FAHAD BIN SULTAN UNIVERSITY
College of Engineering
Department of Mechanical Engineering

Mechanical Design

MECH 330

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Chapter 5

Theory of Failure: Static Loading

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Failure under static loading

- Topics
  - Failures from static loading
    - Ductile Failures
      - Maximum Shear Stress
      - Maximum Distortion Energy
    - Brittle Failures
      - Maximum Normal Stress
      - Coulomb-Mohr
      - Modified Mohr
- Reading --- Chapter 5

Failures under static loading

Introduction

- **Strength** is a property or characteristic of a mechanical element.
- the **strength** of a **material** is its **ability to withstand** an applied load without failure or plastic deformation.
- This property results from the material identity, the treatment and processing incidental to creating its geometry, and the loading, and it is at the controlling or critical location.

Failures under static load

- **Static load–** a stationary load that is gradually applied having an unchanging **magnitude** and **direction**.
  - A static load can produce axial tension or compression, a shear load, a bending load, a torsional load, or any combination of these.
- **Failure–** a part is permanently distorted and will not function properly. Thus it has had reliability downgraded.
  - A part has been separated into two or more pieces.

- Under any load combination, there is always a combination of **normal** and **shearing** stresses in the material.

- Generally, mechanical components **fail** because the **applied stresses exceeds the material's strength**.

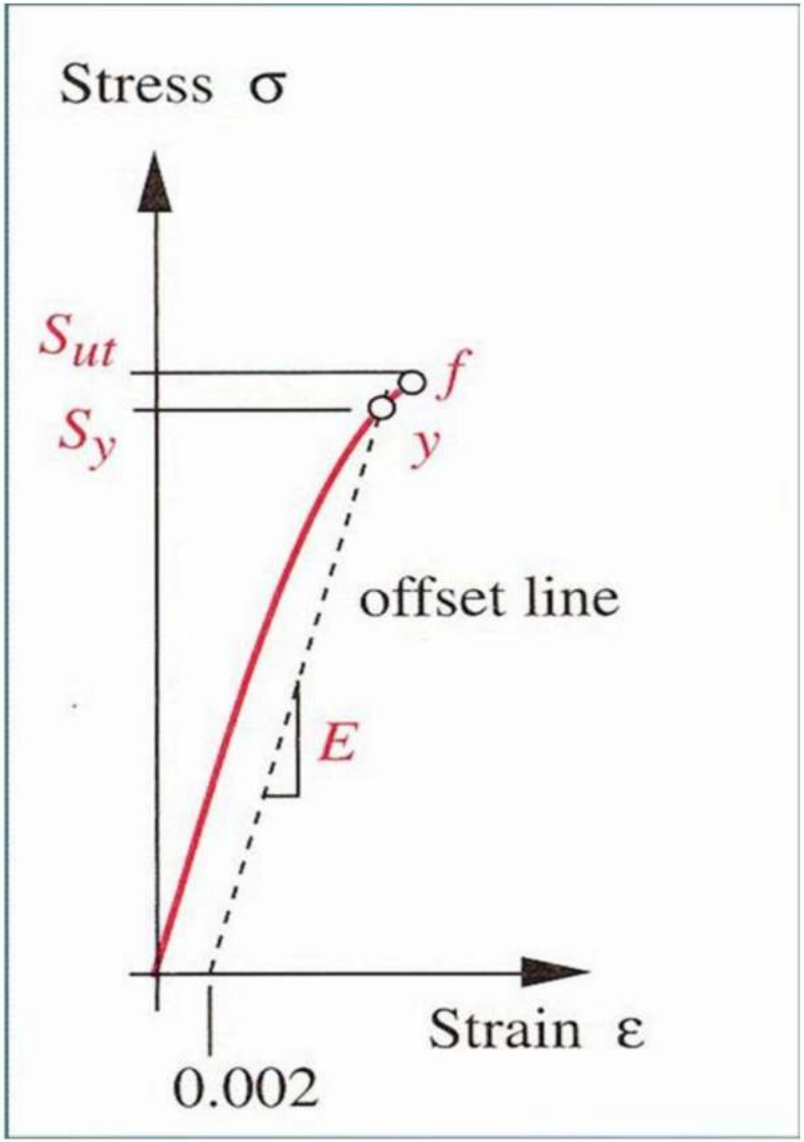
- Failure – any change in a machine part which makes it unable to perform its intended function.(From Spotts M. F. and Shoup T. E.)

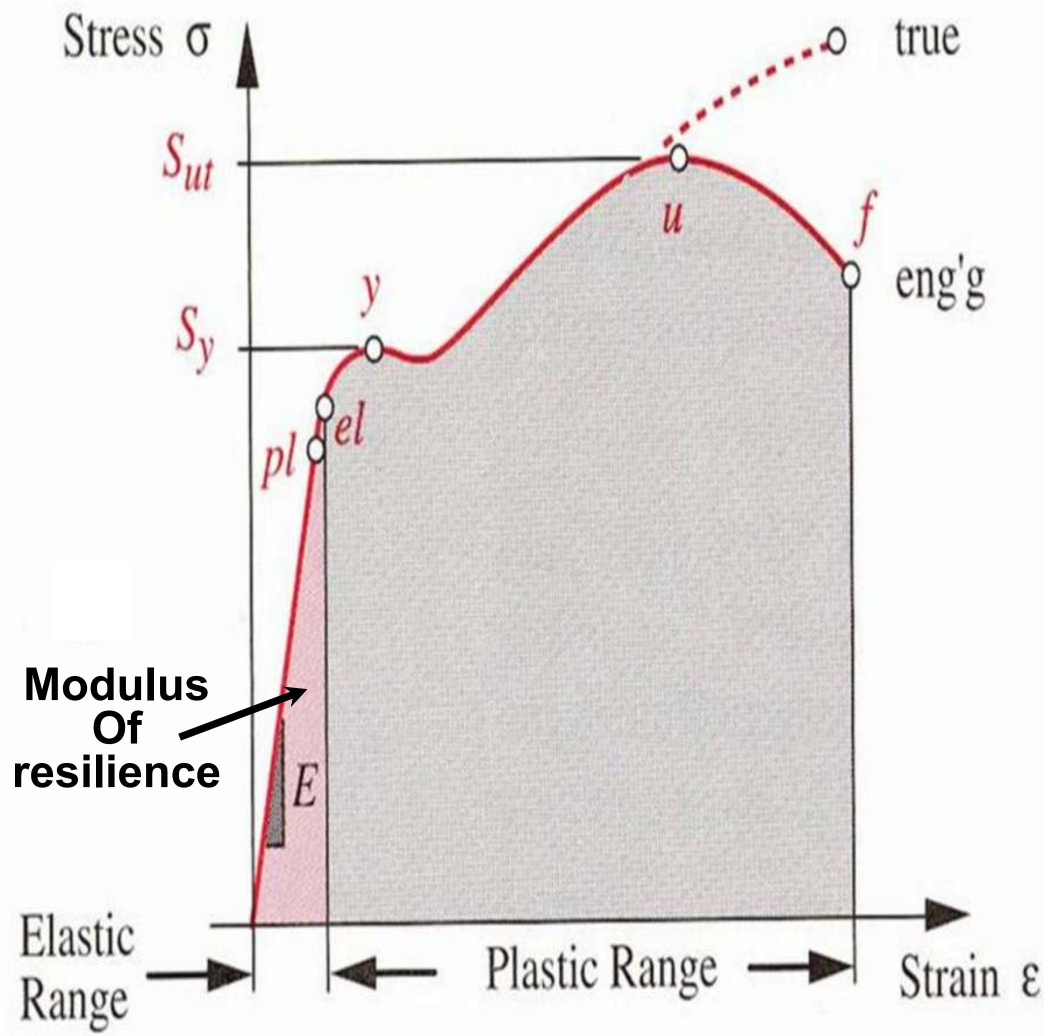
- We will normally use a <u>**yield failure criteria** for **ductile materials**</u>. The ductile failure theories presented are based on yield.

- We will use a <u>**fracture failure criteria** for **brittle materials**</u>. The brittle failure theories presented are based on <u>**ultimate strength**</u>.

❑ Ductile and Brittle Materials

- **A ductile material** deforms significantly before fracturing.
- Ductility is measured by % elongation at the fracture point.
- Materials with 5% or more elongation are considered ductile.
- The limiting strength of ductile materials is the stress at yield point.
- **A brittle material** yields very little before fracturing.
- The yield strength is approximately the same as the ultimate strength in tension.
- The ultimate strength in compression is much larger than the ultimate strength in tension.
- The limiting strength of brittle materials is the ultimate stress.





Tensile test diagrams of brittle and ductile materials

Tensile Test

3.3 ■ *Implications of the "Engineering" Stress–Strain Curve*

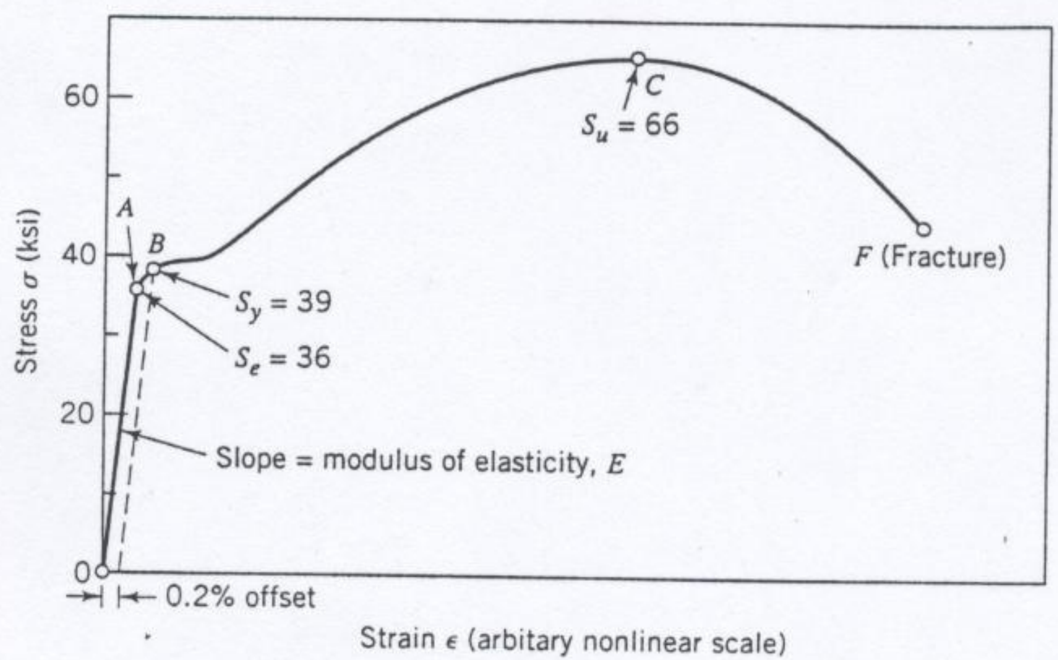


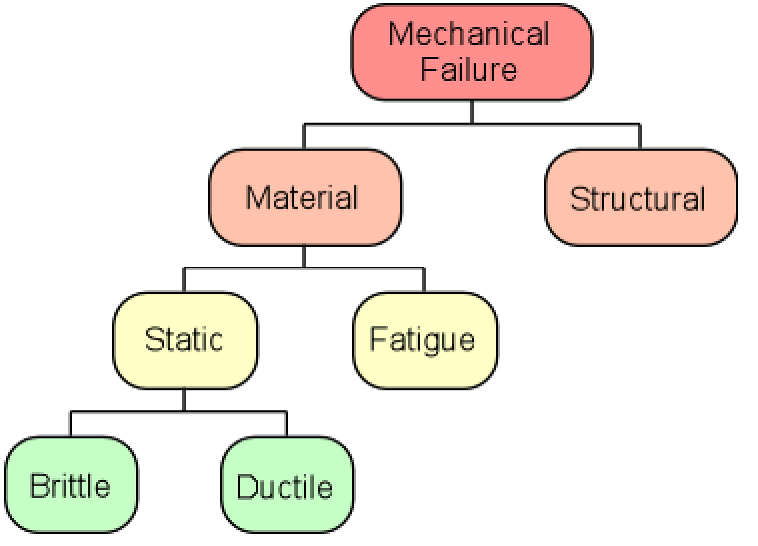
FIGURE 3.1
Engineering stress–strain curve—hot-rolled 1020 steel.

AISI 1020 is a low hardenability and low tensile carbon steel with Brinell hardness of 119 - 235 and tensile strength of 410 - 790 MPa. This standard has high machinability, high ductility, high strength, and good weldability.

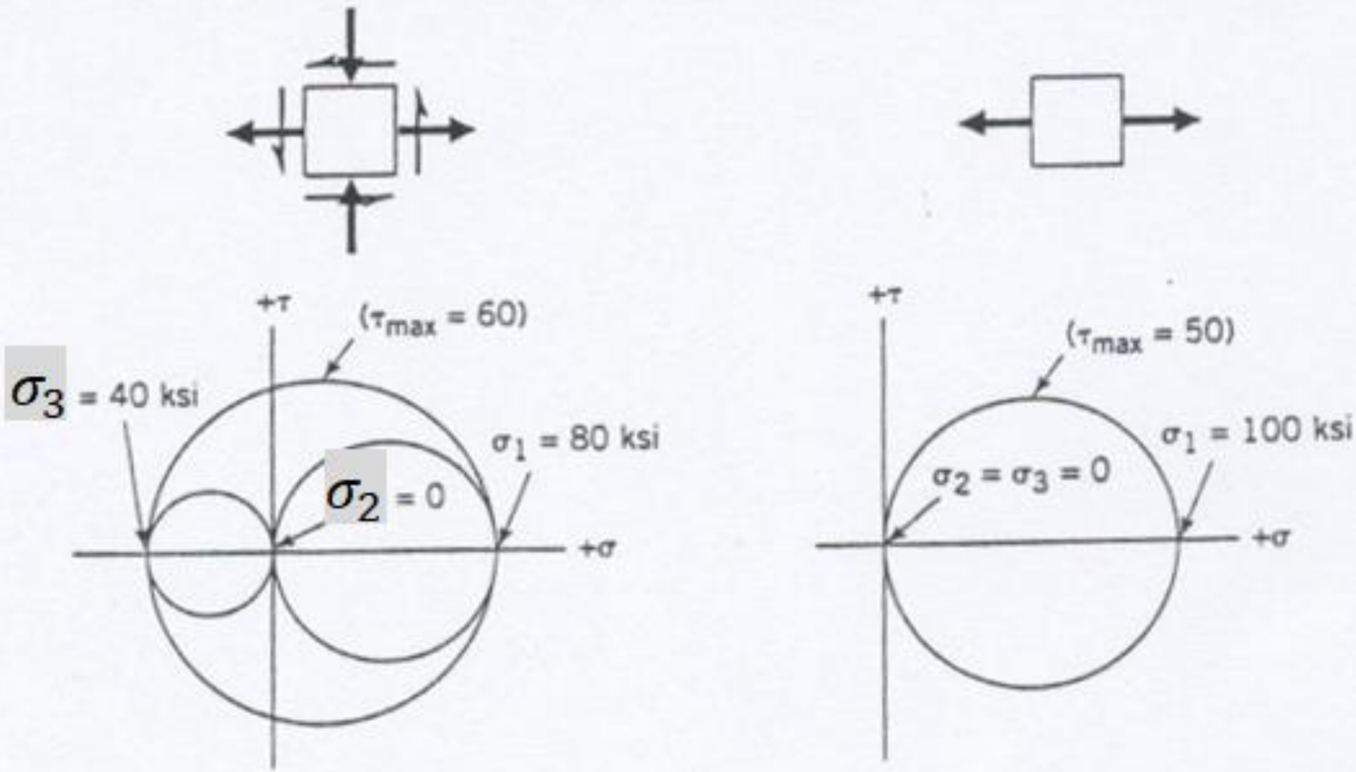
Failure Theories

- Static failure
  - Ductile
  - Brittle
  - Stress concentration

- Recall
  - Ductile
    - Significant plastic deformation between yield and fracture
  - Brittle
    - Yield ~= fracture



Mohr's Circle for Tensile Test



(*a*) Proposed application involving

$$\sigma_1 = 80, \sigma_2 = 0, \sigma_3 = -40\ ksi$$

(*a*) Standard tensile test of proposed material. Tensile strength, $S = 100$ ksi

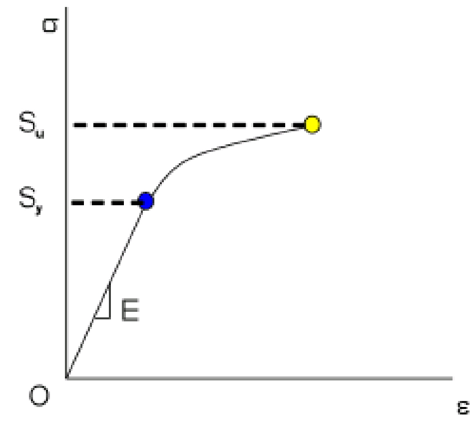
FIGURE 6.6
Typical situation requiring a failure theory.

