ME 8043114 Automatic Control

Lecture 8: Frequency Response
Bode Diagram

Dr. Shadi M. Munshi
E-mail: smmunshi@uqu.edu.sa

Dr. Badr AlOufi
E-mail: baawfey@uqu.edu.sa

Lecture Outline

- Frequency Response
- Bode Diagram

Course roadmap

Modeling

- ✓ Laplace transform
- ✓ Transfer function
- ✓ Models for systems
  - ✓ electrical
  - ✓ mechanical
  - ✓ electromechanical
- ✓ Linearization, delay

Analysis

- ✓ Time response
  - ✓ Transient
  - ✓ Steady state
- Frequency response
  - Bode plot
- ✓ Stability
  - ✓ Routh-Hurwitz
  - Nyquist

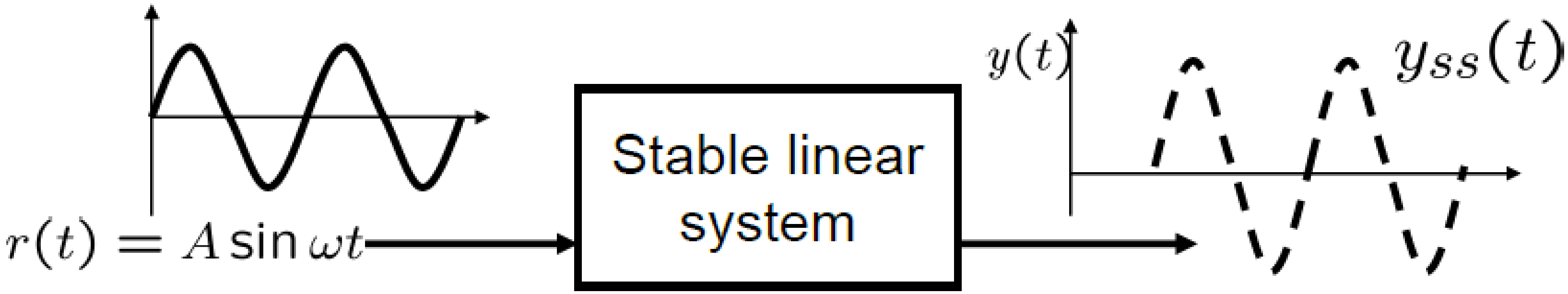
Design

- Design specs
- ✓ Root locus
- Frequency domain
- ✓ PID & Lead-lag
- Design examples

Summary up to now & *from now on*

- Modeling: How to represent systems with transfer functions (s-domain).
- Analysis: How to extract time-response information from s-function.
  - Steady-state error depends on TF evaluated at s=0.
  - Stability and transient depends on pole locations.
  - *Frequency responses contain all these information.*
- Design: How to obtain nice closed-loop system.
  - Poles can be placed by the root-locus technique.
  - *System's freq. responses can be shaped in Bode plot.*

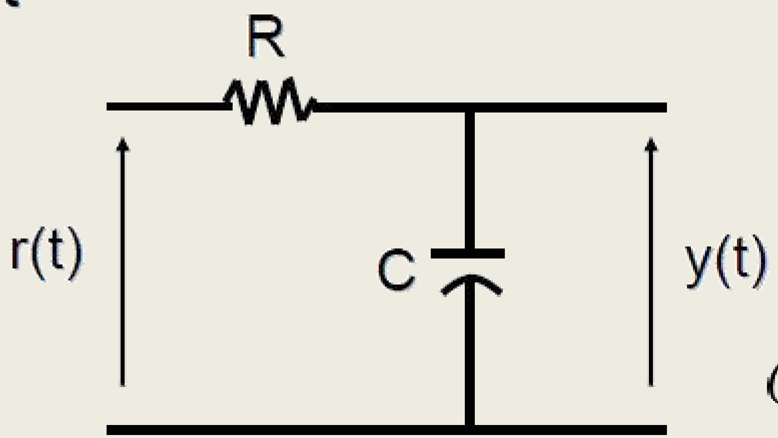
What is frequency response?



- We would like to analyze a system property by applying a *test sinusoidal input* r(t) and observing a response y(t).
- Steady state response $y_{ss}(t)$ (after transient dies out) of a system to sinusoidal inputs is called ***frequency response***.

A simple example

- RC circuit



$$G(s) = \frac{1}{RCs + 1}$$

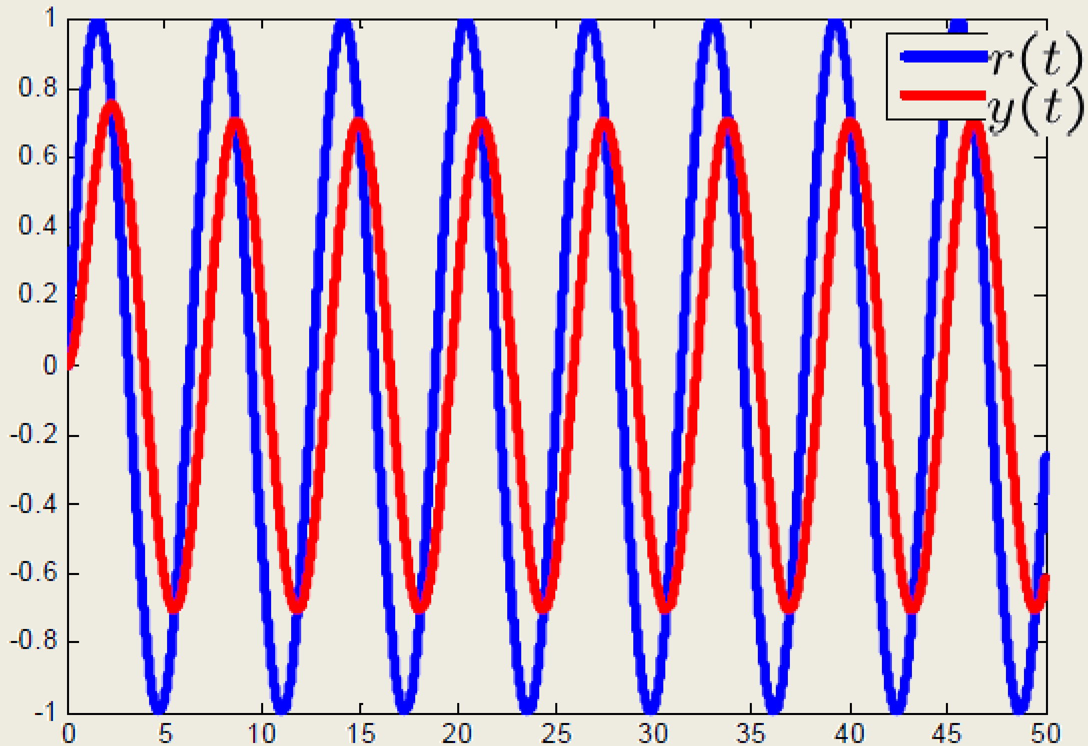
- Input a sinusoidal voltage r(t)
- What is the output voltage y(t)?

