CHAPTER 3

The Simplex Method

The Simplex Method

- The simplex method is an iterative procedure.
- Beginning at a vertex of the feasible region S, each iteration brings us to another vertex of S with an improved value of the objective function.
- The iteration ends when the optimal solution is reached.

Simplex Method: Standard Maximization Problem

- A standard maximization problem is one in which
  - *The objective function is to be maximized.*
  - *All the variables involved in the problem are nonnegative.*
  - *All other linear constraints may be written so that the expression involving the variables is less than or equal to a nonnegative constant.*

Simplex Method: Standard Maximization Problem – An Example

Maximize

$$P = x + 6/5\, y$$

Subject to

$$2x + y \leq 180$$

$$x + 3y \leq 300$$

$$x, y \geq 0$$

Setting Up the Initial Simplex Tableau

- Rewrite the objective function where all the variables are on the left and the coefficient of the objective function is +1.

- Transform the system of linear inequalities into a system of linear equations by introducing slack variables.

- Build the initial Simplex Tableau

Example

- Rewrite the objective function

  $P - x - 6/5\, y = 0$

- Transform the system of linear inequalities into a system of linear equations by introducing slack variables, s1 and s2.

$$
\begin{aligned}
2x + y + s_1 &= 180 \\
x + 3y + s_2 &= 300
\end{aligned}
$$

Example

- Build the initial Simplex tableau

Basic	**P**	**x**	**y**	**s1**	**s2**	**Solution**
P	1	–1	– 6/5	0	0	0
S1	0	2	1	1	0	180
S2	0	1	3	0	1	300

Simplex Method: Computational Details

1. Set up the initial simplex tableau.
2. Determine whether the optimal solution has been reached by examining all entries in the objective function row
   1. *If all the entries are nonnegative, the optimal solution has been reached. Proceed to step 4.*
   2. *If there are one or more negative entries, the optimal solution has not been reached. Proceed to step 3.*
3. Perform the pivot operation
   1. *Decide pivot column, pivot row, and pivot element*
   2. *Apply Gauss Jordan Elimination*
   3. *Return to step 2.*
4. Determine the optimal solution.

Example

- Step 1: Set up the initial Simplex tableau

Basic	**P**	**x**	**y**	**s1**	**s2**	**Solution**
P	1	–1	– 6/5	0	0	0
S1	0	2	1	1	0	180
S2	0	1	3	0	1	300

- Step 2: Determine whether the optimal solution has been reached.
  - Since there are negative entries in the objective function row of the tableau, the initial solution is not optimal.

Example

- Step 3: Perform the pivot operation (Pivot column).
  - *Since the entry –6/5 is the most negative entry in the objective function row of the tableau, the y column in the tableau is the pivot column (y is the entering variable).*

Basic	**P**	**x**	**y**	**s1**	**s2**	**Solution**
P	1	–1	–6/5	0	0	0
S1	0	2	1	1	0	180
S2	0	1	3	0	1	300

Example

- Step 3: Perform the pivot operation (Pivot row).
  - *For each positive number in the pivot column, divide the corresponding entry in the solution column by that number and compare the ratios thus obtained.*
  - *We see that the ratio 300/3 = 100 is less than the ratio 180/1 = 180, so s2 row is the pivot row (s2 is the leaving variable).*

Basic	**P**	**x**	**y**	**s1**	**s2**	**Solution**
P	1	–1	– 6/5	0	0	0
S1	0	2	1	1	0	180
S2	0	1	3	0	1	300

Example

- Step 3: Perform the pivot operation (Pivot Element).
  - *The entry 3 lying in the pivot column and the pivot row is the pivot element.*

Basic	**P**	**x**	**y**	**s1**	**s2**	**Solution**
P	1	–1	–6/5	0	0	0
S1	0	2	1	1	0	180
S2	0	1	3	0	1	300

Example

- Step 3: Perform the pivot operation (Gauss Jordan Elimination).
  - *Convert the pivot element into a 1:*

R3 = R3/3

Basic	**P**	**x**	**y**	**s1**	**s2**	**Solution**
P	1	–1	–6/5	0	0	0
S1	0	2	1	1	0	180
y	0	1/3	1	0	1/3	100