Static Ductile Failure

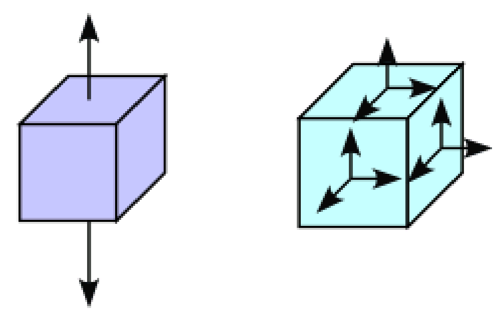
- Two primary theories for static ductile failure
  - Maximum Shear Stress criterion
    - MSS
  - Von Mises criterion
    - Maximum Distortion-energy Theory
    - MDE

Failure Theory Problem Statement

- Given:
  - Stress-strain data for simple uniaxial tension

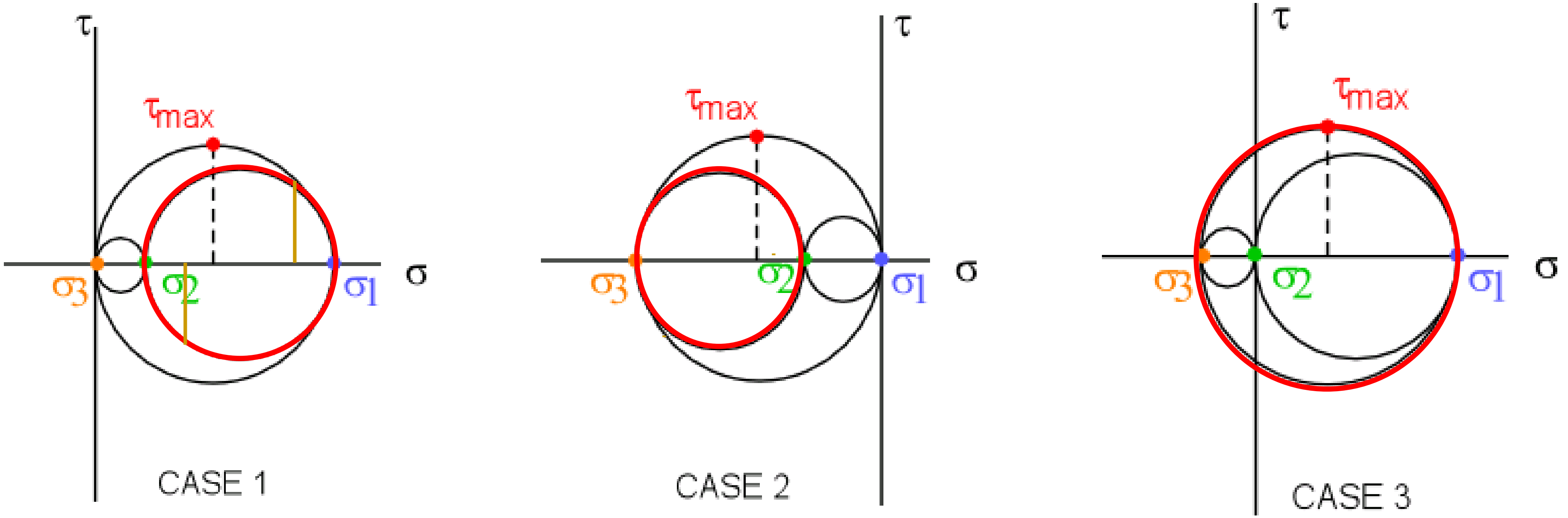


- Find:
  - When failure occurs for general state of stress



Out of Plane Maximum Shear for Biaxial State of Stress

- Case 1
  - $\sigma_{1,2} > 0$
  - $\sigma_3 = 0$
  - $\tau_{max} = \dfrac{\sigma_1}{2}$
- Case 2
  - $\sigma_{2,3} < 0$
  - $\sigma_1 = 0$
  - $\tau_{max} = \dfrac{|\sigma_3|}{2}$
- Case 3
  - $\sigma_1 > 0,\ \sigma_3 < 0$
  - $\sigma_2 = 0$
  - $\tau_{max} = \dfrac{|\sigma_1 - \sigma_3|}{2}$



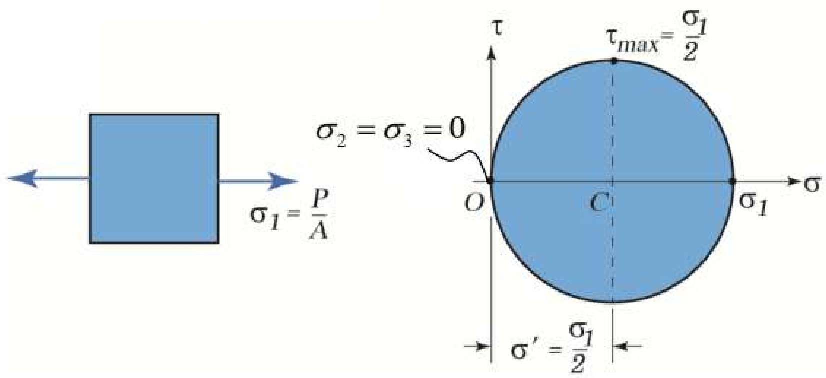
Maximum-Shear-Stress Theory for Ductile Materials

- The **maximum-shear-stress theory** predicts that..

  "*yielding begins whenever the maximum shear stress in any element equals or exceeds the maximum shear stress in a tension test specimen of the same material when that specimen begins to yield".*

- The MSS theory is also referred to as the **Tresca** or **Guest theory**.

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} \geq \frac{S_y}{2} \quad \text{or} \quad \sigma_1 - \sigma_3 \geq S_y \qquad \text{Eq (5-1)}$$



simple tension test

It can be modified to incorporate a **factor of safety, *n*** by

$$\tau_{max} = \frac{S_y}{2n} \quad \text{or} \quad \sigma_1 - \sigma_3 = \frac{S_y}{n}$$

Static Ductile Failure

- Max Shear Stress criterion
  - Material yields (fails) when:

1) $\tau_{\max} \geq \dfrac{S_y}{2}$ or

2) $(\sigma_1 - \sigma_3) \geq S_y$

  - Factor of Safety:

$$\eta = \frac{S_y}{(\sigma_1 - \sigma_3)} = \frac{S_y}{2\tau_{\max}}$$

Maximum Shear Stress Criteria

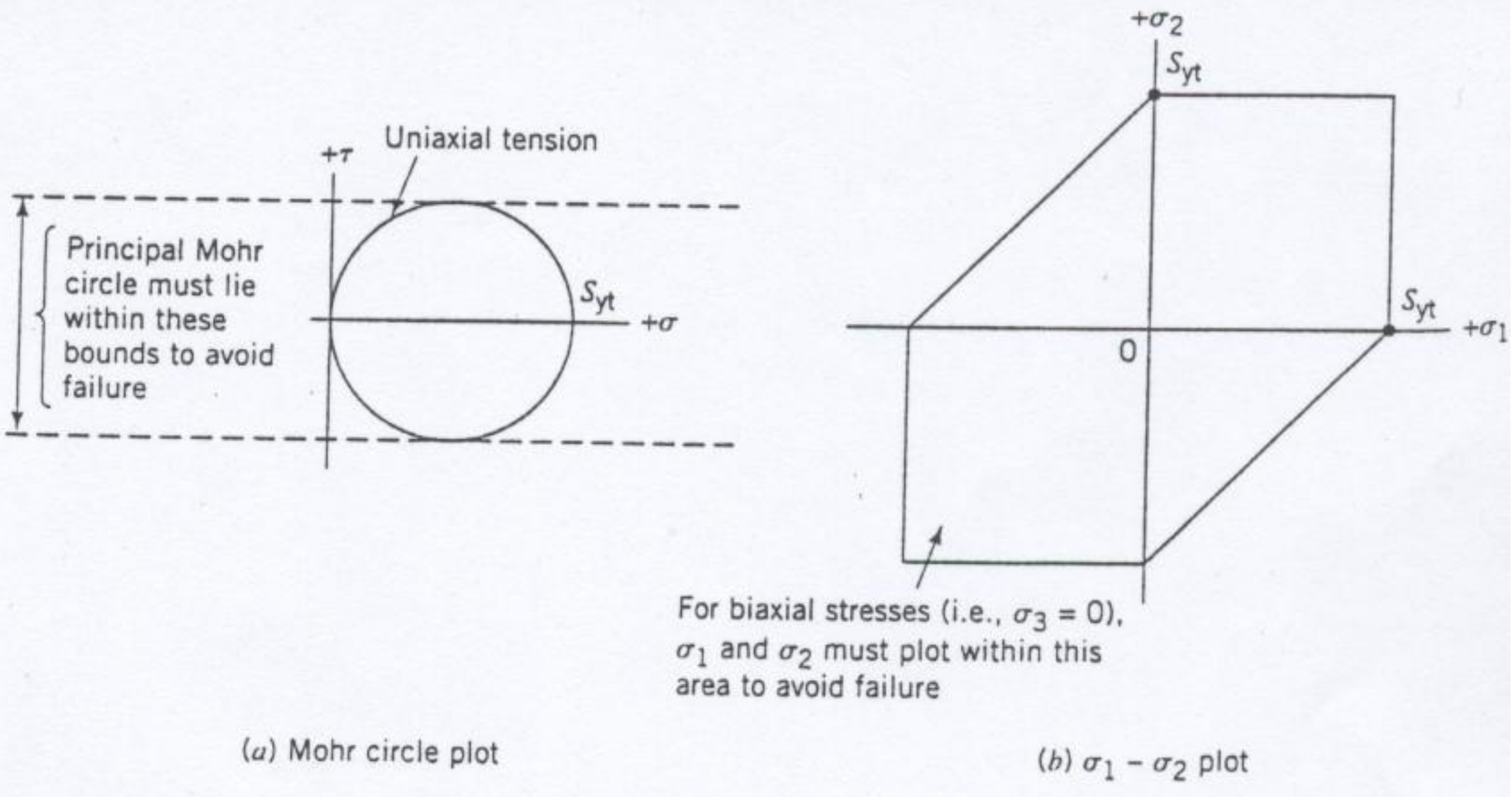
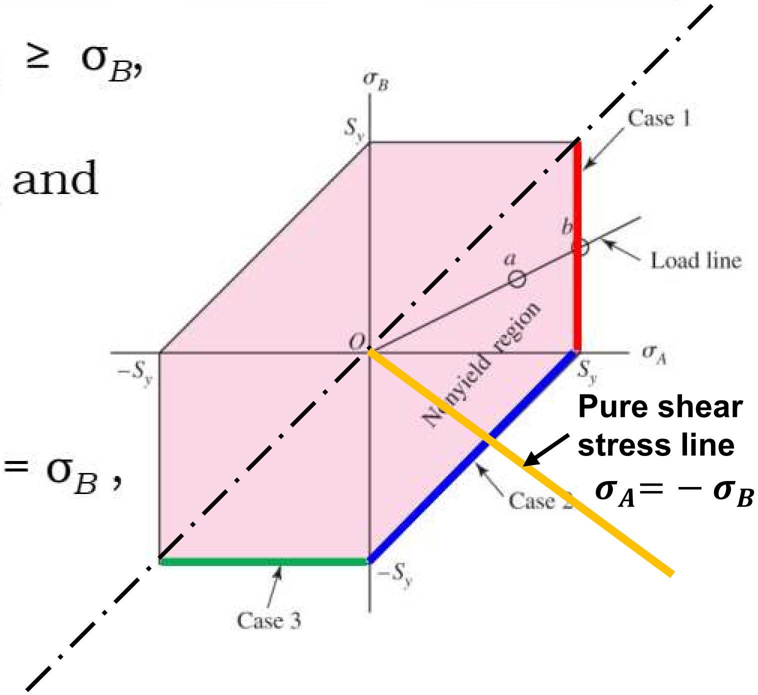
- 

FIGURE 6.8
Two graphical representations of the maximum-shear-stress theory.

Maximum-Shear-Stress Theory for Ductile Materials

Assuming a *plane stress problem* with $\sigma_A \geq \sigma_B$, there are three cases to consider



✓ **Case 1**: $\sigma_A \geq \sigma_B \geq 0$. For this case, $\sigma_1 = \sigma_A$ and $\sigma_3 = 0$. Equation (5–1)reduces to a yield condition of

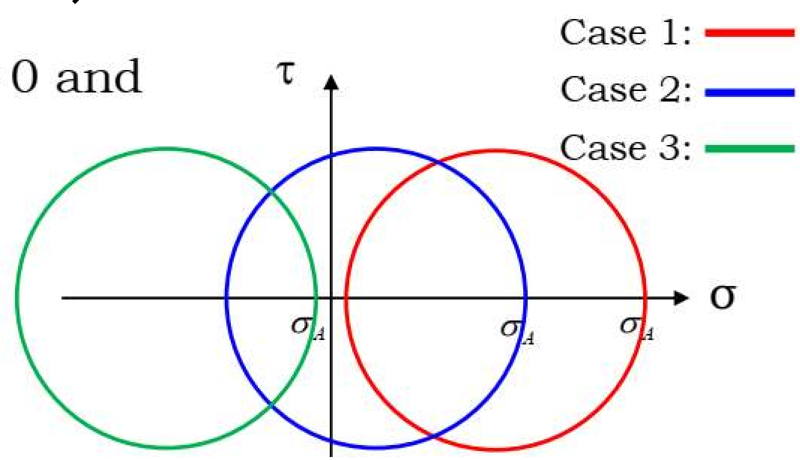
$$\sigma_A \geq S_y$$

✓ **Case 2**: $\sigma_A \geq 0 \geq \sigma_B$. Here, $\sigma 1 = \sigma_A$ and $\sigma_3 = \sigma_B$, and Eq. (5–1) becomes

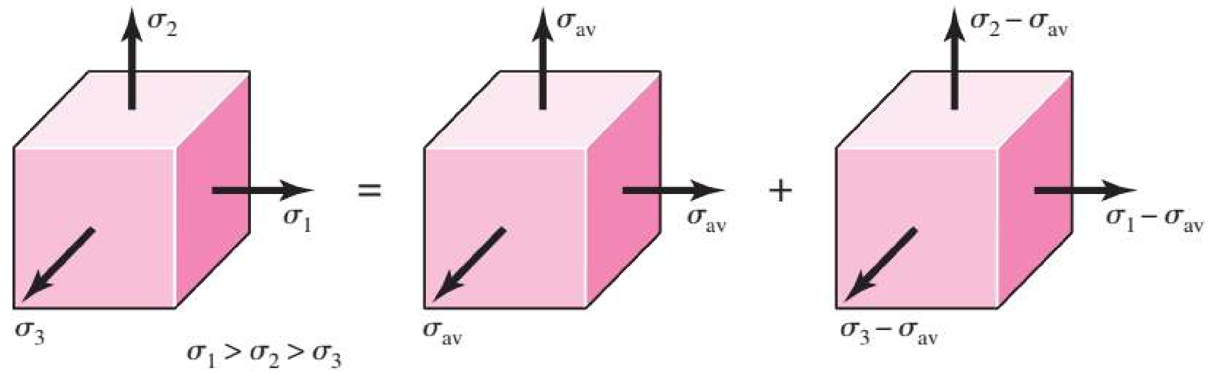
$$\sigma_A - \sigma_B \geq S_y$$

✓ **Case 3**: $0 \geq \sigma_A \geq \sigma_B$. For this case, $\sigma_1 = 0$ and $\sigma_3 = \sigma_B$, and Eq. (5–1) gives

$$\sigma_B \leq -S_y$$



Maximum Distortion-energy Theory MED



(*a*) Triaxial stresses (*b*) Hydrostatic component (*c*) Distortional component

$$\epsilon_1 = \frac{1}{E}\left[\sigma_1 - \nu(\sigma_2 + \sigma_3)\right]$$

$$u = \tfrac{1}{2}\left[\epsilon_1\sigma_1 + \epsilon_2\sigma_2 + \epsilon_3\sigma_3\right]$$

$$u = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right]$$

$$u_v = \frac{3\sigma_{av}^2}{2E}(1 - 2\nu)$$

$$u_v = \frac{1 - 2\nu}{6E}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\sigma_1\sigma_2 + 2\sigma_2\sigma_3 + 2\sigma_3\sigma_1\right)$$

$$\sigma_{av} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$$

$$u_d = u - u_v = \frac{1 + \nu}{3E}\left[\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}\right]$$

Static Ductile Failure

Maximum Distortion-energy Theory

The *distortion-energy theory* predicts that *yielding occurs when the distortion strain energy per unit volume reaches or exceeds the distortion strain energy per unit volume for yield in simple tension or compression of the same material.*

$$u_d = u - u_v = \frac{1+\nu}{3E}\left[\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}\right] \geq u_d = \frac{1+\nu}{3E}S_y^2$$

$$\sigma' = \sigma_e = \left[\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}\right]^{1/2} \geq S_y$$

Static Ductile Failure

- Max Shear Stress criterion
  - Material yields (fails) when:

1) $\tau_{\max} \geq \dfrac{S_y}{2}$ or

2) $\left(\sigma_1 - \sigma_3\right) \geq S_y$

  - Factor of Safety:

$$\eta = \frac{S_y}{\left(\sigma_1 - \sigma_3\right)} = \frac{S_y}{2\tau_{\max}}$$

Static Ductile Failure

Maximum Distortion-energy Theory

- Von Mises criterion
  - Let the Mises stress (σ_e, equivalent stress) be:

$$\sigma_e = \sqrt{\frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2\right]}$$

  - Then failure (yield) occurs when: $\sigma_e \geq S_y$
  - Factor of Safety: $\eta = \frac{S_y}{\sigma_e}$
    - Typically, $1.25 \leq \eta \leq 4$
    - Want a margin of error but not completely overdesigned