An example (cont'd)

- TF (R=C=1)

$$G(s) = \frac{1}{s+1}$$

- r(t)=sin(t)



At steady-state, r(t) and y(t) has same frequency, but different amplitude and phase!

An example (cont'd)

- Derivation of y(t)

$$Y(s) = G(s)R(s) = \frac{1}{s+1}\cdot\frac{1}{s^2+1} = \frac{1}{2}\left(\frac{1}{s+1} + \frac{-s+1}{s^2+1}\right)$$

Partial fraction expansion

- Inverse Laplace

$$y(t) = \frac{1}{2}\left(e^{-t} - \cos t + \sin t\right)$$

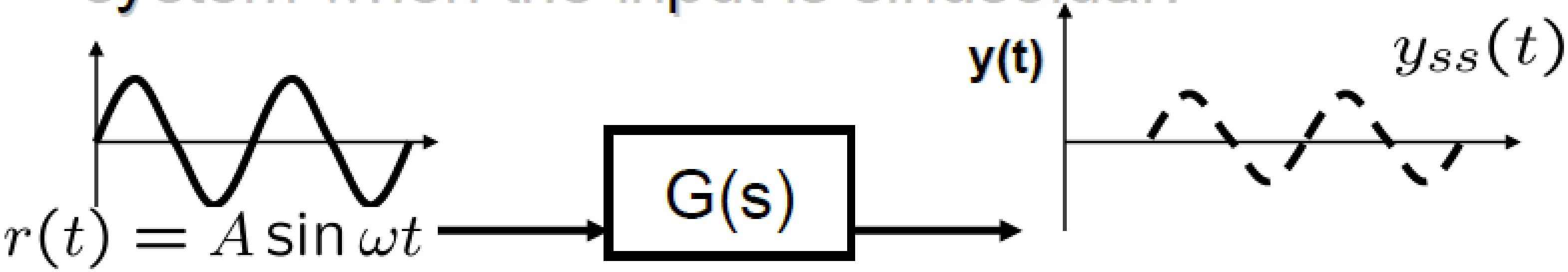
0 as t goes to infinity.

$$\Rightarrow \quad y_{ss}(t) = \frac{1}{2}(-\cos t + \sin t) = \frac{1}{\sqrt{2}}\sin(t - 45^\circ)$$

(Derivation for general G(s) is given at the end of lecture slide.)

Response to sinusoidal input

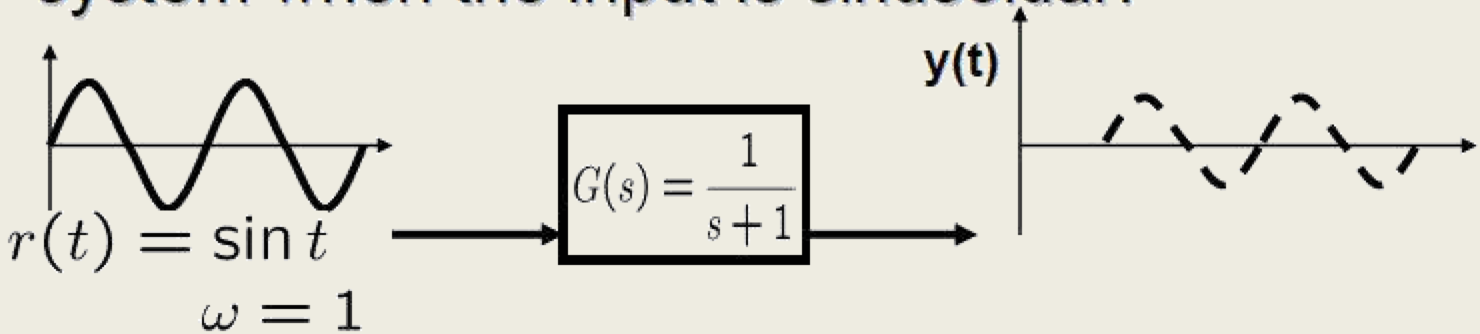
- How is the steady state output of a stable linear system when the input is sinusoidal?



- Steady state output $y_{ss}(t) = A|G(j\omega)|\sin(\omega t + \angle G(j\omega))$
  - Frequency is same as the input frequency ω
  - Amplitude is that of input (A) multiplied by $|G(j\omega)|$ Gain
  - Phase shifts $\angle G(j\omega)$

Revisit to the example

- How is the steady state output of a stable linear system when the input is sinusoidal?

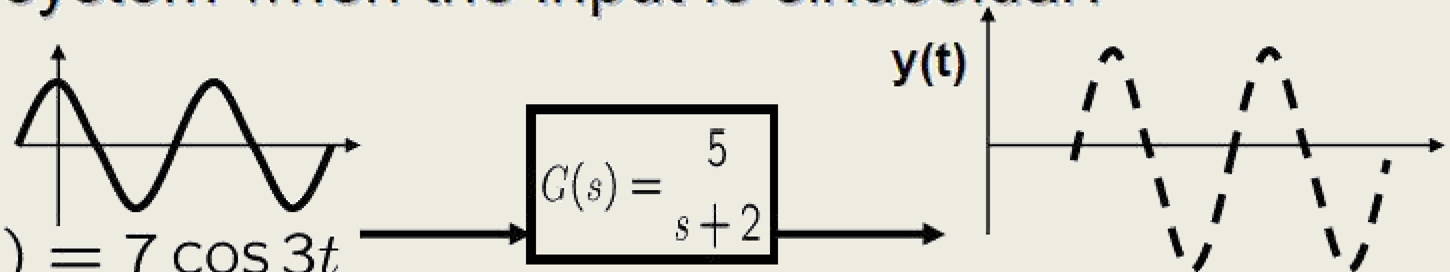


$$y_{ss}(t) = \underbrace{\frac{1}{\sqrt{2}}}_{|G(j\cdot 1)|} \sin(t \underbrace{-45^\circ}_{\angle G(j\cdot 1)})$$

Gain Phase

Another example

- How is the steady state output of a stable linear system when the input is sinusoidal?



$$r(t) = 7\cos 3t$$
$$\omega = 3$$

$$y_{ss}(t) = 7 \cdot \underbrace{\frac{5}{\sqrt{13}}}_{|G(j\cdot 3)|} \cos(t + \underbrace{\theta}_{\angle G(j\cdot 3)})$$

