 - r_1\dot{\theta}_1^2 \sin\theta_1 + r_{DG4}\ddot{\theta}_4 \cos\theta_4 - r_{DG4}\dot{\theta}_4^2 \sin\theta_4)\end{aligned}$$

$$\ddot{\vec{r}}_{G4}^{x} = -r_1\ddot{\theta}_1 \sin\theta_1 - r_1\dot{\theta}_1^2 \cos\theta_1 - r_{DG4}\ddot{\theta}_4 \sin\theta_4 - r_{DG4}\dot{\theta}_4^2 \cos\theta_4$$

$$\ddot{\vec{r}}_{G4}^{y} = r_1\ddot{\theta}_1 \cos\theta_1 - r_1\dot{\theta}_1^2 \sin\theta_1 + r_{DG4}\ddot{\theta}_4 \cos\theta_4 - r_{DG4}\dot{\theta}_4^2 \sin\theta_4$$

$$-m_4\ddot{\vec{r}}_{G4}^{x} = m_4(r_1\ddot{\theta}_1 \sin\theta_1 + r_1\dot{\theta}_1^2 \cos\theta_1 + r_{DG4}\ddot{\theta}_4 \sin\theta_4 + r_{DG4}\dot{\theta}_4^2 \cos\theta_4)$$

$$-m_4\ddot{\vec{r}}_{G4}^{y} = m_4(-r_1\ddot{\theta}_1 \cos\theta_1 + r_1\dot{\theta}_1^2 \sin\theta_1 - r_{DG4}\ddot{\theta}_4 \cos\theta_4 + r_{DG4}\dot{\theta}_4^2 \sin\theta_4)$$

$$\theta_1 = \dot{\theta}_1 = \ddot{\theta}_1 = 0$$

$$-m_4\ddot{\vec{r}}_{G4}^{x} = m_4 r_{DG4}(\ddot{\theta}_4 \sin\theta_4 + \dot{\theta}_4^2 \cos\theta_4)$$

$$-m_4\ddot{\vec{r}}_{G4}^{y} = m_4 r_{DG4}(-\ddot{\theta}_4 \cos\theta_4 + \dot{\theta}_4^2 \sin\theta_4)$$

$$\vec{r}_{G3} = \vec{r}_2 + \vec{r}_{BG3} = r_2 e^{j\theta_2} + r_{BG3} e^{j\theta_3}$$

$$\dot{\vec{r}}_{G3} = r_2 j\dot{\theta}_2 e^{j\theta_2} + r_{BG3} j\dot{\theta}_3 e^{j\theta_3}$$

$$\ddot{\vec{r}}_{G3} = r_2 j\ddot{\theta}_2 e^{j\theta_2} + r_2 j^2\dot{\theta}_2^2 e^{j\theta_2} + r_{BG3} j\ddot{\theta}_3 e^{j\theta_3} + r_{BG3} j^2\dot{\theta}_3^2 e^{j\theta_3}$$

$$\ddot{\vec{r}}_{G3} = r_2 j\ddot{\theta}_2 e^{j\theta_2} - r_2\dot{\theta}_2^2 e^{j\theta_2} + r_{BG3} j\ddot{\theta}_3 e^{j\theta_3} - r_{BG3}\dot{\theta}_3^2 e^{j\theta_3} \qquad (j^2 = -1)$$

$$\ddot{\vec{r}}_{G3} = (r_2 j\ddot{\theta}_2 - r_2\dot{\theta}_2^2) e^{j\theta_2} + (r_{BG3} j\ddot{\theta}_3 - r_{BG3}\dot{\theta}_3^2) e^{j\theta_3}$$

$$\ddot{\vec{r}}_{G3} = (r_2 j\ddot{\theta}_2 - r_2\dot{\theta}_2^2)(\cos\theta_2 + j\sin\theta_2) + (r_{BG3} j\ddot{\theta}_3 - r_{BG3}\dot{\theta}_3^2)(\cos\theta_3 + j\sin\theta_3)$$