Static Ductile Failure

- Max Shear Stress criterion
  - Material yields (fails) when:

1) $\tau_{max} \geq \dfrac{S_y}{2}$ or

2) $(\sigma_1 - \sigma_3) \geq S_y$

  - Factor of Safety:

$$\eta = \frac{S_y}{(\sigma_1 - \sigma_3)} = \frac{S_y}{2\tau_{max}}$$

Distortion-Energy Theory for Ductile Materials

Left hand side is defined as von Mises stress or effective stress σ' Or σ_e

$$\sigma_e = \left[\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}\right]^{1/2} \le S_y$$

For *plane stress* simplifies to (principal stress σ_A , σ_B , 0)

$$\sigma_e = \left(\sigma_A^2 - \sigma_A\sigma_B + \sigma_B^2\right)^{1/2}$$

Using *xyz* components of three-dimensional stress, the von Mises stress can be written as

$$\sigma_e = \frac{1}{\sqrt{2}}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right)\right]^{1/2}$$

$(\sigma_z = \tau_{xz} = \tau_{yz} = 0)$

$$\sigma_e = \left(\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau_{xy}^2\right)^{1/2}$$

Distortion-Energy Theory for Ductile Materials

- Consider a case of pure shear τ_{xy} ,where for plane stress $\sigma_x = \sigma_y = 0$. For yield

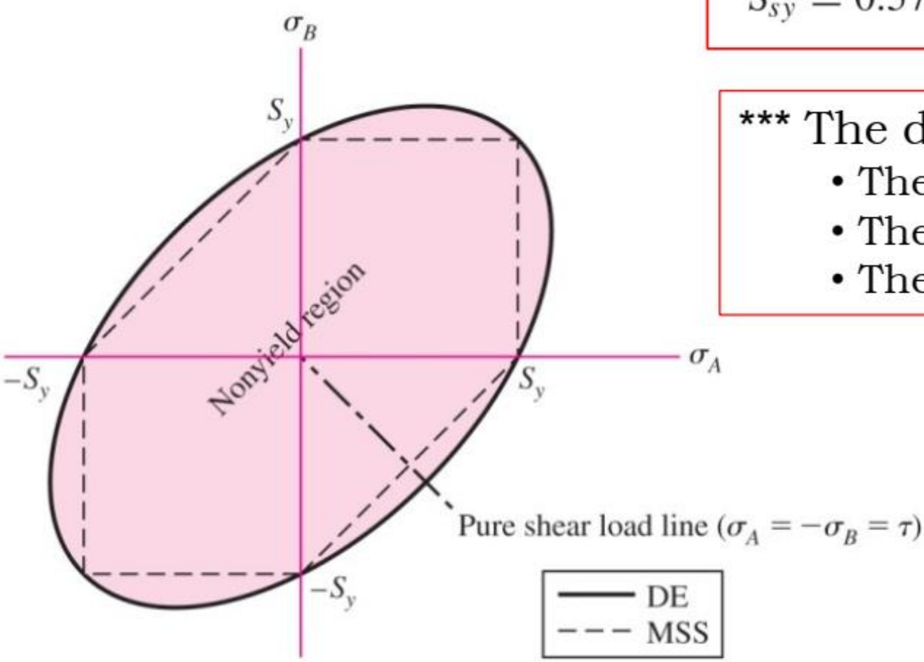
$$\left(3\tau_{xy}^2\right)^{1/2} = S_y \qquad \text{or} \qquad \tau_{xy} = \frac{S_y}{\sqrt{3}} = 0.577S_y$$

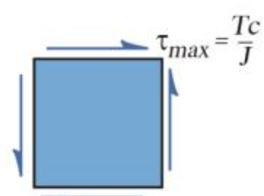
Thus, the *shear yield strength* predicted by the distortion energy theory is

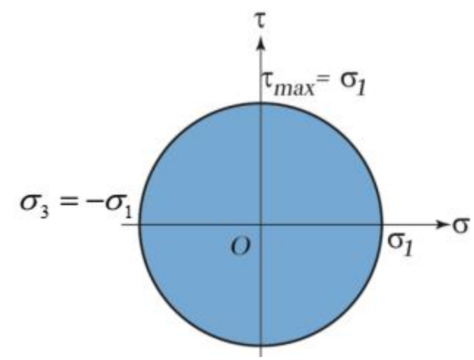
$$S_{sy} = 0.577S_y$$

*** The distortion-energy theory is also called

- The *von Mises* / *von Mises-Hencky* theory
- The *shear-energy* theory
- The *octahedral-shear-stress* theory

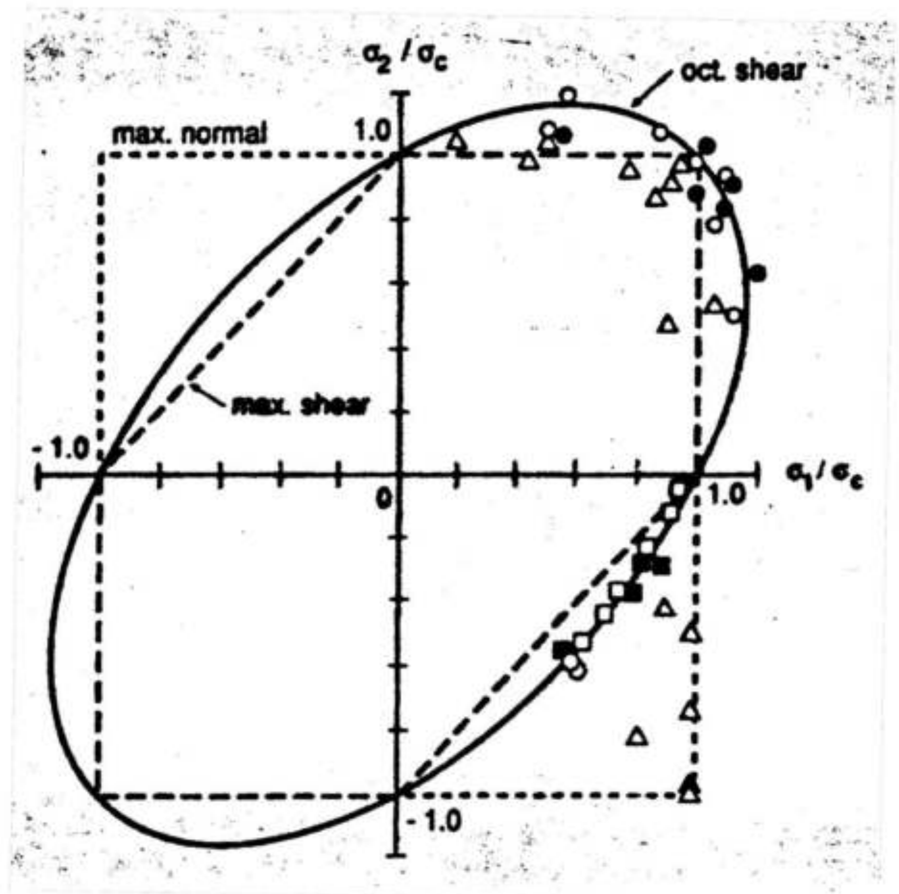






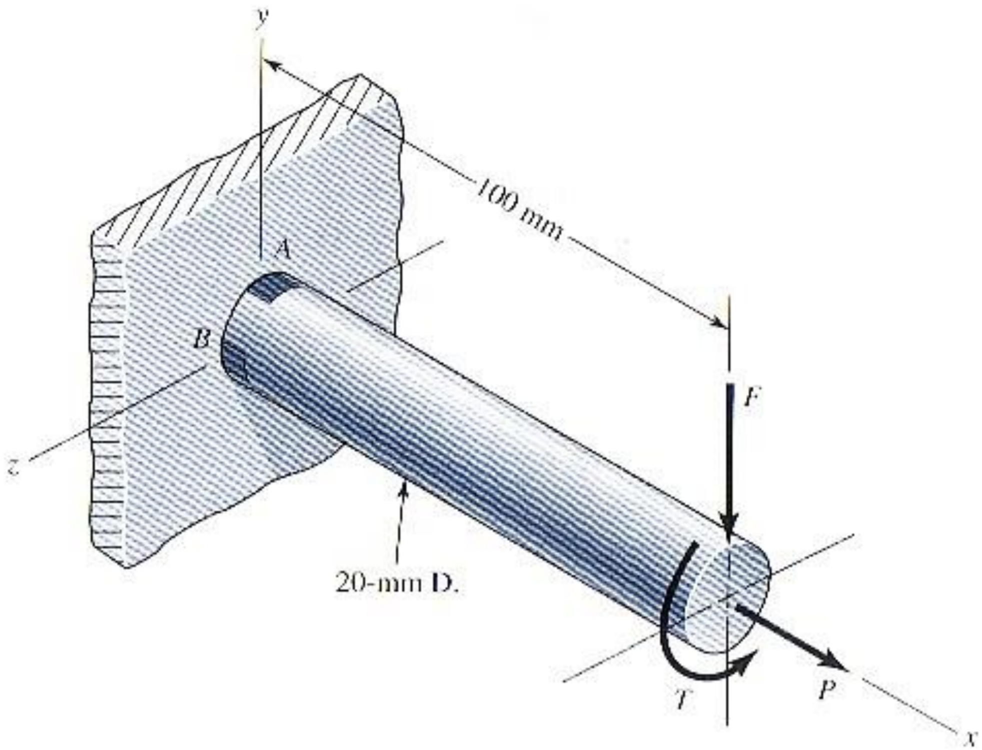
Which theory to use?

- Look at a plot of the principal stresses
  - σ_B vs. σ_A
  - The non-zero principal stresses
- Failure occurs when the principal stresses lie outside the enclosed area
- Shape of area depends on the failure theory
- Data points are experimental results
- MSS
  - Slightly more conservative
  - Easier to calculate
- MDE
  - More accurate
  - If not specified, use this one!



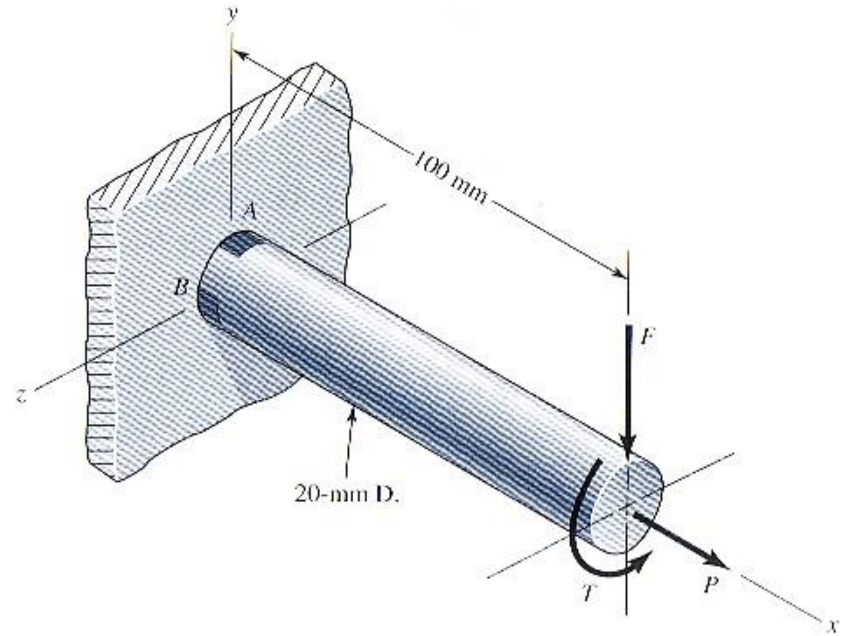
Ductile failure theory example

- Given:
  - Bar is AISI 1020 hot-rolled steel
    - A DUCTILE material
  - F = 0.55 kN
  - P = 8.0 kN
  - T = 30 Nm
- Find:
  - Factor of safety (η)
- Two areas of interest:
  - A
    - Top – where max normal stress is seen (bending!)
  - B
    - Side – where max shear stress is seen



Element A

- Consider the types of loading we have
- Axial?
  - Yes – due to P
- Bending?
  - Recall that bending produces σ and τ, depending on the element of interest
  - Yes – due to M (σ at A, τ at B)
  -
- Torsion?
  - Yes – due to T



Element A

- Calculate stresses due to each load
- Axial:

$$\sigma_x = \frac{P}{A} = \frac{P}{\left(\frac{\pi D^2}{4}\right)} = \frac{4P}{\pi D^2}$$

- Bending:

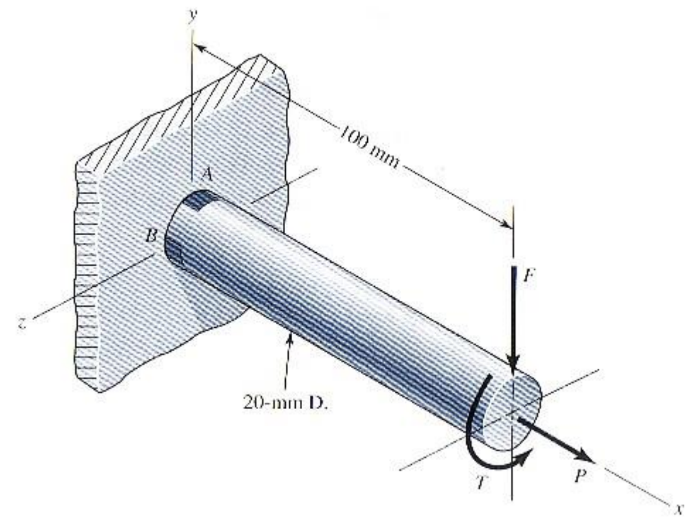
$$\sigma_x = \frac{My}{I} = \frac{(FL)\left(\frac{D}{2}\right)}{\left(\frac{\pi D^4}{64}\right)} = \frac{32FL}{\pi D^3}$$

- Shear:

$$\tau_{xy} = 0$$

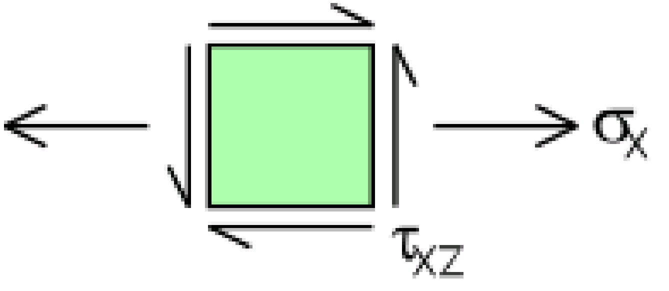
- Torsion:

$$\tau_{xz} = \frac{Tc}{J} = \frac{(T)\left(\frac{D}{2}\right)}{\left(\frac{\pi D^4}{32}\right)} = \frac{16T}{\pi D^3}$$



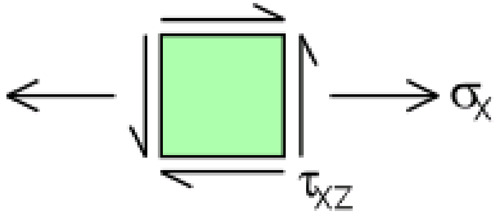
Element A

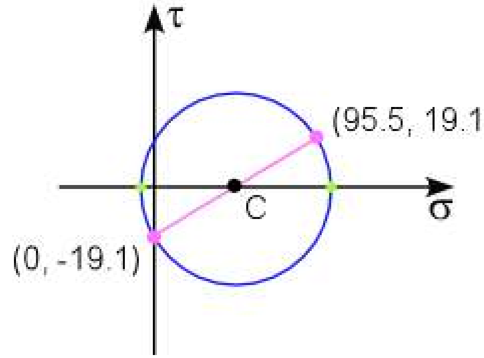
- Look at a stress element
- Sum up stresses due to all the loads
- $$\sigma_x = \frac{4P}{\pi D^2} + \frac{32FL}{\pi D^3} = \frac{4PD + 32FL}{\pi D^3}$$
- $$\tau_{xz} = \frac{16T}{\pi D^3}$$
- σ_x = 95.5 MPa
- τ_{xz} = 19.1 MPa