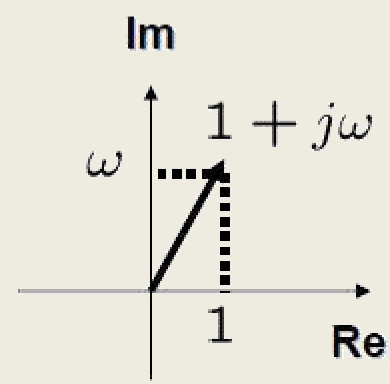
Gain　　　　Phase

Frequency response function

- For a stable system G(s), G(jω) (ω is positive) is called *frequency response function (FRF)*.
- For each ω, FRF takes a complex number G(jω), which has a gain and a phase.
- First order example

$$G(s) = \frac{1}{s+1} \Rightarrow G(j\omega) = \frac{1}{j\omega + 1}$$

$$\Rightarrow \begin{cases} |G(j\omega)| = \dfrac{1}{\sqrt{1+\omega^2}} \\ \angle G(j\omega) = \angle(1) - \angle(j\omega + 1) = -\tan^{-1}\omega \end{cases}$$



First order example revisited

- FRF $G(j\omega) = \dfrac{1}{j\omega + 1}$

frequency ω	amplitude $\lvert G(j\omega)\rvert$	phase $\angle G(j\omega)$
0	1	0°
0.5	0.894	-26.6°
1.0	0.707	-45°
$\vdots$	$\vdots$	$\vdots$
∞	0	-90°

- Two graphs representing FRF
  - Bode diagram (Bode plot)
  - Nyquist diagram (Nyquist plot)

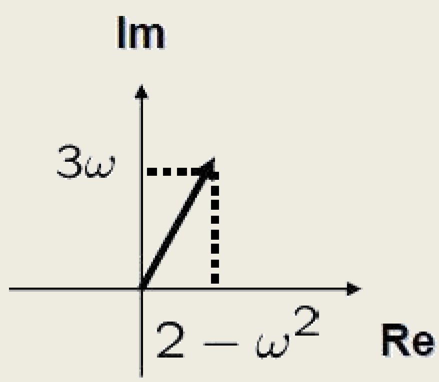
Another example of FRF

- Second order system

$$G(s) = \frac{2}{s^2 + 3s + 2}$$

$$\Rightarrow \quad G(j\omega) = \frac{2}{(j\omega)^2 + 3(j\omega) + 2} = \frac{2}{2 - \omega^2 + j \cdot 3\omega}$$

$$\Rightarrow \quad \begin{cases} |G(j\omega)| = \dfrac{2}{\sqrt{(2-\omega^2)^2 + 9\omega^2}} \\ \angle G(j\omega) = \angle(2) - \angle(2 - \omega^2 + j \cdot 3\omega) \\ \quad = -\tan^{-1} \dfrac{3\omega}{2 - \omega^2} \end{cases}$$



Course roadmap

Modeling

- ✓ Laplace transform
- ✓ Transfer function
- ✓ Models for systems
  - ✓ electrical
  - ✓ mechanical
  - ✓ electromechanical
- ✓ Linearization, delay

Analysis

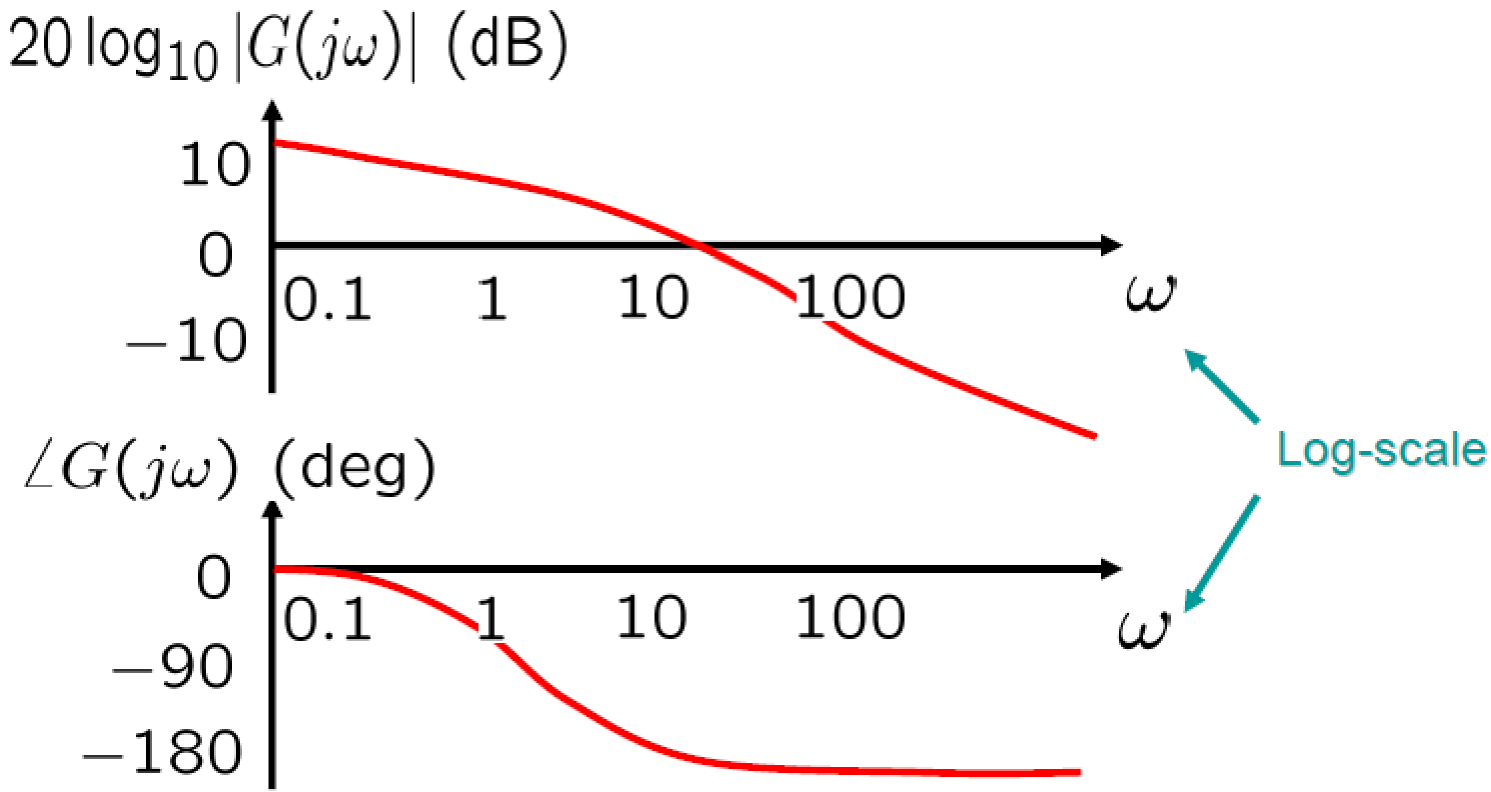
- ✓ Time response
  - ✓ Transient
  - ✓ Steady state
- Frequency response
  - Bode plot
- ✓ Stability
  - ✓ Routh-Hurwitz
  - Nyquist