$$\begin{aligned}\ddot{\vec{r}}_{G3} = {} & jr_2\ddot{\theta}_2\cos\theta_2 - r_2\ddot{\theta}_2\sin\theta_2 - r_2\dot{\theta}_2^2\cos\theta_2 - jr_2\dot{\theta}_2^2\sin\theta_2 \\ & + jr_{BG3}\ddot{\theta}_3\cos\theta_3 - r_{BG3}\ddot{\theta}_3\sin\theta_3 - r_{BG3}\dot{\theta}_3^2\cos\theta_3 - jr_{BG3}\dot{\theta}_3^2\sin\theta_3\end{aligned}$$

$$\begin{aligned}\ddot{\vec{r}}_{G3} = {} & -r_2\ddot{\theta}_2\sin\theta_2 - r_2\dot{\theta}_2^2\cos\theta_2 - r_{BG3}\ddot{\theta}_3\sin\theta_3 - r_{BG3}\dot{\theta}_3^2\cos\theta_3 \\ & + j(r_2\ddot{\theta}_2\cos\theta_2 - r_2\dot{\theta}_2^2\sin\theta_2 + r_{BG3}\ddot{\theta}_3\cos\theta_3 - r_{BG3}\dot{\theta}_3^2\sin\theta_3)\end{aligned}$$

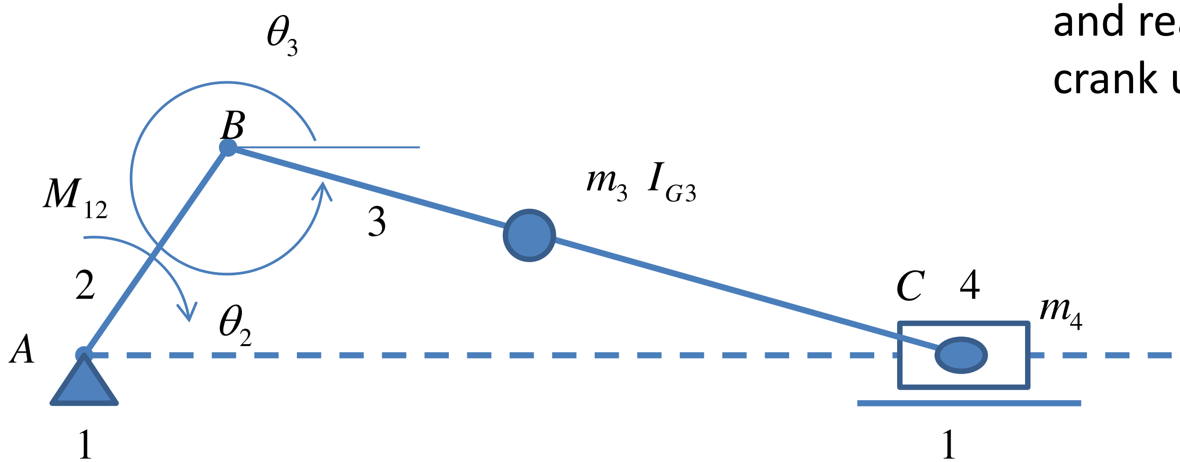
$$\ddot{\vec{r}}_{G3}^{x} = -r_2\ddot{\theta}_2 \sin\theta_2 - r_2\dot{\theta}_2^2 \cos\theta_2 - r_{BG3}\ddot{\theta}_3 \sin\theta_3 - r_{BG3}\dot{\theta}_3^2 \cos\theta_3$$

$$\ddot{\vec{r}}_{G3}^{y} = r_2\ddot{\theta}_2 \cos\theta_2 - r_2\dot{\theta}_2^2 \sin\theta_2 + r_{BG3}\ddot{\theta}_3 \cos\theta_3 - r_{BG3}\dot{\theta}_3^2 \sin\theta_3$$