Element A

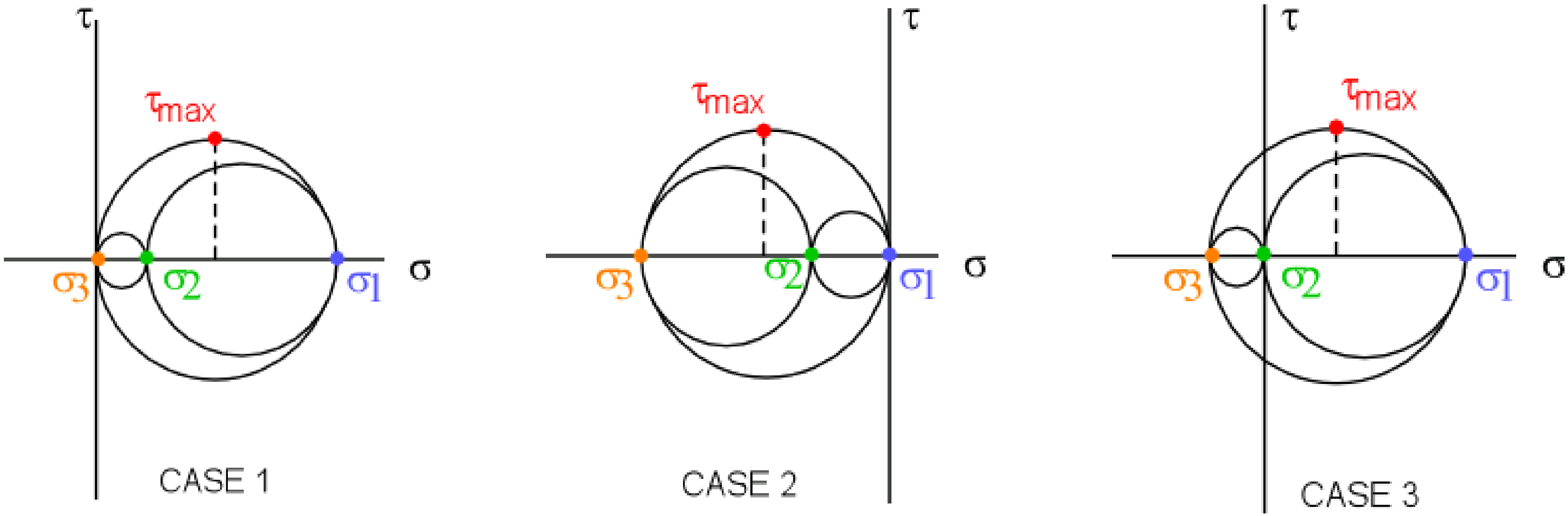
- Draw Mohr's Circle with the stresses that we calculated
  - σ_x = 95.5 MPa
  - τ_{xz} = 19.1 MPa
  - x at (σ_x, τ_{xz})
    - (95.5, 19.1)
  - y at (σ_y, τ_{zx})
    - (σ_y, -τ_{xz})
    - (0, -19.1)
  - Find C
    - $$\left(\frac{\sigma_x+\sigma_y}{2},0\right)=\left(\frac{95.5-0}{2},0\right)=(47.8,0)$$
  - Find radius
    - $$R=\sqrt{(\sigma_x-C_x)^2+\tau_{xz}{}^2}=\sqrt{(95.5-47.8)^2+19.1^2}=51.4$$





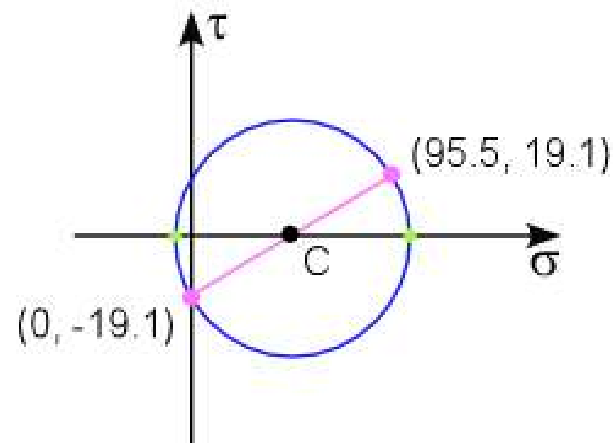
Out of Plane Maximum Shear for Biaxial State of Stress

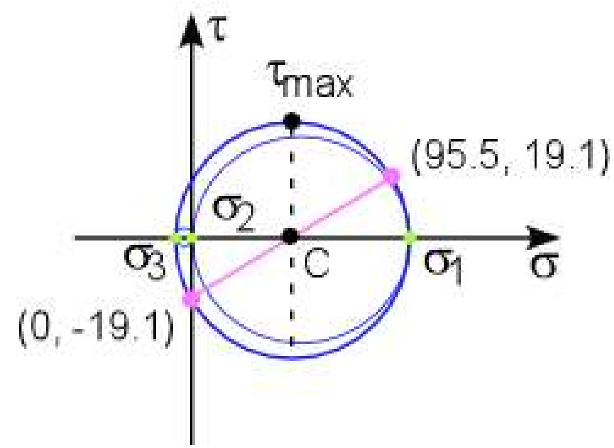
- Case 1
  - $\sigma_{1,2} > 0$
  - $\sigma_3 = 0$
  - $\tau_{max} = \dfrac{\sigma_1}{2}$
- Case 2
  - $\sigma_{2,3} < 0$
  - $\sigma_1 = 0$
  - $\tau_{max} = \dfrac{|\sigma_3|}{2}$
- Case 3
  - $\sigma_1 > 0, \sigma_3 < 0$
  - $\sigma_2 = 0$
  - $\tau_{max} = \dfrac{|\sigma_1 - \sigma_3|}{2}$



Element A

- Find principal stresses
  - $\sigma_A = C + R$
    - 99.2 MPa
  - $\sigma_B = C - R$
    - -3.63 MPa
  - Think about 3-D Mohr's Circle!
  - This is Case #3…
  - We want $\sigma_1 > \sigma_2 > \sigma_3$
  - Assign $\sigma_2 = 0$ and $\sigma_3 = -3.63$ MPa
- No failure theory was given, so use MDE





Element A

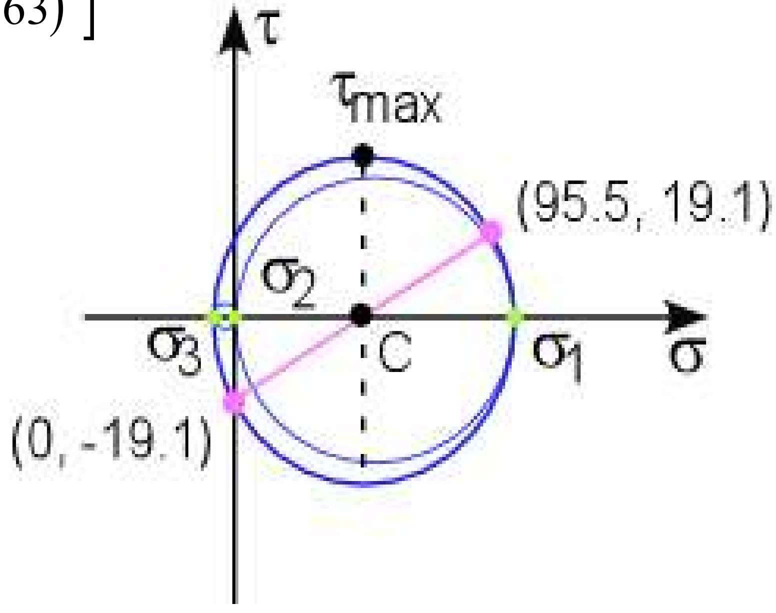
- Find the von Mises stress (σ_e)

$$\sigma_e = \sqrt{\frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2\right]}$$

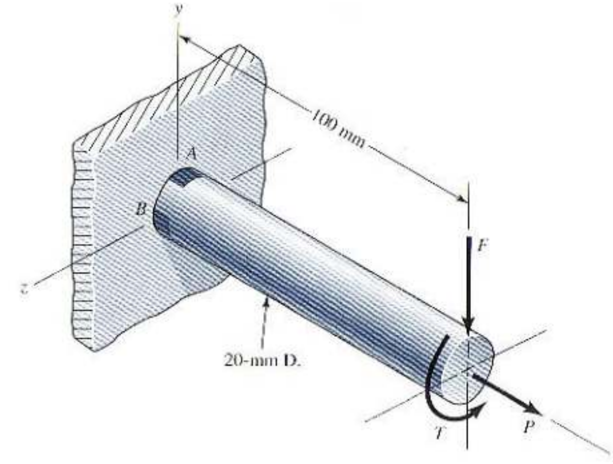
$$\sigma_e = \sqrt{\frac{1}{2}\left[(99.2 - 0)^2 + (0 + 3.63)^2 + (99.2 + 3.63)^2\right]}$$

$$\sigma_e = 101 MPa$$

- S_y for our material = 331 MPa
- Calculate the factor of safety
  - $$\eta = \frac{S_y}{\sigma_e} = \frac{331}{101} = 3.28$$ For yield



Element B



- Consider the types of loading we have
- Axial?
    - Yes – due to P
- Bending?
    - Recall that bending produces σ and τ, depending on the element of interest
    - Yes – due to M (τ at B)
- Torsion?
    - Yes – due to T

Element B

- Calculate stresses due to each load
- Axial:

$$\sigma_x = \frac{P}{A} = \frac{P}{\left(\frac{\pi D^2}{4}\right)} = \frac{4P}{\pi D^2}$$

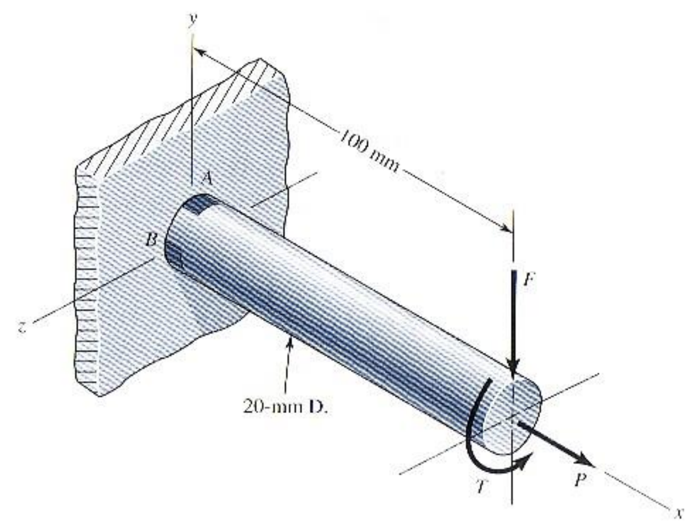
- Bending:
  - Use equation for round solid cross-section

$$\tau_{xy} = \frac{VQ}{Ib} = \frac{4V}{3A} = \frac{(4F)}{3\left(\frac{\pi D^2}{4}\right)} = \frac{16F}{3\pi D^2} = \frac{4F}{3A}$$

- Shear: $\tau_{xy} = 0$
- Torsion:

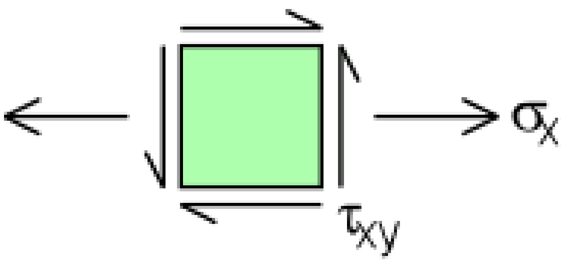
$$\tau_{xy} = \frac{Tc}{J} = \frac{(T)\left(\frac{D}{2}\right)}{\left(\frac{\pi D^4}{32}\right)} = \frac{16T}{\pi D^3}$$

τ is the (transverse) shear stress acting at a distance, y, from the neutral axis.
V is the value of the shear force at the section
Q is the first moment of the area between the location where the shear stress is being calculated and the location where the shear stress is zero about the neutral (centroidal) axis
I is the moment of inertia of the entire cross-section about the neutral axis
b is the width of the cross-section at the location where the shear stress is being calculated (D)

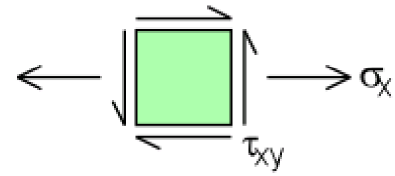


Element B

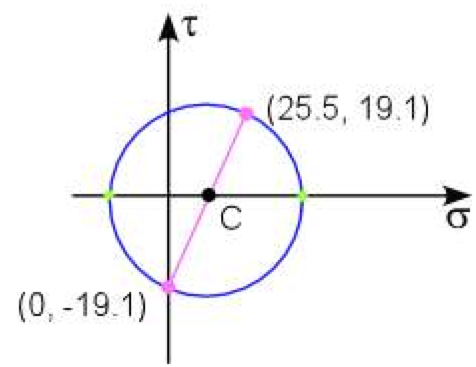
- Look at a stress element
- Sum up stresses due to all the loads
- $$\sigma_x = \frac{4P}{\pi D^2}$$
- $$\tau_{xy} = \frac{16F}{3\pi D^2} + \frac{16T}{\pi D^3} = 19.1 + .002$$
- σ_x = 25.5 MPa
- τ_{xy} = 19.1 MPa
  - Note small contribution of shear stress due to bending



Element B

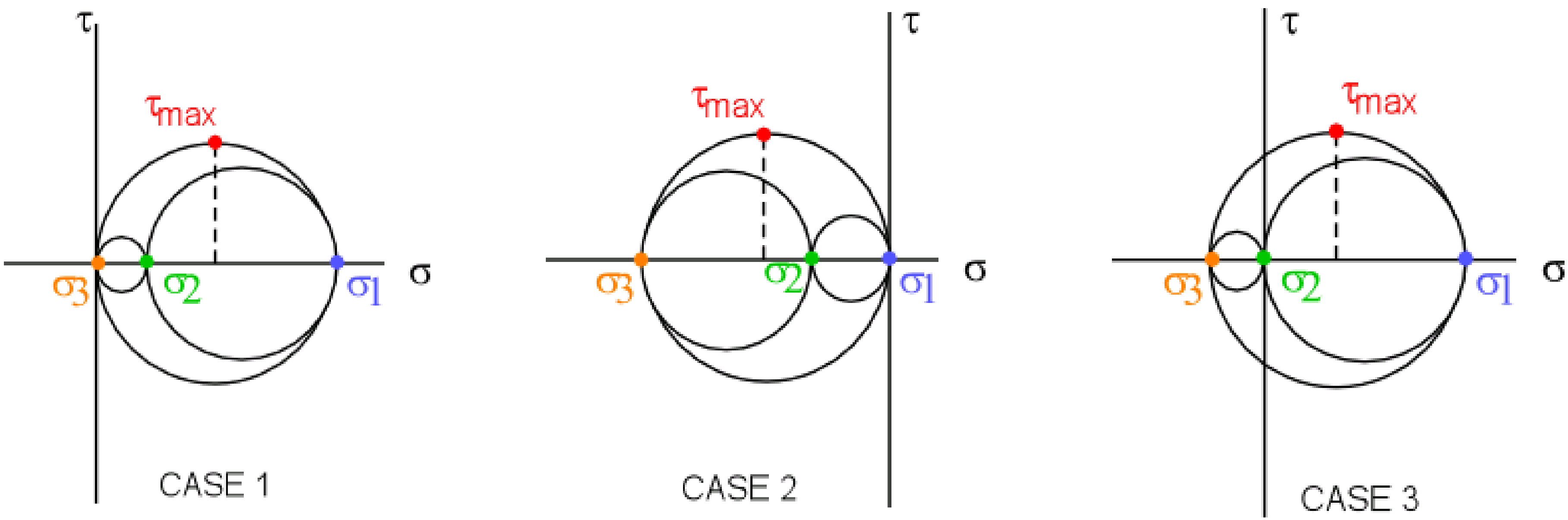


- Draw Mohr's Circle with the stresses that we calculated
  - σ_x = 25.5 MPa
  - τ_{xy} = 19.1 MPa
  - x at (σ_x, τ_{xy})
    - (25.5, 19.1)
  - y at (σ_y, τ_{yx})
    - (σ_y, -τ_{xy})
    - (0, -19.1)
  - Find C
    - $$\left(\frac{\sigma_x + \sigma_y}{2}, 0\right) = \left(\frac{25.5 - 0}{2}, 0\right) = (12.8, 0)$$
  - Find radius
    - $$R = \sqrt{(\sigma_x - C_x)^2 + \tau_{xz}^{\ 2}} = \sqrt{(25.5 - 12.8)^2 + 19.1^2} = 22.96$$



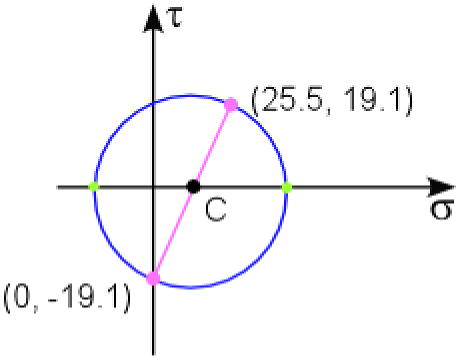
Out of Plane Maximum Shear for Biaxial State of Stress

- Case 1
  - $\sigma_{1,2} > 0$
  - $\sigma_3 = 0$
  - $\tau_{\max} = \dfrac{\sigma_1}{2}$
- Case 2
  - $\sigma_{2,3} < 0$
  - $\sigma_1 = 0$
  - $\tau_{\max} = \dfrac{|\sigma_3|}{2}$
- Case 3
  - $\sigma_1 > 0$, $\sigma_3 < 0$
  - $\sigma_2 = 0$
  - $\tau_{\max} = \dfrac{|\sigma_1 - \sigma_3|}{2}$



Element B

- Find principal stresses
  - $\sigma_1 = C + R$
    - 35.8 MPa
  - $\sigma_2 = C - R$
    - -10.2 MPa
  - Think about 3-D Mohr's Circle!
  - This is Case #3…
  - We want $\sigma_1 > \sigma_2 > \sigma_3$
  - Assign $\sigma_2 = 0$ and $\sigma_3 = -10.2$ MPa
- No failure theory was given, so again use MDE



Element B

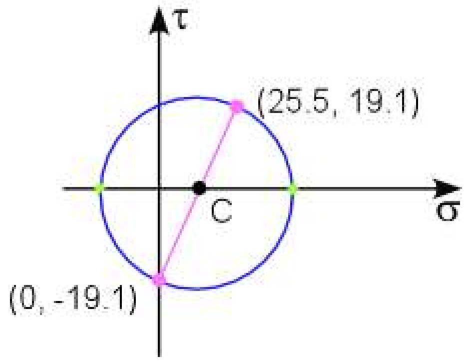
- Find the von Mises stress (σ_e)

$$\sigma_e = \sqrt{\frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2\right]}$$

$$\sigma_e = \sqrt{\frac{1}{2}\left[(35.8 - 0)^2 + (0 + 10.2)^2 + (35.8 + 10.2)^2\right]}$$