Design

- Design specs
- ✓ Root locus
- Frequency domain
- ✓ PID & Lead-lag
- Design examples

Bode diagram (Bode plot) of G(jω)

- Bode diagram consists of gain plot & phase plot



Bode plot of a 1st order system

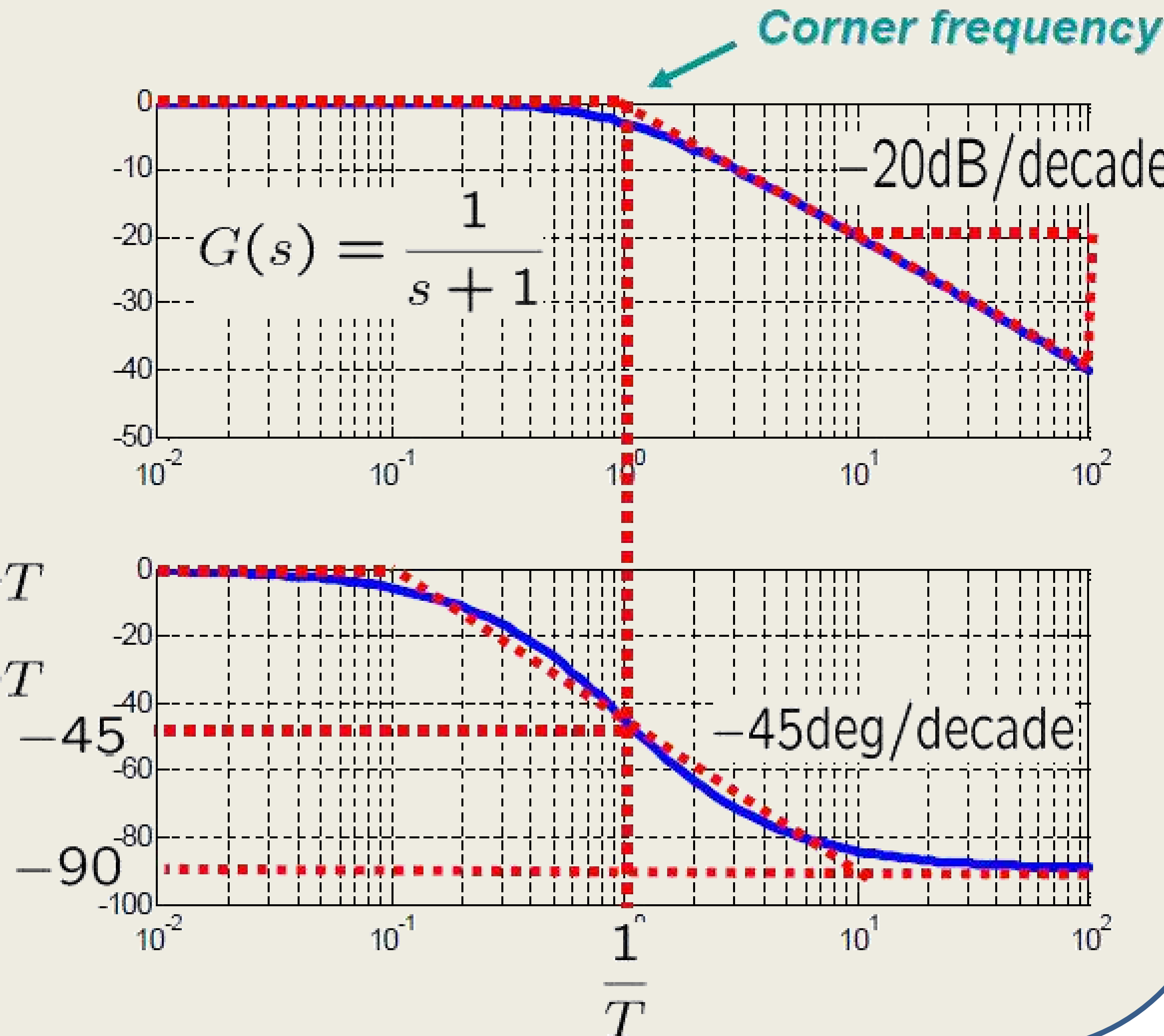
- TF

$$G(s) = \frac{1}{Ts+1}$$

↓

$$G(j\omega) = \frac{1}{j\omega T + 1}$$

$$\approx \begin{cases} 1 & \text{if } 1 \gg \omega T \\ \frac{1}{j\omega T} & \text{if } 1 \ll \omega T \end{cases}$$



Exercises of sketching Bode plot

- First order system

$$G(s) = \frac{1}{s+1} \qquad G(s) = \frac{1}{0.1s+1} \qquad G(s) = \frac{1}{10s+1}$$

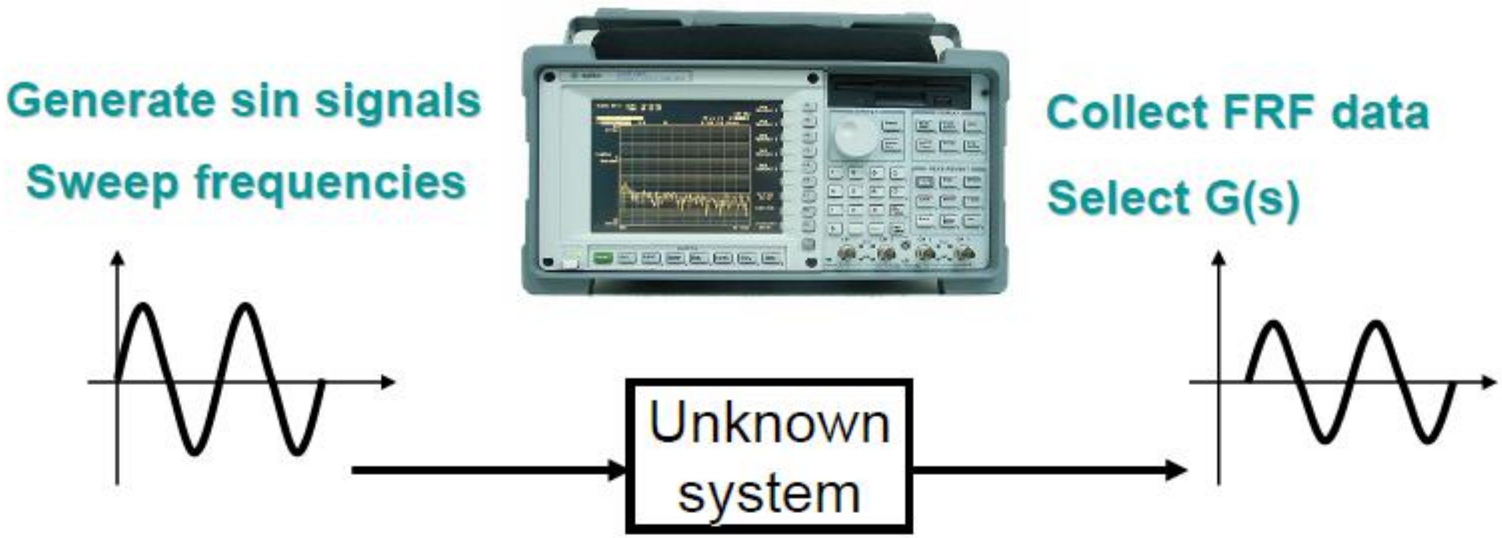
Remarks on Bode diagram

- Bode diagram shows amplitude and phase shift of a system output for sinusoidal inputs with various frequencies.
- Bode diagram is very useful and important in analysis and design of control systems.
- The shape of Bode plot contains information of stability, time responses, and much more!
- It can also be used for system identification. (Given FRF experimental data, obtain a transfer function that matches the data.)