Example

- Step 3: Perform the pivot operation (Gauss Jordan Elimination).
  - *Convert the pivot column into a unit column:*

    *R1 = R1 +6/5*R3*

    R2 = R2 – R3

Basic	**P**	**x**	**y**	**s1**	**s2**	**Solution**
P	1	–3/5	0	0	2/5	120
S1	0	5/3	0	1	-1/3	80
y	0	1/3	1	0	1/3	100

Example

- Step 2: The Objective function row of the tableau contains a negative number, so an optimal solution has not been reached.
  - *Therefore, we repeat step 3.*

Basic	**P**	**x**	**y**	**s1**	**s2**	**Solution**
P	1	–3/5	0	0	2/5	120
S1	0	5/3	0	1	-1/3	80
y	0	1/3	1	0	1/3	100

Example

- Step 3: Perform the pivot operation (Pivot column).
  - *Since the entry –3/5 is the most negative entry in the objective function row of the tableau, the x column in the tableau is the pivot column (x is the entering variable).*

Basic	**P**	**x**	**y**	**s1**	**s2**	**Solution**
P	1	–3/5	0	0	2/5	120
S1	0	5/3	0	1	-1/3	80
y	0	1/3	1	0	1/3	100

Example

- Step 3: Perform the pivot operation (Pivot row).
  - *For each positive number in the pivot column, divide the corresponding entry in the solution column by that number and compare the ratios thus obtained.*
  - *We see that the ratio 80/(5/3) = 48 is less than the ratio 100/(1/3) = 300, so s1 row is the pivot row (s1 is the leaving variable).*

Basic	**P**	**x**	**y**	**s1**	**s2**	**Solution**
P	1	–3/5	0	0	2/5	120
S1	0	5/3	0	1	-1/3	80
y	0	1/3	1	0	1/3	100

Example

- Step 3: Perform the pivot operation (Pivot Element).
  - *The entry 5/3 lying in the pivot column and the pivot row is the pivot element.*

Basic	**P**	**x**	**y**	**s1**	**s2**	**Solution**
P	1	–3/5	0	0	2/5	120
S1	0	5/3	0	1	-1/3	80
y	0	1/3	1	0	1/3	100

Example

- Step 3: Perform the pivot operation (Gauss Jordan Elimination).
  - *Convert the pivot element into a 1:*

R2 = R2/(5/3)

Basic	**P**	**x**	**y**	**s1**	**s2**	**Solution**
P	1	–3/5	0	0	2/5	120
x	0	1	0	3/5	-1/5	48
y	0	1/3	1	0	1/3	100

Example

- Step 3: Perform the pivot operation (Gauss Jordan Elimination).
  - *Convert the pivot column into a unit column:*

    *R1 = R1 + 3/5*R2*

    *R3 = R3 – 1/3*R2*

Basic	**P**	**x**	**y**	**s1**	**s2**	**Solution**
P	1	0	0	9/25	7/25	148.8
X	0	1	0	3/5	-1/5	48
Y	0	0	1	-1/5	2/5	84

Example

- Step 2: The Objective function row of the tableau contains no negative numbers, so an optimal solution has been reached.

Basic	**P**	**x**	**y**	**s1**	**s2**	**Solution**
P	1	0	0	9/25	7/25	148.8
X	0	1	0	3/5	-1/5	48
Y	0	0	1	-1/5	2/5	84

Example

- Step 4: Determine the optimal solution.
  - *The optimal value for x is 48.*
  - *The optimal value for y is 84.*
  - *The optimal value for P is 148.8.*

- *Check with Excel or Tora graphical solution*

Basic	**P**	**x**	**y**	**s1**	**s2**	**Solution**
P	1	0	0	9/25	7/25	148.8
X	0	1	0	3/5	-1/5	48
Y	0	0	1	-1/5	2/5	84

Practice and Exercises

- Work out Problem set 3.3B on Pages 100-103:

Simplex Method: Standard Minimization Problem – Method 1

- Follow the same steps, as the maximization problem, with the following changes
  - *The optimal solution is reached if all the entries in the objective function row are* ***nonpositive***
  - *In the pivot column selection, select the column with the* ***most positive*** *value*

Simplex Method: Standard Minimization Problem – Method 2

- Convert the objective function from minimize Obj to maximize (–Obj).
- Solve as a standard maximization problem
- Remember to negate the value of the objective function after getting it.
- Practice:

Minimize $C = -2x - 3y$

Subject to $5x + 4y \leq 32$

$x + 2y \leq 10$

$x, y \geq 0$

M Method

- M method allow us to solve maximization and minimization problems with $\geq$, =, and negative constants
- $\geq$ inequalities could be transformed into equalities by subtracting a positive surplus variable from the L.H.S

$x1 + x2 > 800$ ☾ $x1 + x2 - s1 = 800$

- Negative constants could be transformed into positive by multiplying the whole inequality by –1 (Notice: the inequality sign is changed)

$-x1 + x2 \leq -3$ ☾ $x1 - x2 \geq 3$ ☾ $x1 - x2 - s1 = 3$

M Method: Problems with surplus variables and equality constraints

- Using surplus and the existence of equality constraint cause the simplex method to produce infeasible solution
- Check

Maximize $P = 2x1 + x2$

subject to

$x1 + x2 \leq 10$

$-x1 + x2 \geq 2$

$x1, x2 \geq 0$

M Method: Problems with surplus and equality constraint

- For both $\geq$ or = constraints, add an artificial variables (Keep the surplus as well)

- Add these artificial variables to the objective function and multiply them with a very big penalty M (for minimization problems) or –M (for maximization problem)

- M is selected according to the values in the problem (between 10x and 100x). The same M is used with all artificial variables

Example

Maximize P = 2x1 + x2 ☾ P = 2x1 + x2 – 100R1

subject to

x1 + x2 ≤ 10 ☾ x1 + x2 + s1 = 10

–x1 + x2 ≥ 2 ☾ –x1 + x2 – s2 + R1 = 2

x1, x2 ≥ 0 ☾ x1, x2, s1, s2, R1 ≥ 0

M Method: Computational Details

1. Set up the initial simplex tableau.
   - *For $\geq$ inequality, the artificial variable is used as a basic variable not the surplus*

2. If the solution obtained from the table is not feasible, adjust it
   - *New objective function row = old objective function row $\pm$ M*(row2 + row3 + ….)*

3. Continue with the usual simplex process

4. If any artificial variable is nonzero in the final solution, the original problem has no optimal solution.

Practice and Exercises

- Work out Problem set 3.4A on Pages 106-108

Two-phase Method

- Drawback of M Method
  - *The use of a very large penalty M causes round off errors*

- The two-phase method does not use a Big M, but requires solving two LP problems
  - *Phase 1: Find a feasible starting solution*
  - *Phase 2: Solve the original problem*

- All commercial packages use the two-phase method

Two-phase Method: Computational Details

1. Put the problem in the equality form as the M method (use slack, surplus, and artificial variables)
2. Phase 1: Minimize r = sum (R1, R2,….)
   i. *ignore the original objective function*
   ii. *For both minimization and maximization problems, solve a LP whose objective is to minimize the sum of all artificial variables*
   iii. *After setting up the initial simplex tableau, adjust it as in the M method*
   iv. *In the solution, if any of the artificial variables has a non-zero value, the original problem has no feasible solution*
   v. *In the solution, if any of the artificial variables is a basic variable with zero value, replace it with any other variable*
3. Phase 2: Solve the original LP problem
   i. *Drop columns of all artificial variables*
   ii. *Use the final solution of phase 1 as a starting basic solution to the original problem*

Practice and Exercises

- Work out Problem set 3.4B on Pages 111-112

Sensitivity Analysis – Changes in resources

- After solving the problem, the final tableau could be used to analyze changes in the model

- Shadow price, Dual price, or Unit worth of a resource
  - *Values in the z row under the slack variables*

- Feasibility range
  - *The range at which a resource could change without the need to re-solve the problem*
  - *The range is obtained by manipulating the rows of the optimal tableau ☾ Each row ≥ 0*

Practice and Exercises

- Work out Problem set 3.6C on Pages 134-139

Sensitivity Analysis – Changes in Objective Function Coefficients

- Optimality range
  - *The range at which an objective function coefficient could change without the need to re-solve the problem*
  - *The range is obtained by manipulating the columns of the optimal tableau ☾ Each non-zero column ≥ 0*

Practice and Exercises

- Work out Problem set 3.6D on Pages 144-146