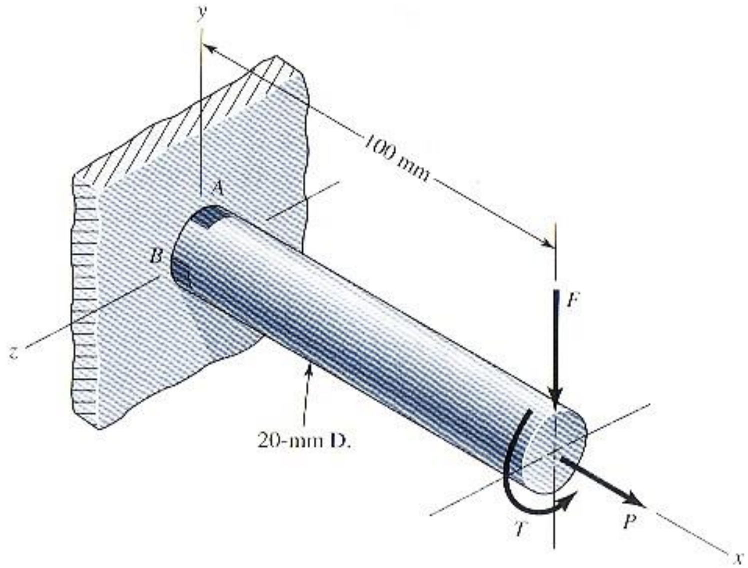
$$\sigma_e = 41.8 MPa$$

- S_y for our material = 331 MPa
- Calculate the factor of safety
  - $$\eta = \frac{S_y}{\sigma_e} = \frac{331}{41.8} = 7.91$$ For yield



Example, concluded

- We found the factors of safety relative to each element, A and B
  - A – 3.28
  - B – 7.91
- A is the limiting factor of safety
  - $\eta = 3.3$



Static Brittle Failure

- Three primary theories for static brittle failure
  - Maximum Normal Stress (MNS)
  - Coulomb-Mohr Theory
  - Modified-Mohr Theory

Mohr's Circle for MNS

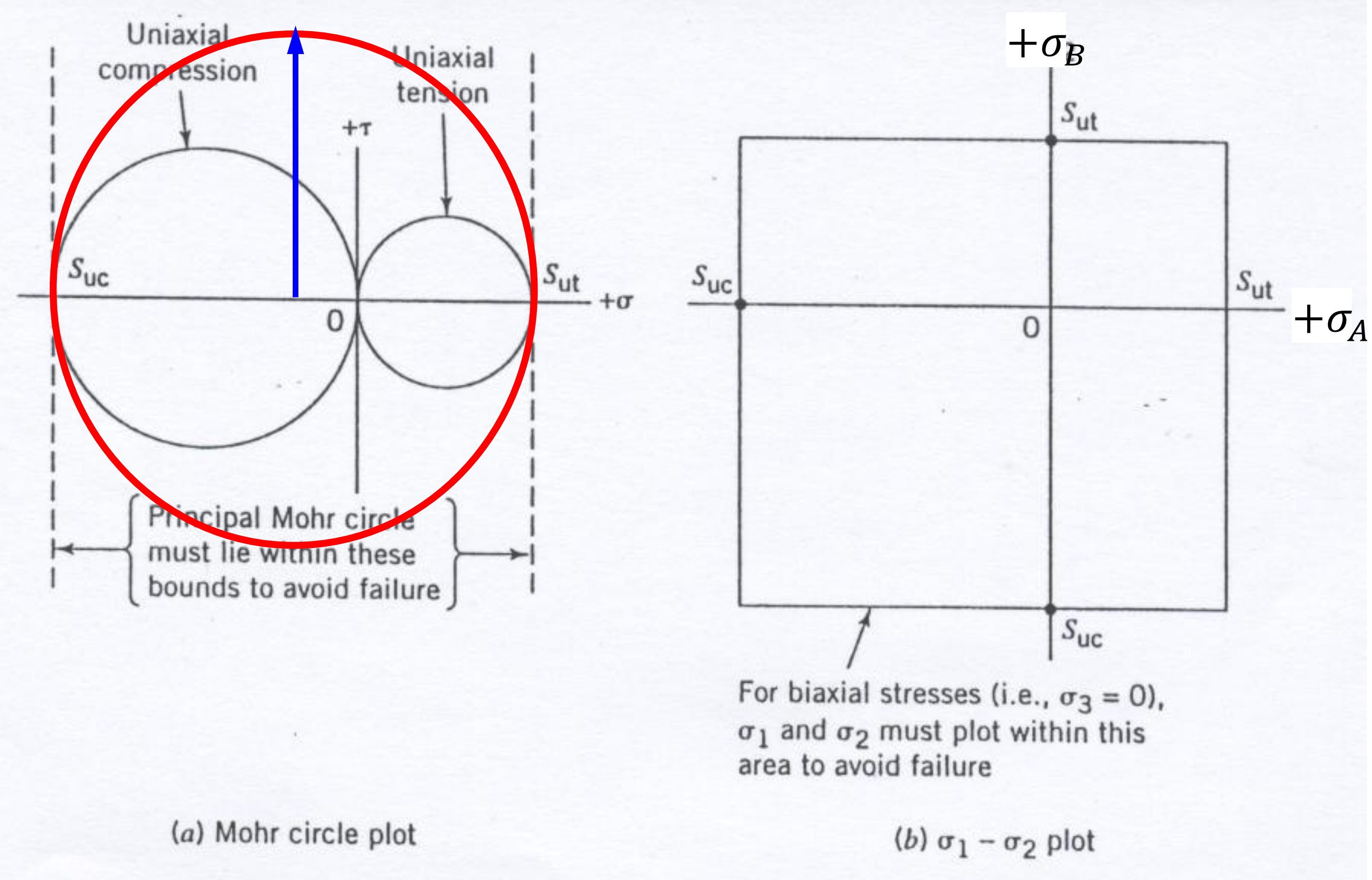
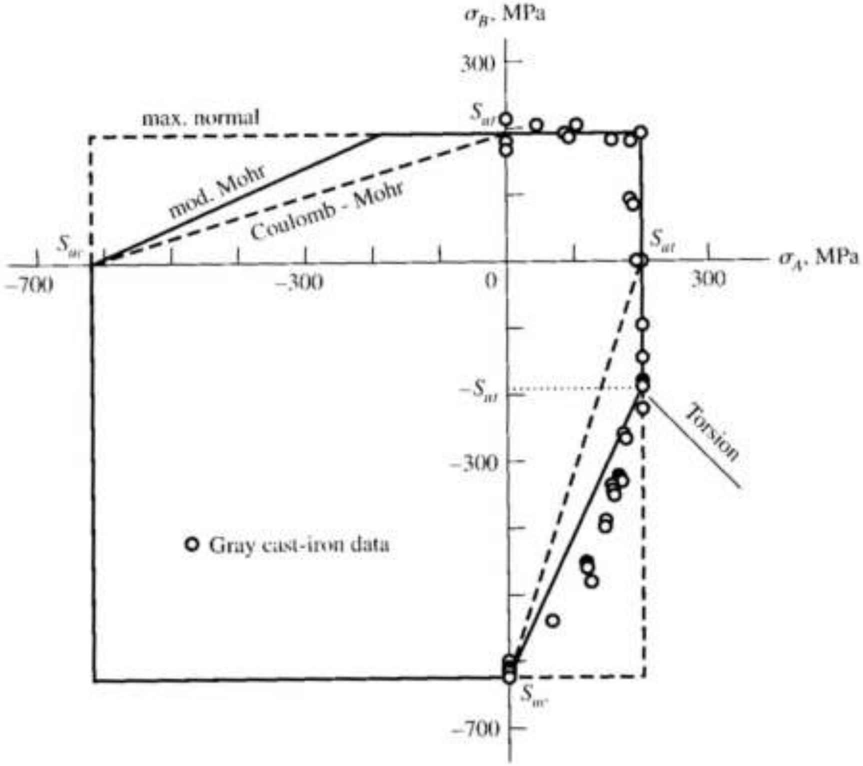
- 

FIGURE 6.7
Two graphical representations of the maximum-normal-stress theory.

Static Brittle Failure

- Maximum Normal Stress (MNS)
  - Oldest failure hypothesis, attributed to Rankine
  - Failure occurs whenever one of the three principal stresses equals the yield strength
    - Say $\sigma_1 > \sigma_2 > \sigma_3$ (as we typically do…)
    - Failure occurs when either
      - $\sigma_1 = S_t$ or $\sigma_3 = -S_c$
    - Note – brittle materials have both a tensile and compressive strength
    - $\eta = S_{ut} / \sigma_1$ or $\eta = -S_{uc} / \sigma_3$



Plot of σ_B vs. σ_A

Mohr Theory

- Some materials have compressive strengths different from tensile strengths
- *Mohr theory* is based on three simple tests: tension, compression, and shear
- Plotting Mohr's circle for each, bounding curve defines failure envelope

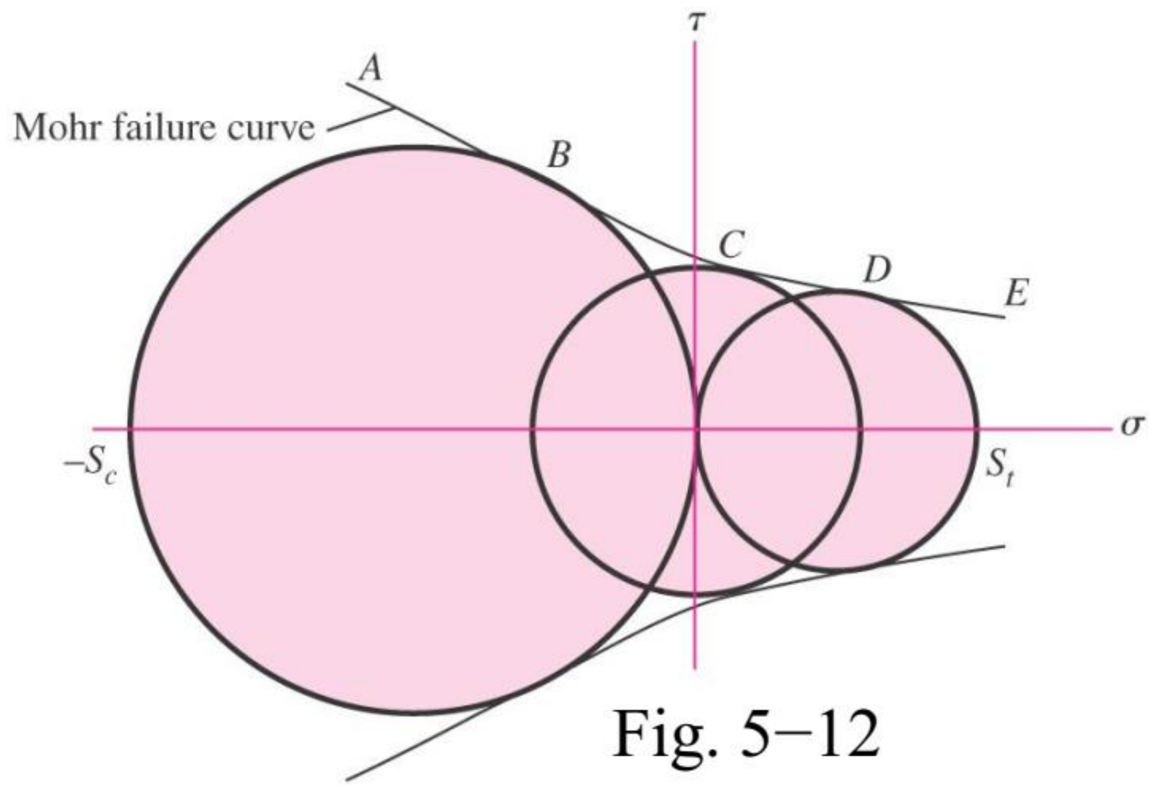


Fig. 5−12

Coulomb-Mohr Theory

- Curved failure curve is difficult to determine analytically
- *Coulomb-Mohr theory* simplifies to linear failure envelope using only tension and compression tests (dashed circles)

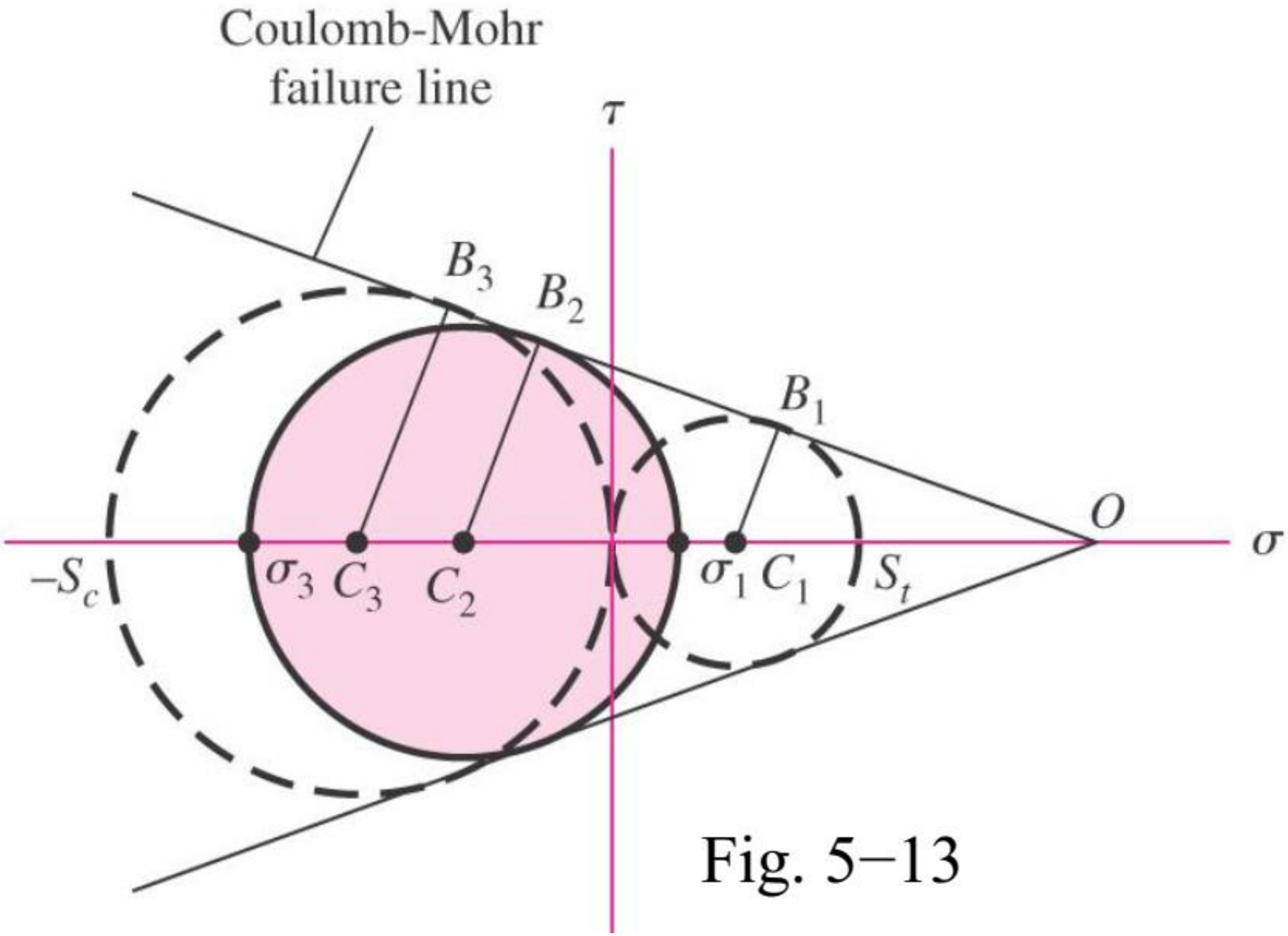


Fig. 5−13

Coulomb-Mohr Theory

$$\frac{\sigma_1}{S_t} - \frac{\sigma_3}{S_c} = 1 \qquad (5\text{–}22)$$

- To plot on principal stress axes, consider three cases
- Case 1: $\sigma_A \geq \sigma_B \geq 0$ For this case, $\sigma_1 = \sigma_A$ and $\sigma_3 = 0$
  - Eq. (5−22) reduces to

$$\sigma_A \geq S_t \qquad (5\text{–}23)$$

- Case 2: $\sigma_A \geq 0 \geq \sigma_B$ For this case, $\sigma_1 = \sigma_A$ and $\sigma_3 = \sigma_B$
  - Eq. (5-22) reduces to

$$\frac{\sigma_A}{S_t} - \frac{\sigma_B}{S_c} \geq 1 \qquad (5\text{–}24)$$

- Case 3: $0 \geq \sigma_A \geq \sigma_B$ For this case, $\sigma_1 = 0$ and $\sigma_3 = \sigma_B$
  - Eq. (5−22) reduces to

$$\sigma_B \leq -S_c \qquad (5\text{–}25)$$

Coulomb-Mohr Theory

- Plot three cases on principal stress axes
- Similar to MSS theory, except with different strengths for compression and tension