System identification

- Sweep frequencies of sinusoidal signals and obtain FRF data (i.e., gain and phase).
- Select G(s) so that G(jω) fits the FRF data.

Agilent Technologies: FFT Dynamic Signal Analyzer

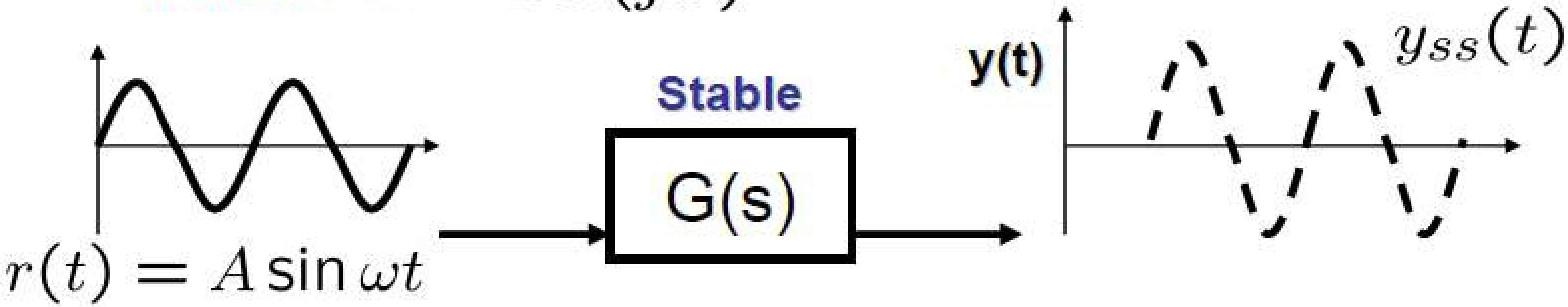


Course roadmap

Modeling	*Analysis*	*Design*
✓ Laplace transform	✓ Time response	Design specs
✓ Transfer function	✓ Transient	✓ Root locus
✓ Models for systems	✓ Steady state	Frequency domain
✓ electrical	✓ Frequency response	✓ PID & Lead-lag
✓ mechanical	➤ Bode plot	Design examples
✓ electromechanical	✓ Stability	
✓ Linearization	✓ Routh-Hurwitz	
	• Nyquist	

Frequency response (review)

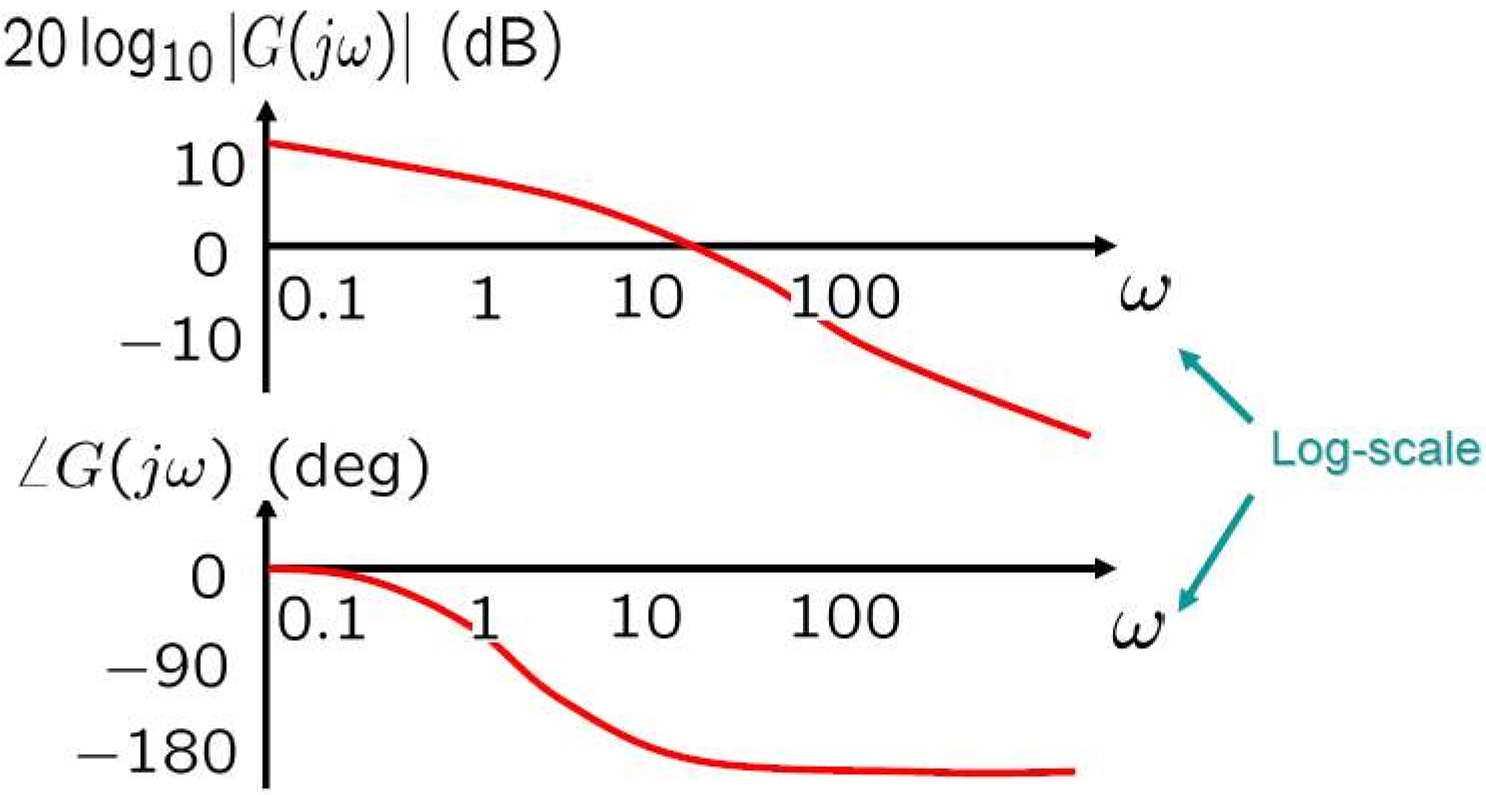
- Steady state output $y_{ss}(t) = A|G(j\omega)|\sin(\omega t + \angle G(j\omega))$
  - Frequency is same as the input frequency ω
  - Amplitude is that of input (A) multiplied by $|G(j\omega)|$ Gain
  - Phase shifts $\angle G(j\omega)$