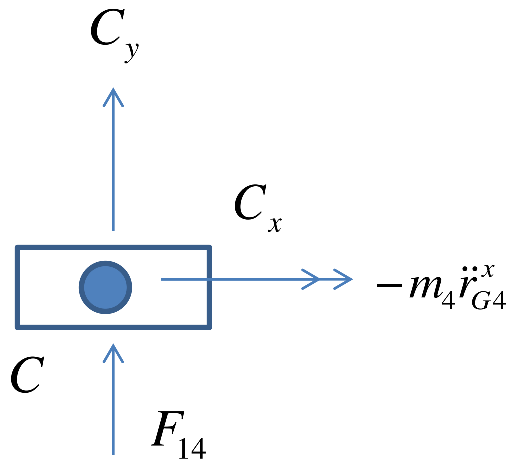
$$-m_3\ddot{\vec{r}}_{G3}^{x} = m_3(r_2\ddot{\theta}_2 \sin\theta_2 + r_2\dot{\theta}_2^2 \cos\theta_2 + r_{BG3}\ddot{\theta}_3 \sin\theta_3 + r_{BG3}\dot{\theta}_3^2 \cos\theta_3)$$

$$-m_3\ddot{\vec{r}}_{G3}^{y} = m_3(-r_2\ddot{\theta}_2 \cos\theta_2 + r_2\dot{\theta}_2^2 \sin\theta_2 - r_{BG3}\ddot{\theta}_3 \cos\theta_3 + r_{BG3}\dot{\theta}_3^2 \sin\theta_3)$$

Dynamic forces for a slider-crank mechanism



Determine reactions at links and reaction moment at crank under dynamic loads.