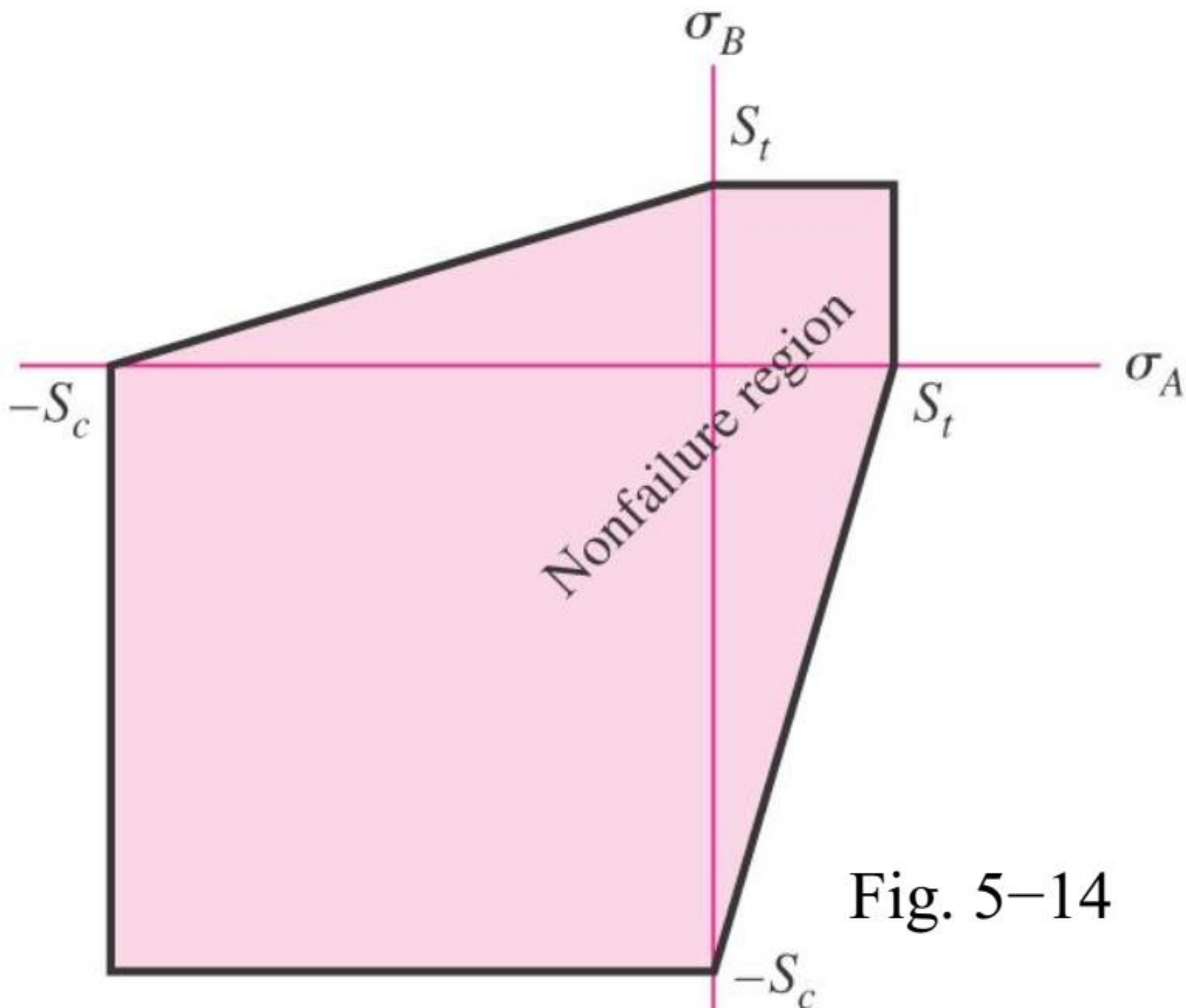


Fig. 5−14

Coulomb-Mohr Theory

- Incorporating factor of safety

$$\frac{\sigma_1}{S_t} - \frac{\sigma_3}{S_c} = \frac{1}{n} \qquad (5\text{–}26)$$

- For ductile material, use tensile and compressive yield strengths
- For brittle material, use tensile and compressive ultimate strengths

Brittle Failure Experimental Data

- Coulomb-Mohr is conservative in 4th quadrant
- *Modified Mohr* criteria adjusts to better fit the data in the 4th quadrant

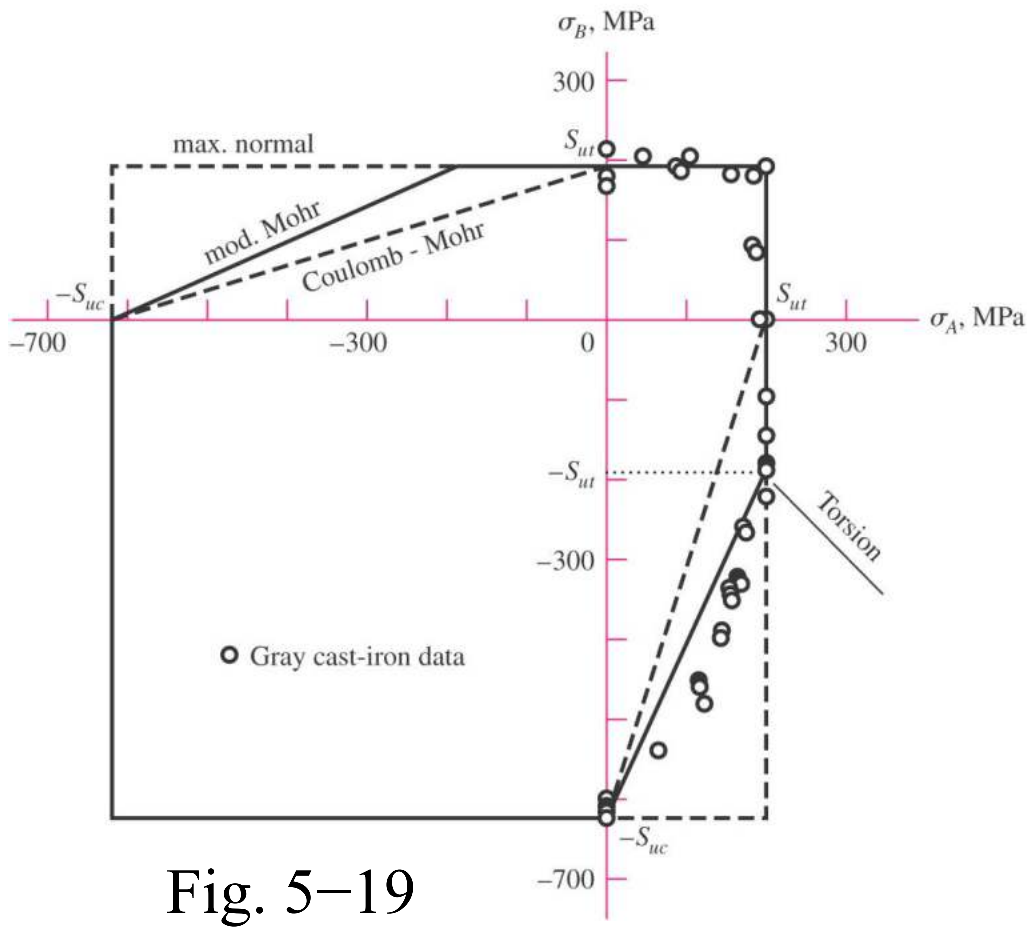
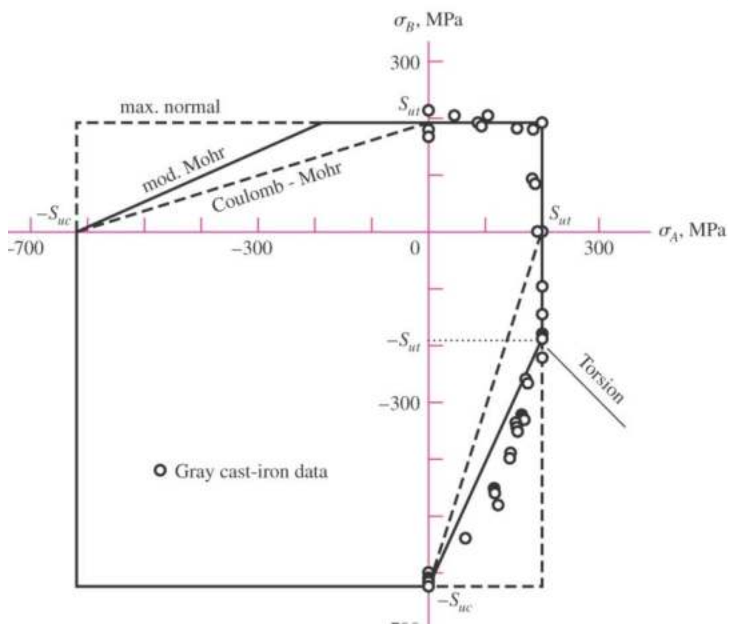


Fig. 5−19

Modified-Mohr

Quadrant condition	Failure criteria	
$\sigma_A \geq \sigma_B \geq 0$	$\sigma_A = \dfrac{S_{ut}}{n}$	(5–32*a*)
$\sigma_A \geq 0 \geq \sigma_B$ and $\left\lvert \dfrac{\sigma_B}{\sigma_A} \right\rvert \leq 1$	$\sigma_A = \dfrac{S_{ut}}{n}$	(5–32*a*)
$\sigma_A \geq 0 \geq \sigma_B$ and $\left\lvert \dfrac{\sigma_B}{\sigma_A} \right\rvert > 1$	$\dfrac{(S_{uc} - S_{ut})\,\sigma_A}{S_{uc} S_{ut}} - \dfrac{\sigma_B}{S_{uc}} = \dfrac{1}{n}$	(5–32*b*)
$0 \geq \sigma_A \geq \sigma_B$	$\sigma_B = -\dfrac{S_{uc}}{n}$	(5–32*c*)



Modified Mohr Failure Theory

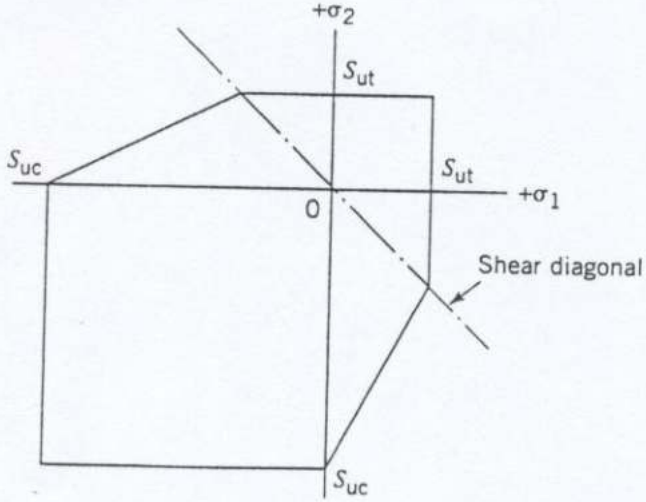
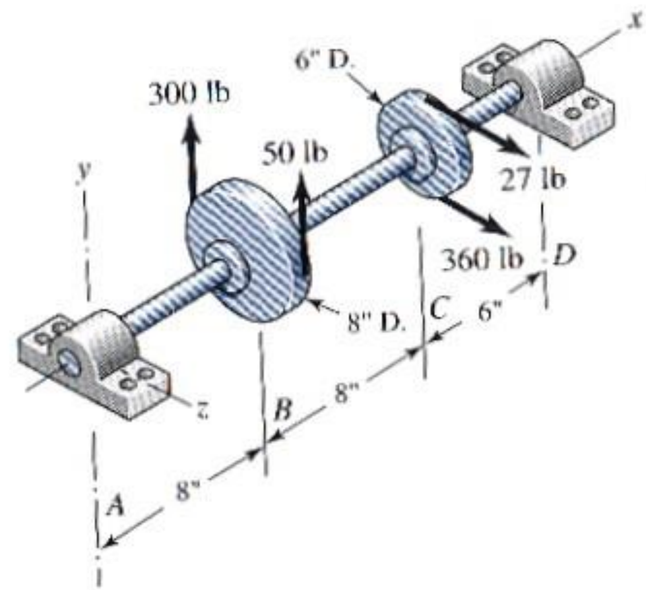
- 

FIGURE 6.12
Graphical representation of the modified Mohr theory for biaxial stresses ($\sigma_3 = 0$).

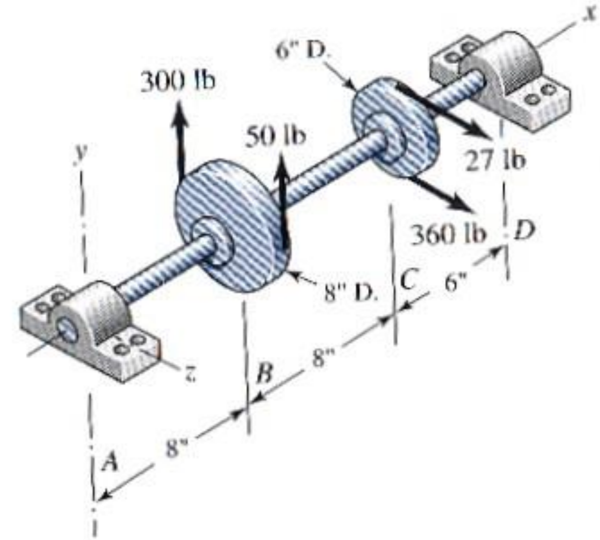
Brittle Failure example

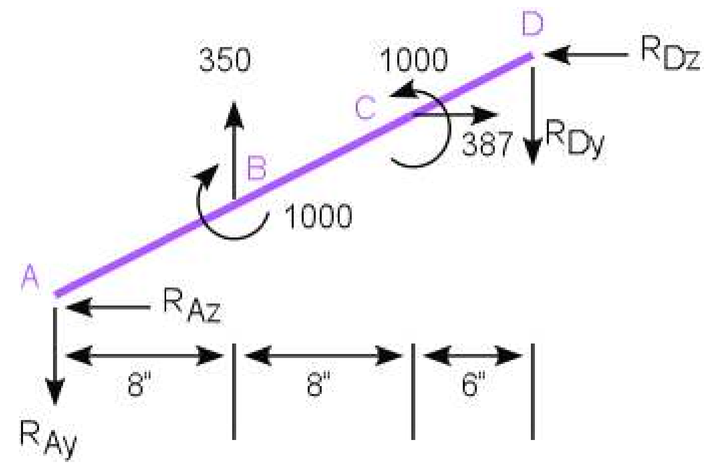
- Given:
  - Shaft of ASTM G25 cast iron subject to loading shown
  - From Table A-24
    - S_{ut} = 26 kpsi
    - S_{uc} = 97 kpsi
- Find:
  - For a factor of safety of η = 2.8, what should the diameter of the shaft (d) be?



Brittle Failure example

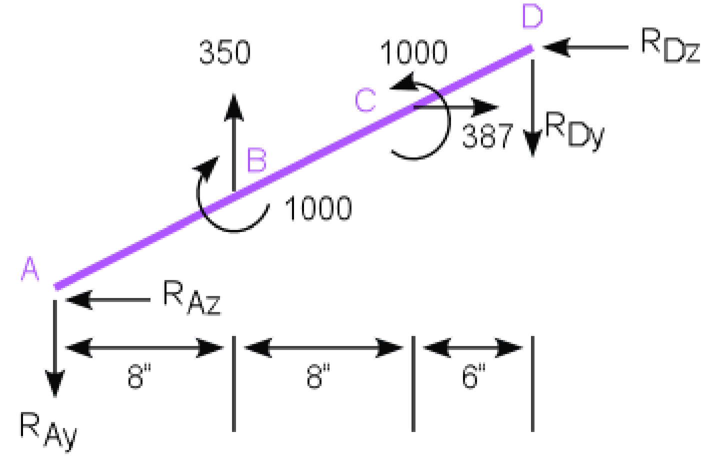
- First, we need to find the forces acting on the shaft
  - Torque on shaft from pulley at B
    - T_B = (300-50)(4) = 1000 in·lb
  - Torque on shaft from pulley at C
    - T_C = (360-27)(3) = 1000 in·lb
  - Shaft is in static equilibrium
  - Note that shaft is free to move along the x-axis (bearings)
- Draw a FBD
  - Reaction forces at points of attachment to show constrained motion