- Frequency response function (FRF): G(jω)
- Bode plot: Graphical representation of G(jω)

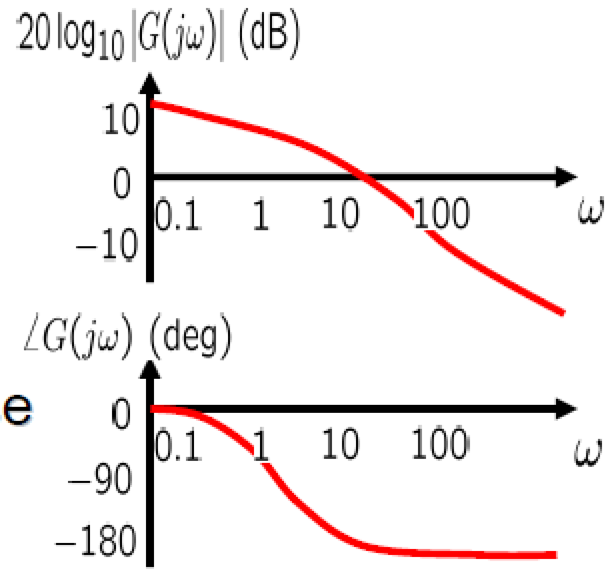
Bode plot of G(jω) (review)

- Bode plot consists of gain plot & phase plot



Sketching Bode plot

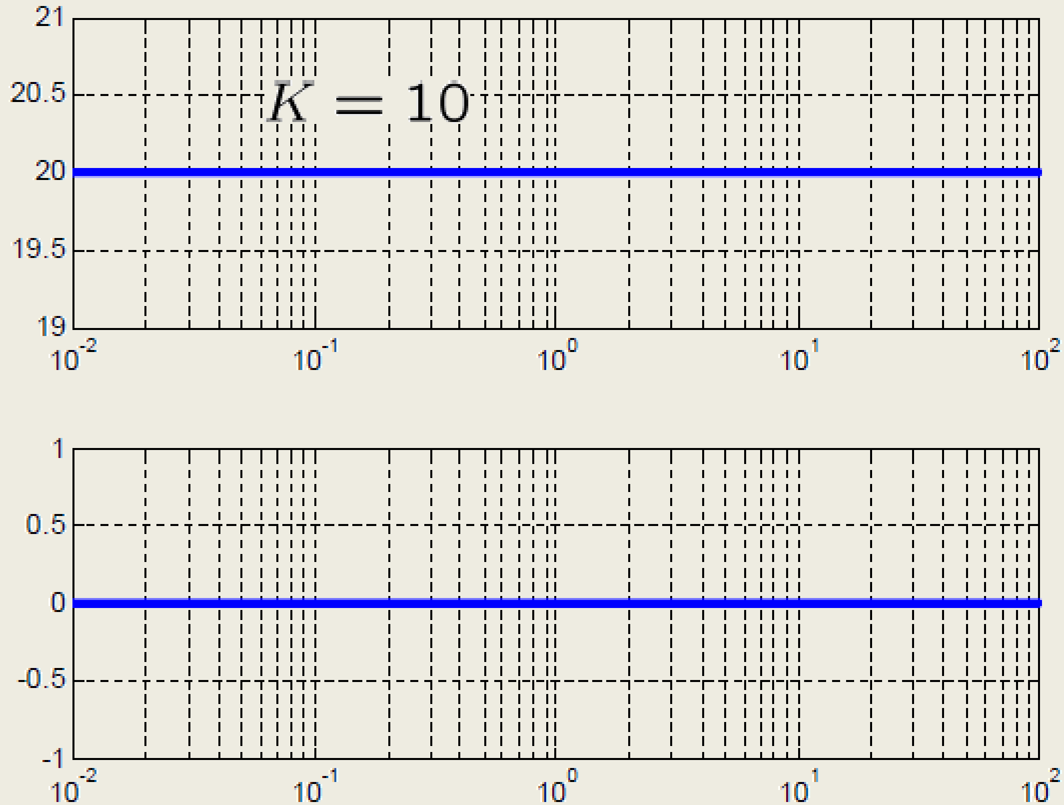
- Basic functions
  - Constant gain
  - Differentiator and integrator
  - Double integrator
  - First order system and its inverse
  - Second order system
  - Time delay



- Product of basic functions
  1. Sketch Bode plot of each factor, and
  2. Add the Bode plots graphically.

Bode plot of a constant gain

- TF $G(s) = K \Rightarrow |G(j\omega)| = K, \ \angle G(j\omega) = 0^{\circ}, \ \forall \omega$