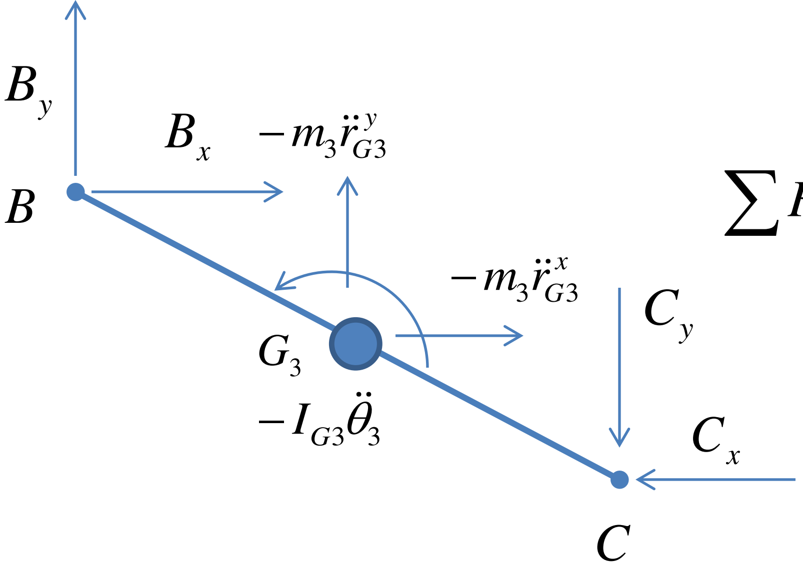


$$\sum F_x = 0 \quad \sum F_y = 0$$

$$C_x - m_4\ddot{r}_{G4}^x = 0$$

$$C_x = m_4\ddot{r}_{G4}^x \quad \#1$$

$$F_{14} + C_y = 0 \quad \#2$$



$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum M_B = 0$$

$$B_x - C_x - m_3\ddot{r}_{G3}^{x} = 0$$

$$B_x - C_x = m_3\ddot{r}_{G3}^{x} \quad \#3$$

$$B_y - C_y - m_3\ddot{r}_{G3}^{y} = 0$$

$$B_y - C_y = m_3\ddot{r}_{G3}^{y} \quad \#4$$

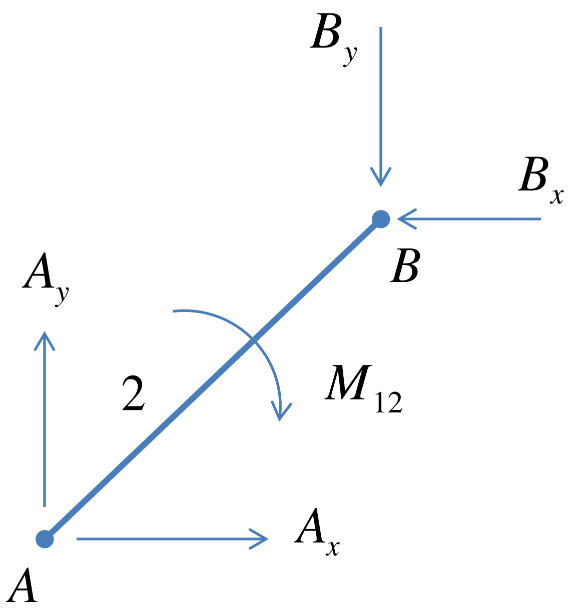
$$\vec{r}_{BG3} \times (-m_3\ddot{r}_{G3}^{x} - jm_3\ddot{r}_{G3}^{y}) + \vec{r}_3 \times (-C_x - jC_y) - I_{G3}\ddot{\theta}_3 = 0$$

$$r_{BG3}e^{\theta_3} \times (-m_3\ddot{r}_{G3}^{x} - jm_3\ddot{r}_{G3}^{y}) + r_3 e^{\theta_3} \times (-C_x - jC_y) - I_{G3}\ddot{\theta}_3 = 0$$

$$r_{BG3}(\cos\theta_3 + j\sin\theta_3) \times (-m_3\ddot{r}_{G3}^{x} - jm_3\ddot{r}_{G3}^{y}) + r_3(\cos\theta_3 + j\sin\theta_3) \times (-C_x - jC_y) - I_{G3}\ddot{\theta}_3 = 0$$

$$r_{BG3}m_3(-\ddot{r}_{G3}^{y}\cos\theta_3 + \ddot{r}_{G3}^{x}\sin\theta_3) - r_3\cos\theta_3 C_y + r_3\sin\theta_3 C_x - I_{G3}\ddot{\theta}_3 = 0$$

$$-r_3\cos\theta_3 C_y + r_3\sin\theta_3 C_x = -r_{BG3}m_3(-\ddot{r}_{G3}^{y}\cos\theta_3 + \ddot{r}_{G3}^{x}\sin\theta_3) + I_{G3}\ddot{\theta}_3 \quad \#5$$



$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum M_A = 0$$

$$A_x - B_x = 0 \quad \#6$$

$$A_y - B_y = 0 \quad \#7$$

$$\vec{r}_2 \times \vec{B} - M_{12} = 0$$

$$r_2 e^{j\theta_2} \times (-B_x - jB_y) - M_{12} = 0$$

$$r_2(\cos\theta_2 + j\sin\theta_2) \times (-B_x - jB_y) - M_{12} = 0$$

$$-r_2\cos\theta_2 B_y + r_2\sin\theta_2 B_x - M_{12} = 0 \quad \#8$$

unkonwns are $A_x, A_y, B_x, B_y, C_x, C_y, D_x, D_y, M_{12}$

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & r_3 \sin\theta_3 & -r_3 \cos\theta_3 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & r_2 \sin\theta_2 & -r_2 \cos\theta_2 & 0 & 0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
A_x \\ A_y \\ B_x \\ B_y \\ C_x \\ C_y \\ F_{14} \\ M_{12}
\end{bmatrix}
=
\begin{bmatrix}
m_4 \ddot{r}_{G4}^{x} \\
0 \\
m_3 \ddot{r}_{G3}^{x} \\
m_3 \ddot{r}_{G3}^{y} \\
-r_{BG3} m_3 (-\ddot{r}_{G3}^{y} \cos\theta_3 + \ddot{r}_{G3}^{x} \sin\theta_3) + I_{G3} \ddot{\theta}_3 \\
0 \\
0 \\
0
\end{bmatrix}
$$