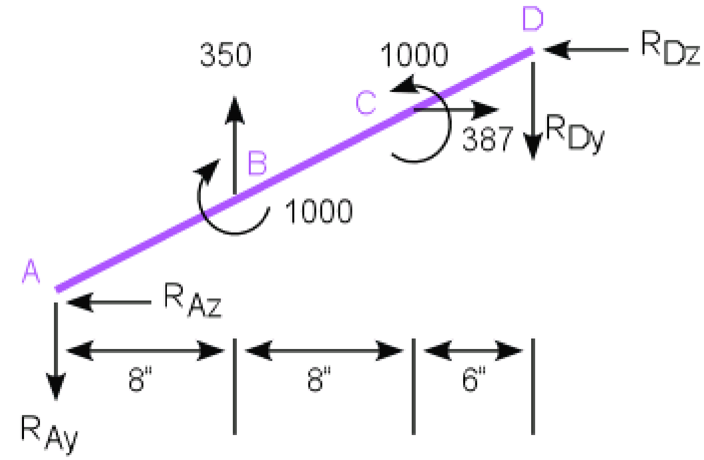
Equilibrium

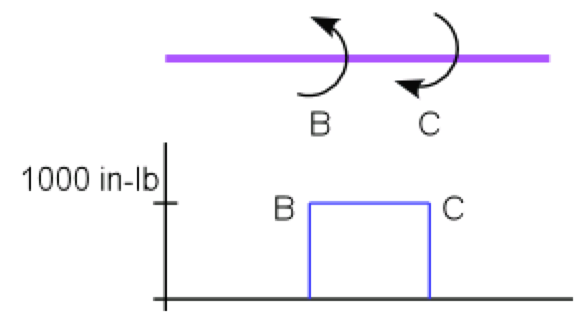
- Use statics to solve for reactions forces
  - R_{Ay} = 222 lb
  - R_{Az} = 106 lb
  - R_{Dy} = 127 lb
  - R_{Dz} = 281 lb
- OK, now we know all the forces. The problem gives us a factor of safety, but unlike our last example, we aren't told specific places (elements) at which to look for failure!
- We are going to have to calculate stresses
- What do we need?
  - Axial forces, bending moments, and torques
  - We need to find our moments… HOW?
  - Shear-Moment diagrams will give us the forces and moments along the shaft.
  - Failure will likely occur where the max values are seen



Torsion and moment diagrams

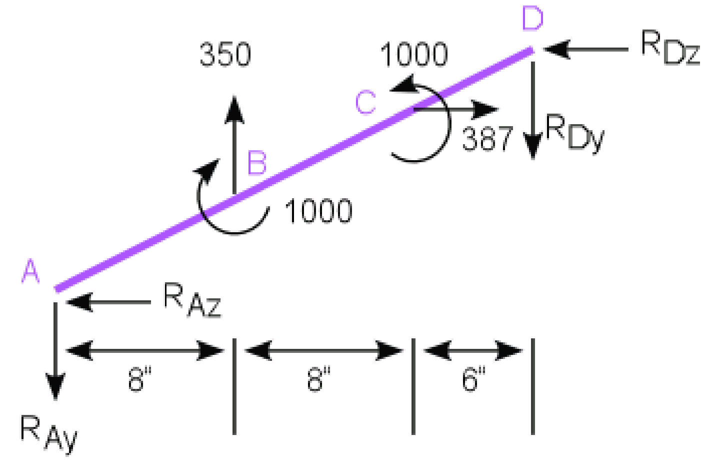
- Let's look at torsion and how it varies across the shaft
  - We calculated the torques at B and C to be 1000 in·lb each
  - Plot that along the shaft and we see that max torque occurs at B and C (and all points between)

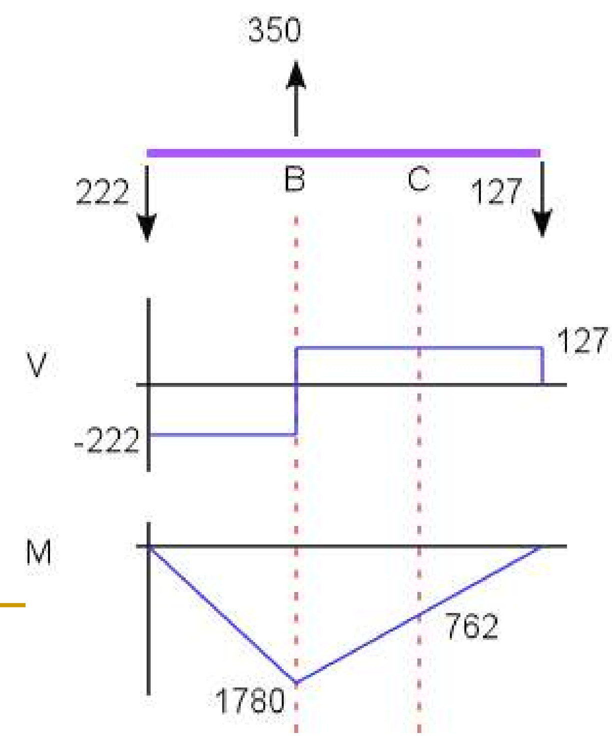




Torsion and moment diagrams

- Now let's look at the moments
- We have a 3-D loading
  - How are we going to do the V-M diagrams?
  - Look at one plane at a time
- Moment in the x-y plane
  - From geometry you can calculate the values of the moment at B and C

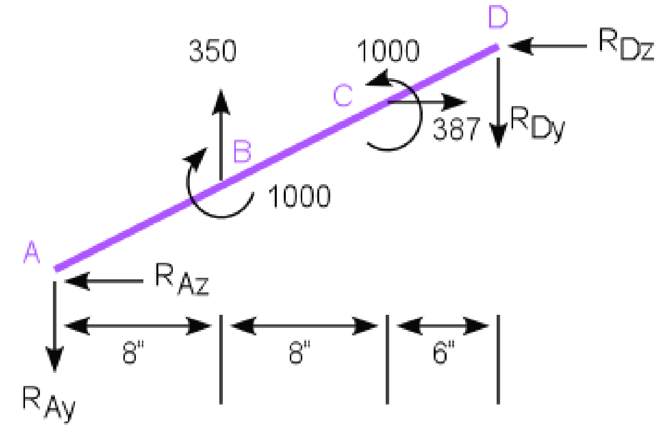


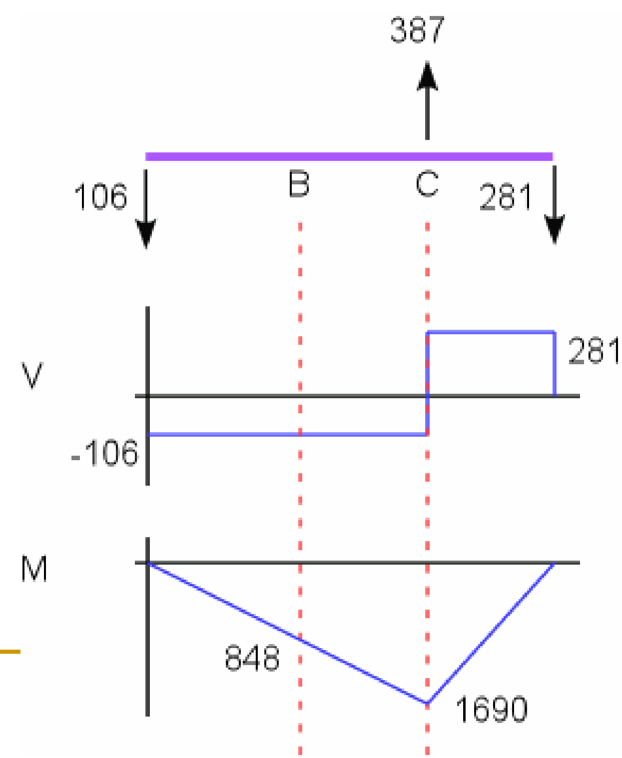


Torsion and moment diagrams

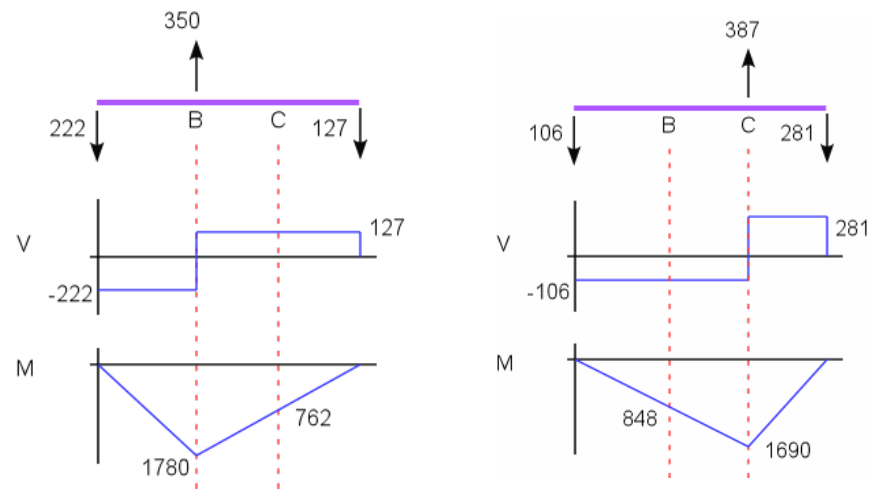
- Moment in the x-z plane
  - Failure is going to occur at either B or C, since these are locations where maximum moments are seen
  - But we have moments in both planes
  - To find the max bending stresses, we must find the total maximum moment
  - Just as we would vectorally add the two force components to find the force magnitude, we can vectorally add the two moment components to find the moment magnitude

$$M = \sqrt{M_{xy}^{2} + M_{xz}^{2}}$$





Calculate the max moment



- We found the following:
  - $M_{B\ x\text{-}y}$ = 1780 in·lb
  - $M_{B\ x\text{-}z}$ = 848 in·lb
  - $M_{C\ x\text{-}y}$ = 762 in·lb
  - $M_{C\ x\text{-}z}$ = 1690 in·lb
- Calculating the magnitudes with $M = \sqrt{M_{xy}^{\ 2} + M_{xz}^{\ 2}}$
  - M_B = 1971.7 in·lb
  - M_C = 1853.8 in·lb
- Since the overall max moment is at B, we will expect failure there, and use M_B in our stress calculations
- If we had been told the location of interest, we would essentially start here.

Calculate the stresses at B

- Bending stress (σ and τ)
  - We know from experience that σ is the predominant stress, so essentially we will look for failure at an element at the top of the shaft
    - M = 1971
    - Plug in known values
    - $\sigma_{max} = (20x10^3)/d^3$
- Torsional stress
  - T = 1000
  - $\tau = (5.1x10^3)/d^3$

$$\sigma = \frac{My}{I}$$

$$y = \frac{d}{2}$$

$$I = \frac{\pi d^4}{64}$$

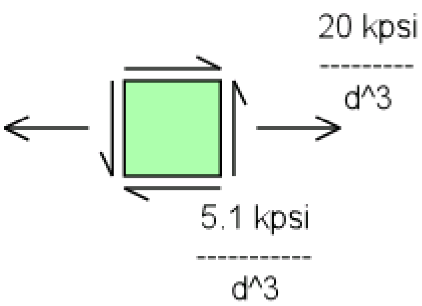
$$\tau = \frac{Tc}{J}$$

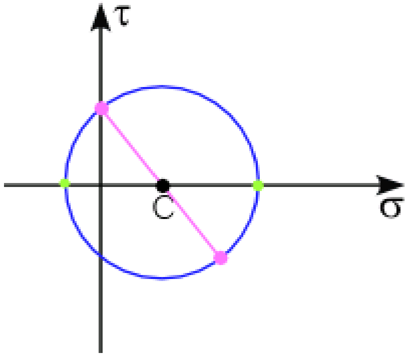
$$c = \frac{d}{2}$$

$$J = \frac{\pi d^4}{32}$$

Mohr's Circle

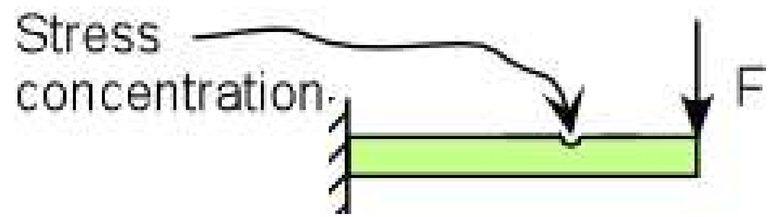
- Let's look at our stress element
- Now construct Mohr's circle
  - C at $(10 \times 10^3)/d^3$
  - $R = (11.2 \times 10^3)/d^3$
  - $\sigma_1 = (21.2 \times 10^3)/d^3$
  - $\sigma_3 = (-1.2 \times 10^3)/d^3$
- Use Coulomb-Mohr theory for brittle failure
  - $$\frac{\sigma_1}{S_{ut}} - \frac{\sigma_3}{S_{uc}} = \frac{1}{\eta}$$
    $$\frac{21.2}{26d^3} + \frac{1.2}{97d^3} = \frac{1}{2.8}$$
    $$d = 1.32''$$
  - If making a design recommendation, you would recommend the next largest standard dimension (16th's)
    - d = 1.375 in





Stress concentration

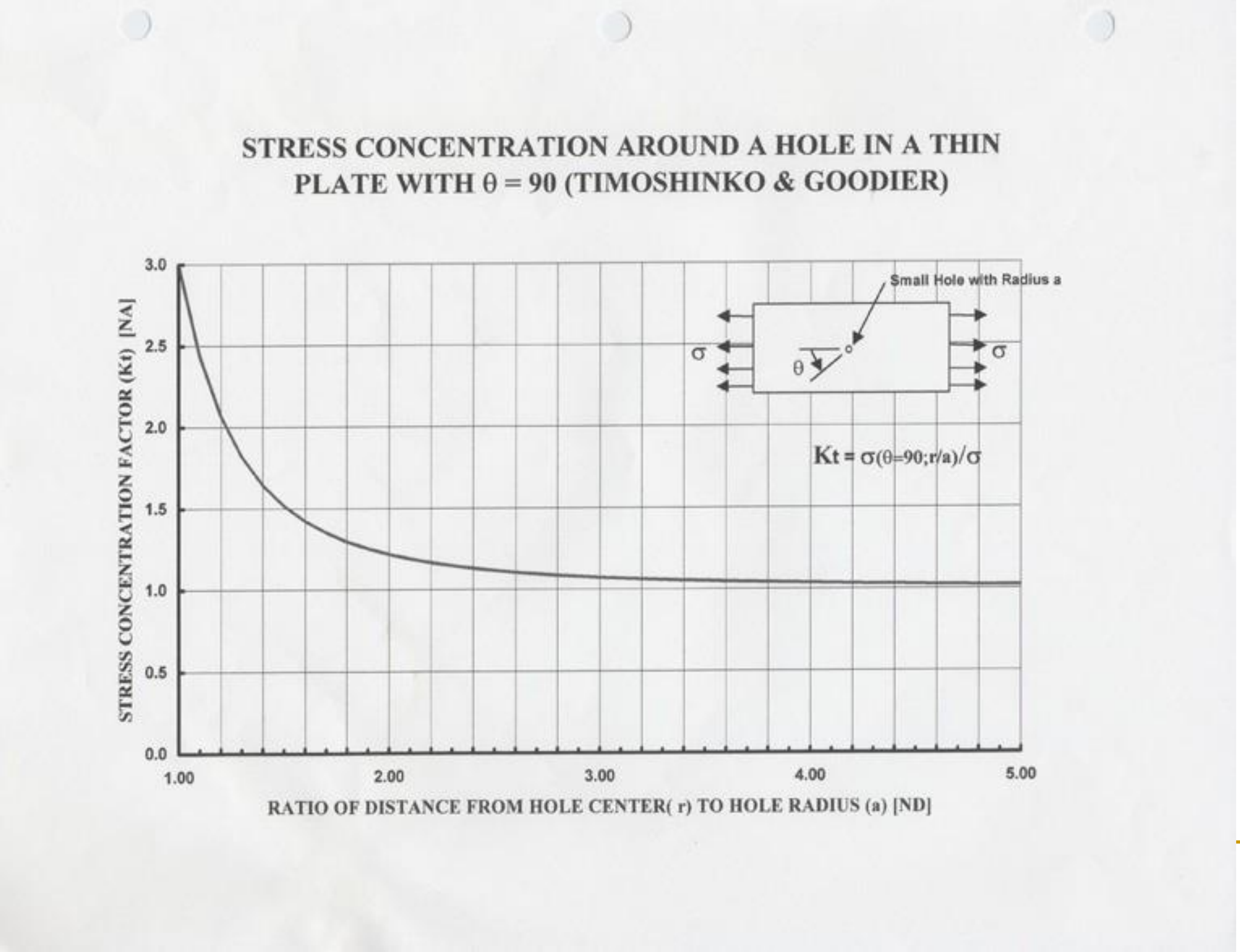
- A stress concentration is any geometric discontinuity in an element that is subjected to stress
- Aside from reducing the cross-sectional area, these stress concentrations do not significantly affect static <u>ductile</u> failure
- Stress concentrations DO, however, have a significant influence on <u>brittle</u> failure