Sketching Bode plot

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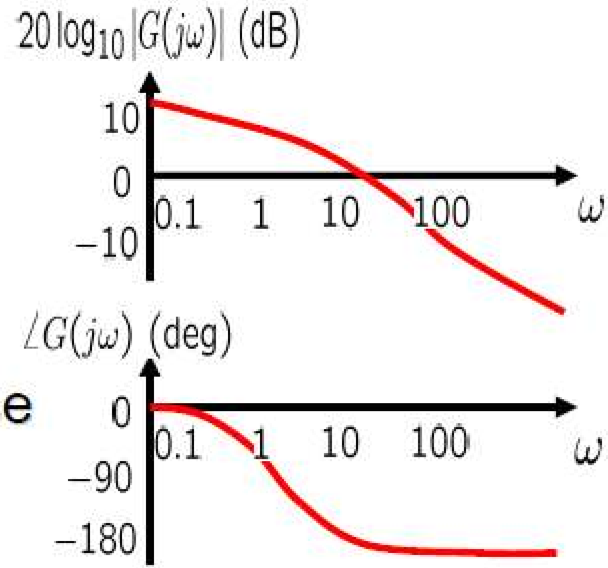
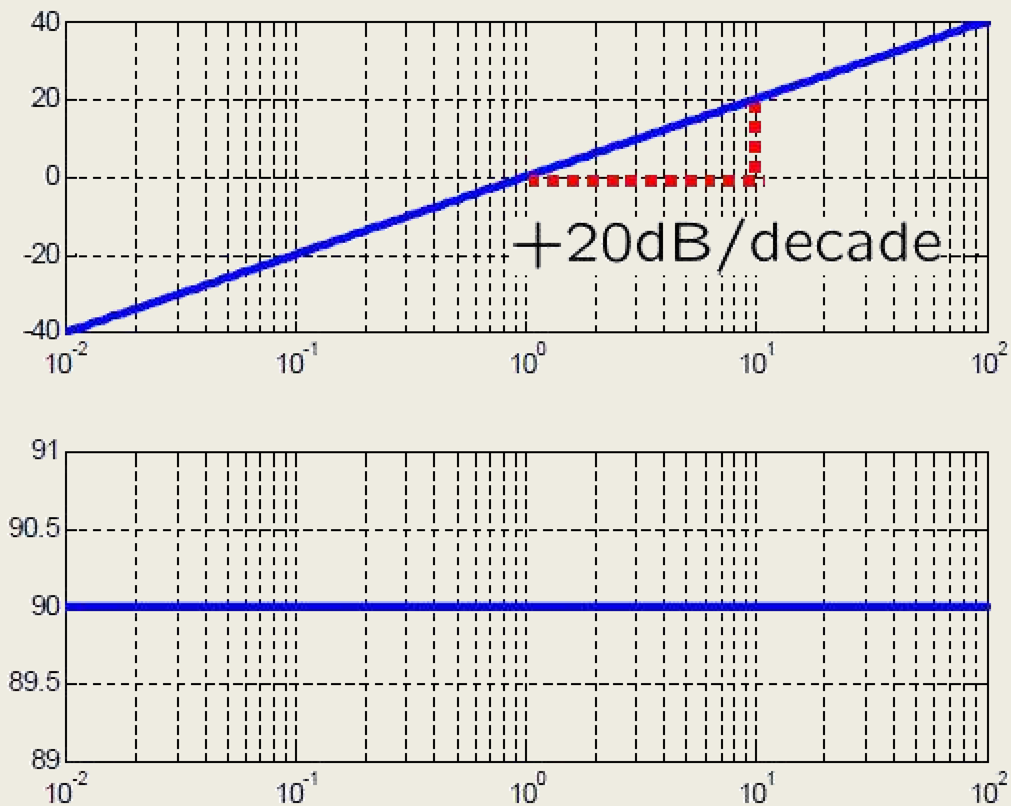


- Product of basic functions
  1. Sketch Bode plot of each factor, and
  2. Add the Bode plots graphically.

Bode plot of a differentiator

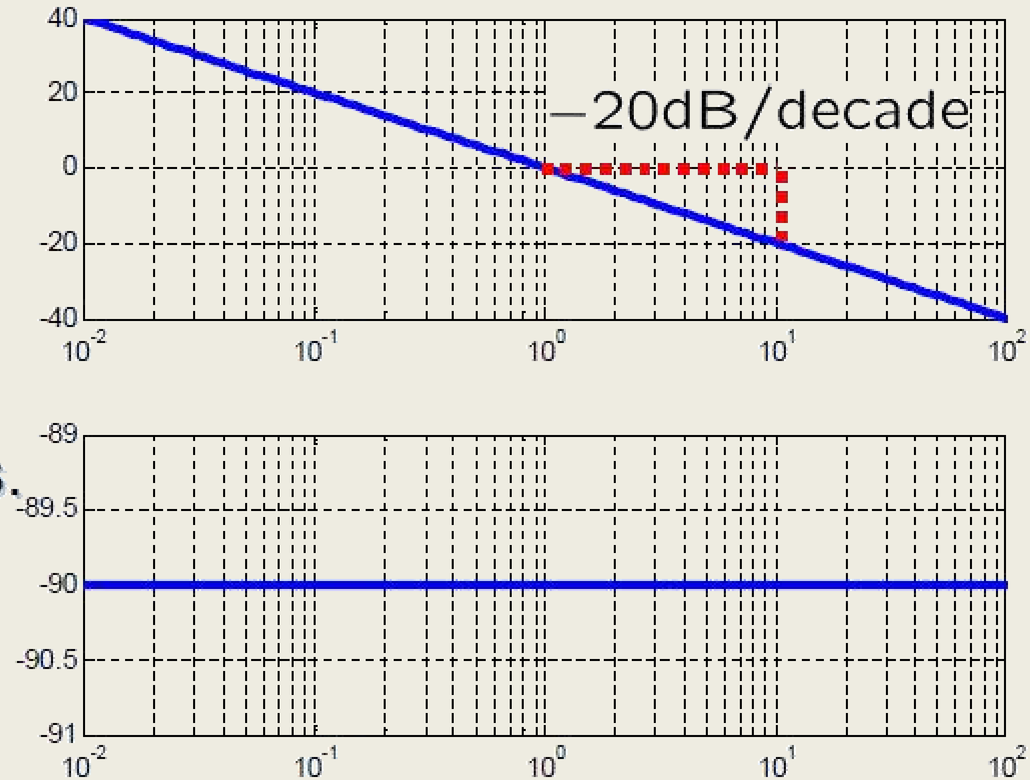
- TF $G(s) = s \Rightarrow |G(j\omega)| = \omega, \ \angle G(j\omega) = 90^{\circ}, \ \forall \omega$



Bode plot of an integrator

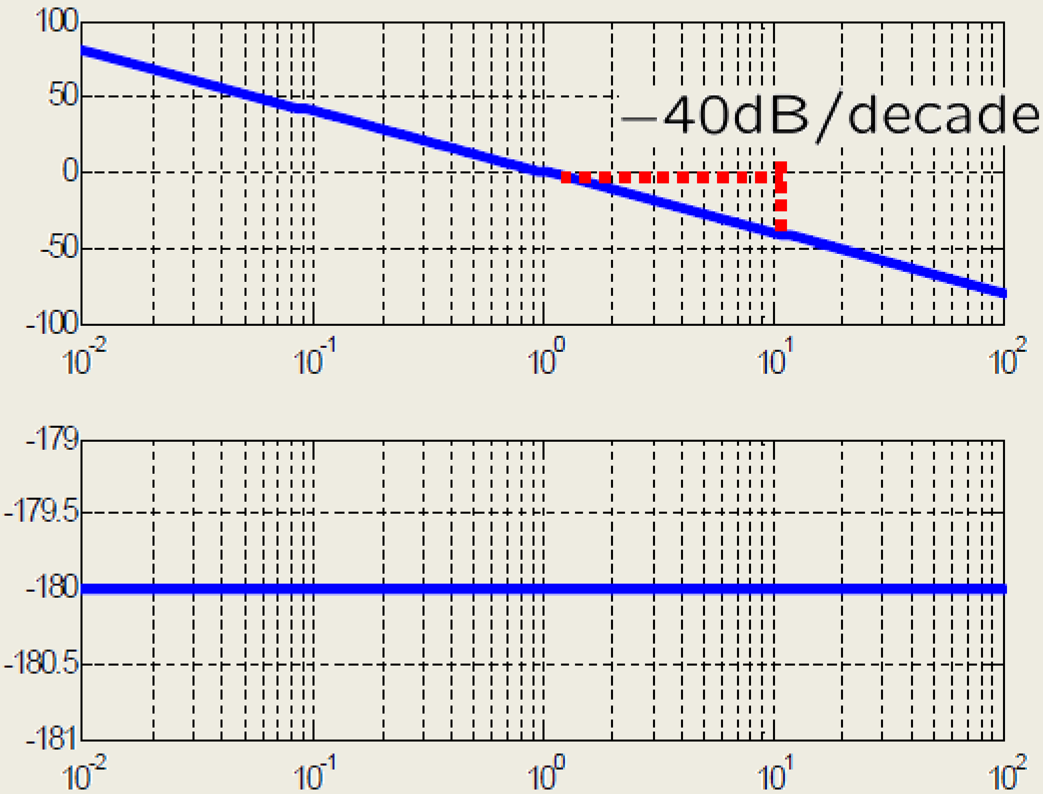
- TF $G(s) = \frac{1}{s} \Rightarrow |G(j\omega)| = \frac{1}{\omega}, \ \angle G(j\omega) = -90^\circ, \ \forall\omega$