$$\mathbf{Ax} = \mathbf{B}$$

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{B}$$

To find inertial forces

$$\vec{r}_{G3} = \vec{r}_2 + \vec{r}_{BG3} = r_2 e^{\theta_2} + r_{BG3} e^{\theta_3}$$

$$\dot{\vec{r}}_{G3} = r_2 j\dot{\theta}_2 e^{j\theta_2} + r_{BG3} j\dot{\theta}_3 e^{j\theta_3}$$

$$\ddot{\vec{r}}_{G3} = r_2 j\ddot{\theta}_2 e^{j\theta_2} + r_2 j^2 \dot{\theta}_2^2 e^{j\theta_2} + r_{BG3} j\ddot{\theta}_3 e^{j\theta_3} + r_{BG3} j^2 \dot{\theta}_3^2 e^{j\theta_3}$$

$$\ddot{\vec{r}}_{G3} = r_2 j\ddot{\theta}_2 e^{j\theta_2} - r_2 \dot{\theta}_2^2 e^{j\theta_2} + r_{BG3} j\ddot{\theta}_3 e^{j\theta_3} - r_{BG3} \dot{\theta}_3^2 e^{j\theta_3} \qquad (j^2 = -1)$$

$$\ddot{\vec{r}}_{G3} = (r_2 j\ddot{\theta}_2 - r_2 \dot{\theta}_2^2) e^{j\theta_2} + (r_{BG3} j\ddot{\theta}_3 - r_{BG3} \dot{\theta}_3^2) e^{j\theta_3}$$

$$\ddot{\vec{r}}_{G3} = (r_2 j\ddot{\theta}_2 - r_2 \dot{\theta}_2^2)(\cos\theta_2 + j\sin\theta_2) + (r_{BG3} j\ddot{\theta}_3 - r_{BG3} \dot{\theta}_3^2)(\cos\theta_3 + j\sin\theta_3)$$

$$\begin{aligned}\ddot{\vec{r}}_{G3} = {} & jr_2\ddot{\theta}_2 \cos\theta_2 - r_2\ddot{\theta}_2 \sin\theta_2 - r_2\dot{\theta}_2^2 \cos\theta_2 - jr_2\dot{\theta}_2^2 \sin\theta_2 \\ & + jr_{BG3}\ddot{\theta}_3 \cos\theta_3 - r_{BG3}\ddot{\theta}_3 \sin\theta_3 - r_{BG3}\dot{\theta}_3^2 \cos\theta_3 - jr_{BG3}\dot{\theta}_3^2 \sin\theta_3\end{aligned}$$

$$\begin{aligned}\ddot{\vec{r}}_{G3} = {} & -r_2\ddot{\theta}_2 \sin\theta_2 - r_2\dot{\theta}_2^2 \cos\theta_2 - r_{BG3}\ddot{\theta}_3 \sin\theta_3 - r_{BG3}\dot{\theta}_3^2 \cos\theta_3 \\ & + j(r_2\ddot{\theta}_2 \cos\theta_2 - r_2\dot{\theta}_2^2 \sin\theta_2 + r_{BG3}\ddot{\theta}_3 \cos\theta_3 - r_{BG3}\dot{\theta}_3^2 \sin\theta_3)\end{aligned}$$

$$\ddot{\vec{r}}_{G3}^{x} = -r_2\ddot{\theta}_2 \sin\theta_2 - r_2\dot{\theta}_2^2 \cos\theta_2 - r_{BG3}\ddot{\theta}_3 \sin\theta_3 - r_{BG3}\dot{\theta}_3^2 \cos\theta_3$$

$$\ddot{\vec{r}}_{G3}^{y} = r_2\ddot{\theta}_2 \cos\theta_2 - r_2\dot{\theta}_2^2 \sin\theta_2 + r_{BG3}\ddot{\theta}_3 \cos\theta_3 - r_{BG3}\dot{\theta}_3^2 \sin\theta_3$$

$$-m_3\ddot{\vec{r}}_{G3}^{x} = m_3(r_2\ddot{\theta}_2 \sin\theta_2 + r_2\dot{\theta}_2^2 \cos\theta_2 + r_{BG3}\ddot{\theta}_3 \sin\theta_3 + r_{BG3}\dot{\theta}_3^2 \cos\theta_3)$$