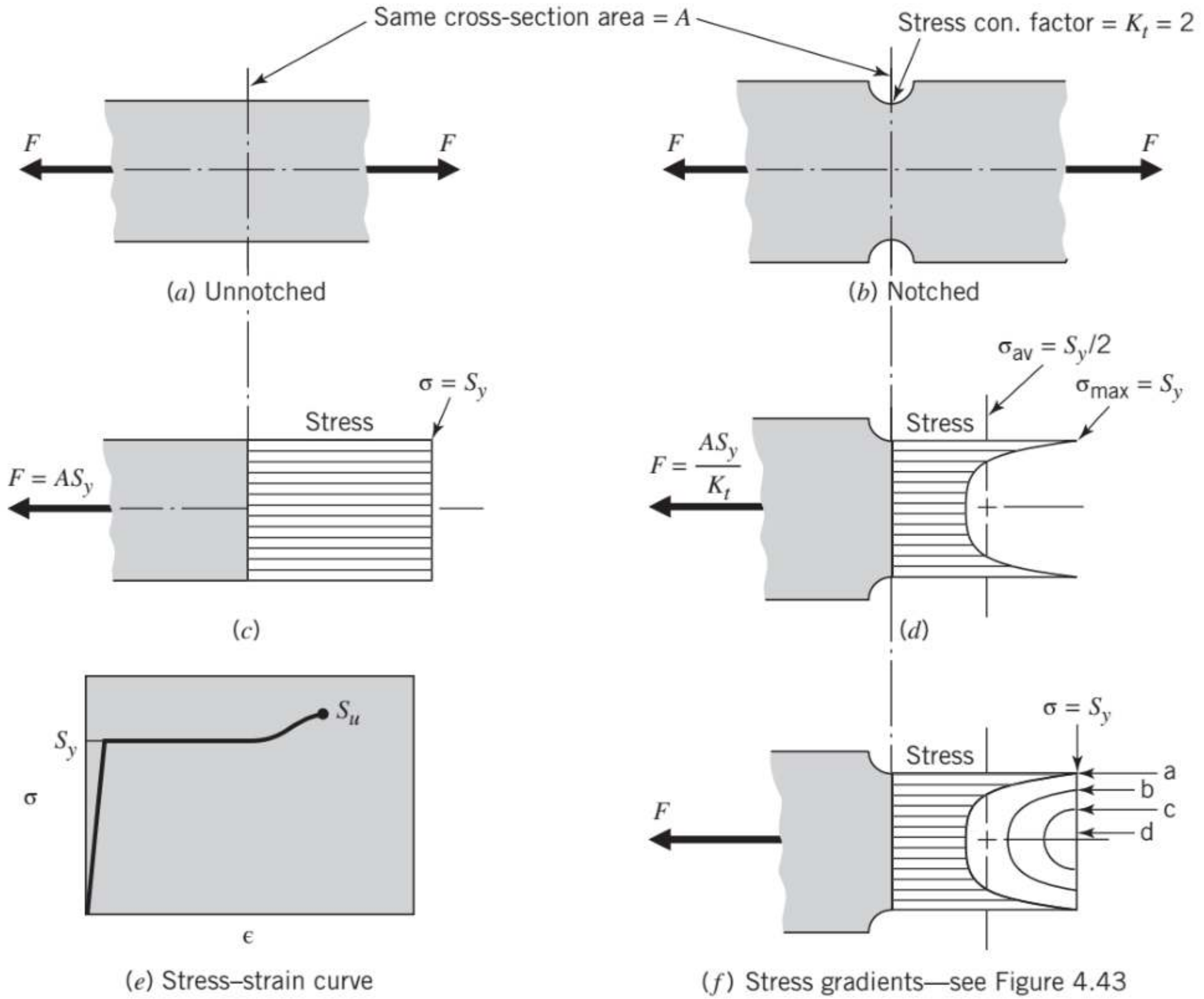
Analytical approach to stress concentrations

- $\sigma_{max} = k_t \sigma_{nom}$
- $\tau_{max} = k_{ts} \tau_{nom}$
- k_t, k_{ts} are **Stress Concentration** (SC) factors
- σ_{nom}, τ_{nom} are nominal stresses
  - Nominal – those stresses that are calculated before taking the SC's into account
- SC factors are given in the text on pages 1034- to -1042 $(10^{th} ed)$
- Equations for the nominal stresses (taking into account geometry change due to the SC's) are given in the same charts

Stresses at a Hole in an Infinite Plate

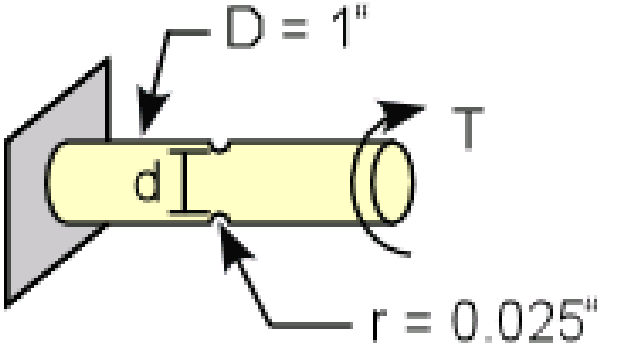


Stress Concentration in Ductile Material



Tensile stress distribution of an unnotched and a notched ductile part.

Brittle failure example



- Given:
  - ASTM 30 cast iron
    - S_{ut} = 31 ksi
    - S_{uc} = 109 ksi
- Find:
  - How much torque before failure with and without the stress concentration?
  - Note – asked to find failure (not given a safety factor)
  - Use η = 1 to find onset of failure

ASTM: American Society Of Testing Materials

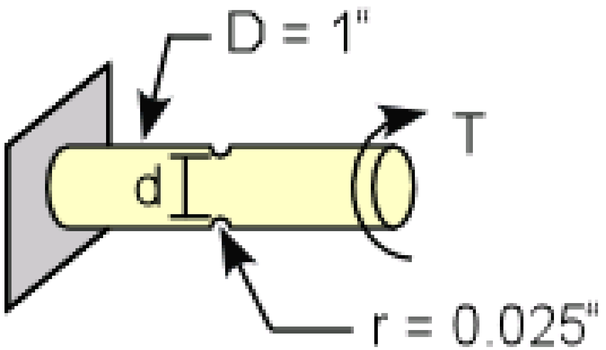
Brittle failure example

- Without the SC

$$\tau = \frac{Tc}{J}$$

$$c = 0.5$$

$$J = \frac{\pi D^4}{32} = \frac{\pi}{32} = 0.098 in^4$$

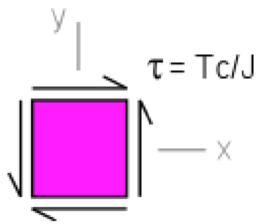


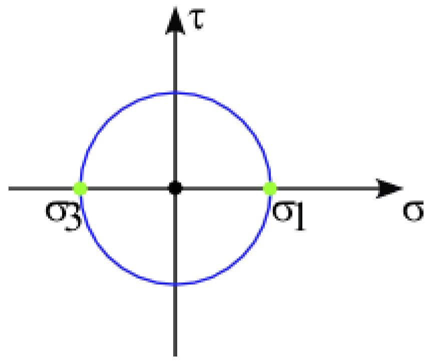
- τ = (5.1)T
- σ_A = 5.1 T, σ_B = -5.1 T
- Use Coulomb-Mohr (easier)

$$\frac{\sigma_A}{S_{ut}} - \frac{\sigma_B}{S_{uc}} \leq 1$$

$$\frac{5.1T}{31\times 10^3} - \frac{-5.1T}{109\times 10^3} \leq 1$$

$$T\left(2.11\times 10^{-4}\right) \leq 1$$

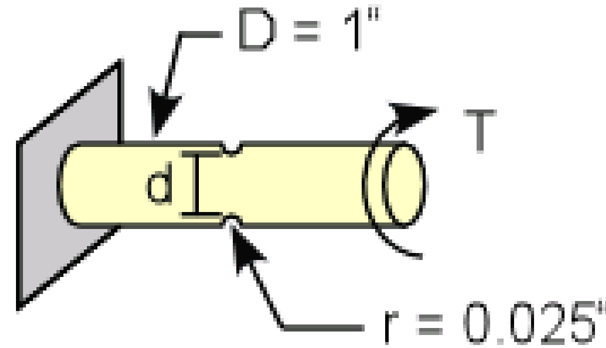


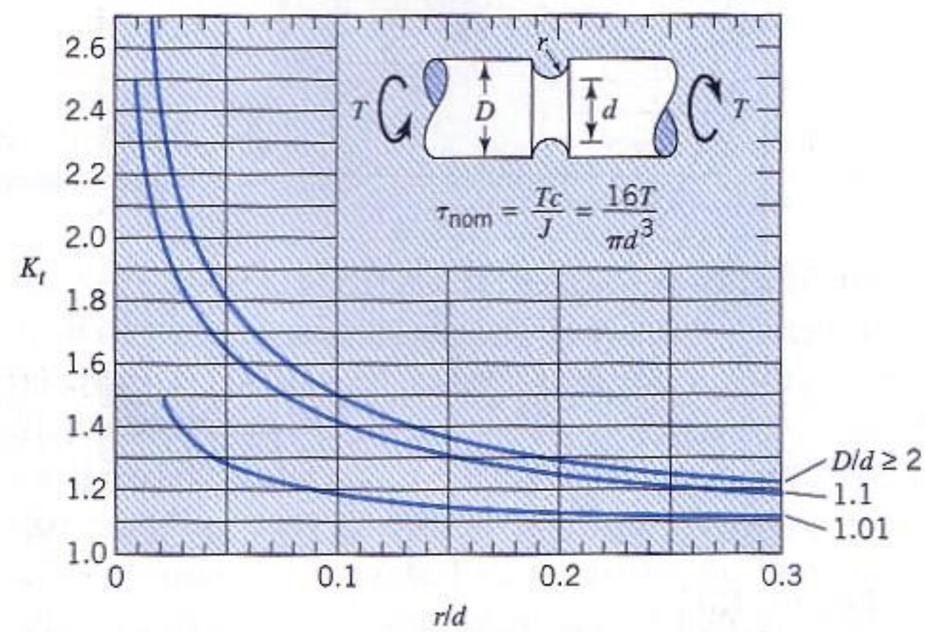


T ≤ 4730 in·lb

Brittle failure example

- With the SC
  - Refer to figure A-15-15, pg. 1038 $(10^{th} ed)$
  - Picture shows us the loading and geometry
  - Equation is given to calculate the nominal stress considering the geometry with the SC
  - Axis and data labels tell us the quantities we need to calculate (using the figure as a guide)

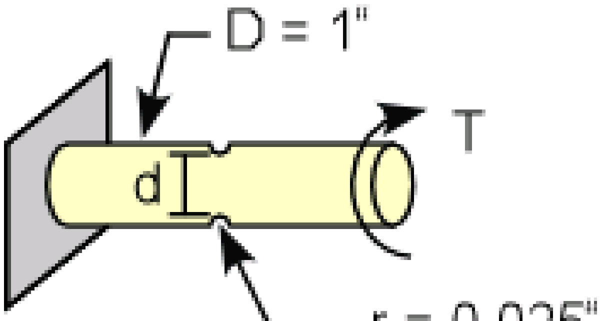


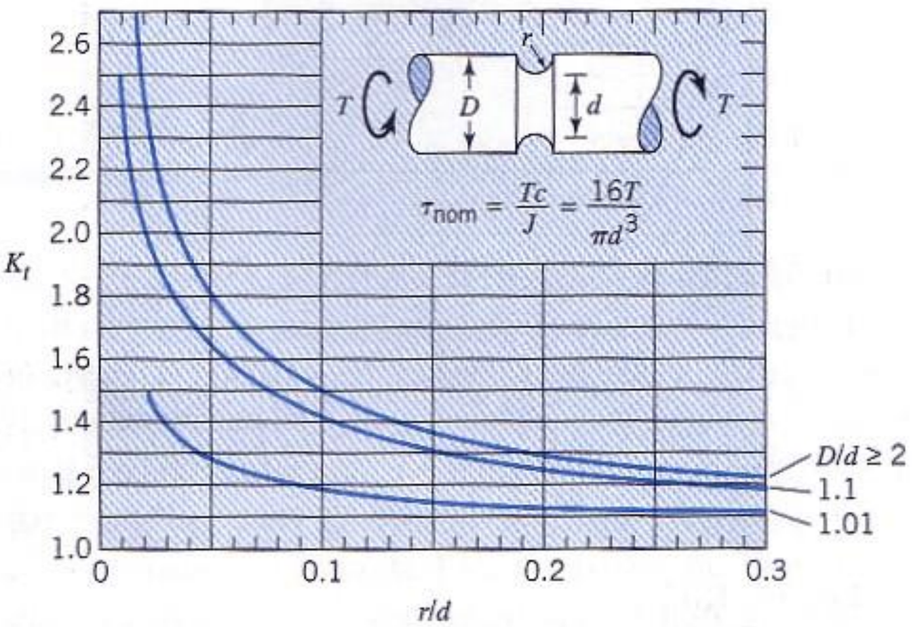


Brittle failure example

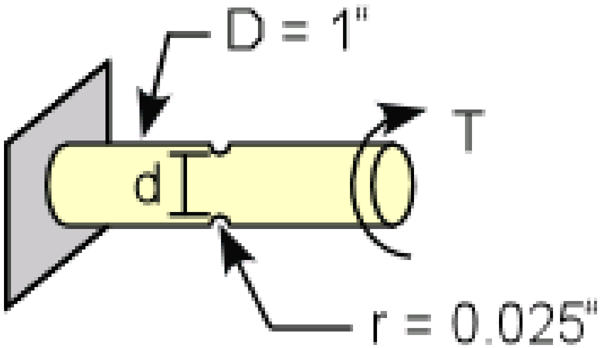
- With the SC
  - d = D – 2r = 1 – 2(.025) = 0.95
  - D/d = 1/0.95 = 1.05
  - r/d = 0.025/0.95 = 0.026
  - $k_t \sim 1.8$
  - $\tau_{max} = k_t \tau_{nom}$

$$\tau_{nom} = \frac{Tc}{J} = \frac{16T}{\pi d^3} = 5.94T$$





Brittle failure example



- $\tau_{max} = k_t \tau_{nom}$

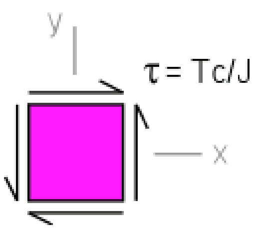
$$\tau_{nom} = \frac{Tc}{J} = \frac{16T}{\pi d^3} = 5.94T$$

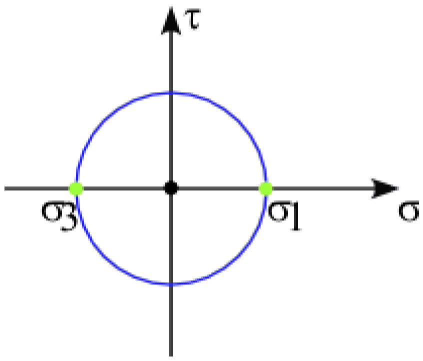
- $\tau_{max} = k_t \tau_{nom}$ = (1.8)(5.94T) = 10.69 T
- Construct stress element and Mohr's Circle as before
- Use Coulomb-Mohr theory

$$\frac{\sigma_A}{S_{ut}} - \frac{\sigma_B}{S_{uc}} \le 1$$

$$\frac{10.69T}{31\times 10^3} - \frac{-10.69T}{109\times 10^3} \le 1$$

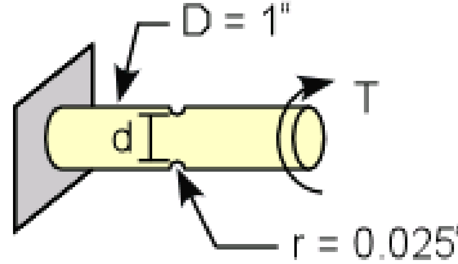
$$T\left(4.43\times 10^{-4}\right) \le 1$$





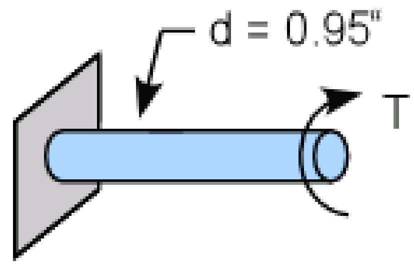
- T ≤ 2258 in·lb
- About half the load that could be withstood in the absence of the SC!

Brittle failure example



- What if we consider a solid shaft (no SC's) with a diameter of d (0.95) ?

$$\tau = \frac{Tc}{J} = \frac{T\left(\frac{d}{2}\right)}{\left(\frac{\pi d^4}{32}\right)} = 5.9T$$

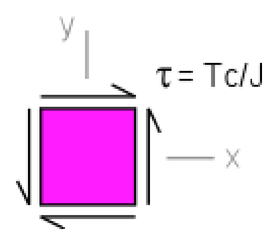


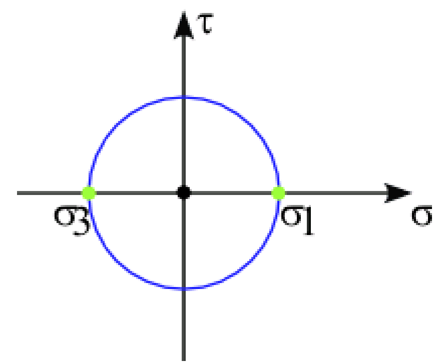
- Again, use Coulomb-Mohr

$$\frac{\sigma_A}{S_{ut}} - \frac{\sigma_B}{S_{uc}} \le 1$$

$$\frac{5.9T}{31\times10^3} - \frac{-5.9T}{109\times10^3} \le 1$$

$$T \le 4090 in \cdot lb$$



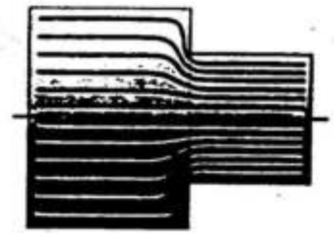


- Note, this is a greater amount of torque than a shaft with larger diameter but with a SC

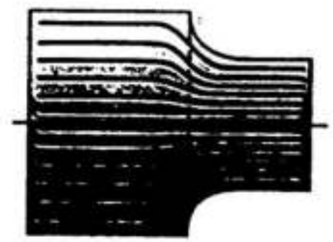
Design to avoid stress concentrations

- Avoid sudden changes in cross-section
- Avoid sharp inside corners
- Force-flow analogy
  - Imagine flow of incompressible fluid through part
  - Sudden curvature in streamlines…
    - High stress concentration!

Design to avoid stress concentrations



(*a*) Force flow around a sharp corner



(*b*) Force flow around a radiused corner

Design to avoid stress concentrations