$$-m_3\ddot{\vec{r}}_{G3}^{y} = m_3(-r_2\ddot{\theta}_2 \cos\theta_2 + r_2\dot{\theta}_2^2 \sin\theta_2 - r_{BG3}\ddot{\theta}_3 \cos\theta_3 + r_{BG3}\dot{\theta}_3^2 \sin\theta_3)$$

$$\vec{r}_C = \vec{r}_2 + \vec{r}_3 = r_2 e^{\theta_2} + r_3 e^{\theta_3}$$

$$\dot{\vec{r}}_C = r_2 j\dot{\theta}_2 e^{j\theta_2} + r_3 j\dot{\theta}_3 e^{j\theta_3}$$

$$\ddot{\vec{r}}_C = r_2 j\ddot{\theta}_2 e^{j\theta_2} + r_2 j^2\dot{\theta}_2^2 e^{j\theta_2} + r_3 j\ddot{\theta}_3 e^{j\theta_3} + r_3 j^2\dot{\theta}_3^2 e^{j\theta_3}$$

$$\ddot{\vec{r}}_C = r_2 j\ddot{\theta}_2 e^{j\theta_2} - r_2\dot{\theta}_2^2 e^{j\theta_2} + r_3 j\ddot{\theta}_3 e^{j\theta_3} - r_3\dot{\theta}_3^2 e^{j\theta_3} \qquad (j^2 = -1)$$

$$\ddot{\vec{r}}_C = (r_2 j\ddot{\theta}_2 - r_2\dot{\theta}_2^2)e^{j\theta_2} + (r_3 j\ddot{\theta}_3 - r_3\dot{\theta}_3^2)e^{j\theta_3}$$

$$\ddot{\vec{r}}_C = (r_2 j\ddot{\theta}_2 - r_2\dot{\theta}_2^2)(\cos\theta_2 + j\sin\theta_2) + (r_3 j\ddot{\theta}_3 - r_3\dot{\theta}_3^2)(\cos\theta_3 + j\sin\theta_3)$$

$$\begin{aligned}\ddot{\vec{r}}_C = {} & jr_2\ddot{\theta}_2\cos\theta_2 - r_2\ddot{\theta}_2\sin\theta_2 - r_2\dot{\theta}_2^2\cos\theta_2 - jr_2\dot{\theta}_2^2\sin\theta_2 \\ & + jr_3\ddot{\theta}_3\cos\theta_3 - r_3\ddot{\theta}_3\sin\theta_3 - r_3\dot{\theta}_3^2\cos\theta_3 - jr_3\dot{\theta}_3^2\sin\theta_3\end{aligned}$$

$$\begin{aligned}\ddot{\vec{r}}_C = {} & -r_2\ddot{\theta}_2\sin\theta_2 - r_2\dot{\theta}_2^2\cos\theta_2 - r_3\ddot{\theta}_3\sin\theta_3 - r_3\dot{\theta}_3^2\cos\theta_3 \\ & + j(r_2\ddot{\theta}_2\cos\theta_2 - r_2\dot{\theta}_2^2\sin\theta_2 + r_3\ddot{\theta}_3\cos\theta_3 - r_3\dot{\theta}_3^2\sin\theta_3)\end{aligned}$$

$$\ddot{\vec{r}}_C^{\,x} = -r_2\ddot{\theta}_2\sin\theta_2 - r_2\dot{\theta}_2^2\cos\theta_2 - r_3\ddot{\theta}_3\sin\theta_3 - r_3\dot{\theta}_3^2\cos\theta_3$$

$$-m_3\ddot{\vec{r}}_C^{\,x} = m_3(r_2\ddot{\theta}_2\sin\theta_2 + r_2\dot{\theta}_2^2\cos\theta_2 + r_3\ddot{\theta}_3\sin\theta_3 + r_3\dot{\theta}_3^2\cos\theta_3)$$