Mirror image of the Bode plot of G(s)=s with respect to ω-axis.



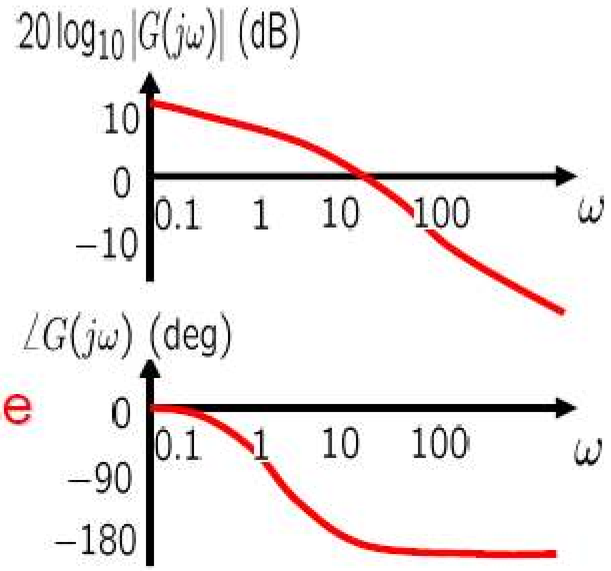
Bode plot of a double integrator

- TF $G(s) = \frac{1}{s^2} \Rightarrow |G(j\omega)| = \frac{1}{\omega^2},\ \angle G(j\omega) = -180^\circ,\ \forall\omega$



Sketching Bode plot

- Basic functions
  - ✓ Constant gain
  - ✓ Differentiator and integrator
  - ✓ Double integrator
  - First order system and its inverse
  - Second order system
  - Time delay



- Product of basic functions
  1. Sketch Bode plot of each factor, and
  2. Add the Bode plots graphically.

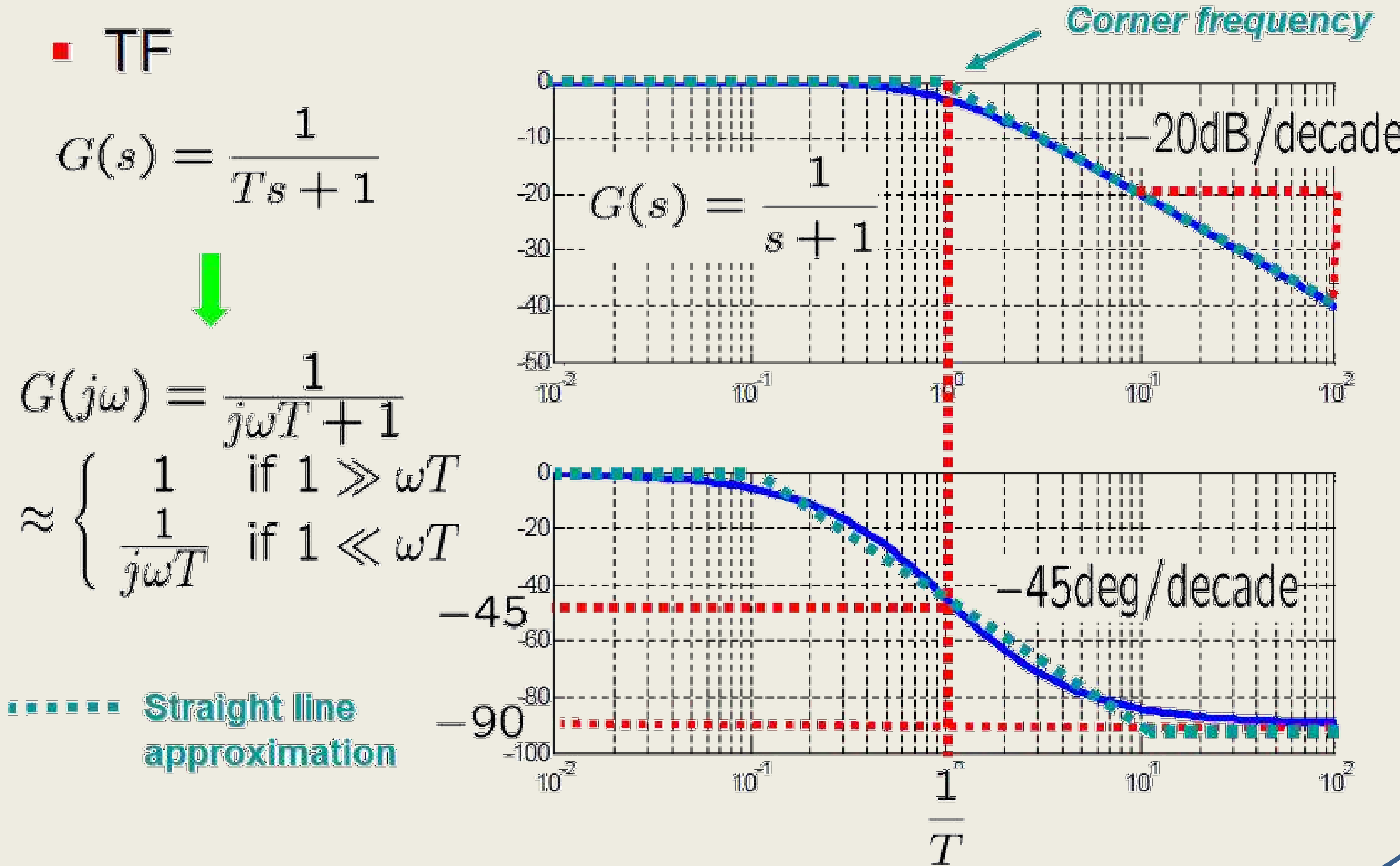
Bode plot of a 1st order system

- TF

$$G(s) = \frac{1}{Ts+1}$$

$$G(j\omega) = \frac{1}{j\omega T + 1}$$

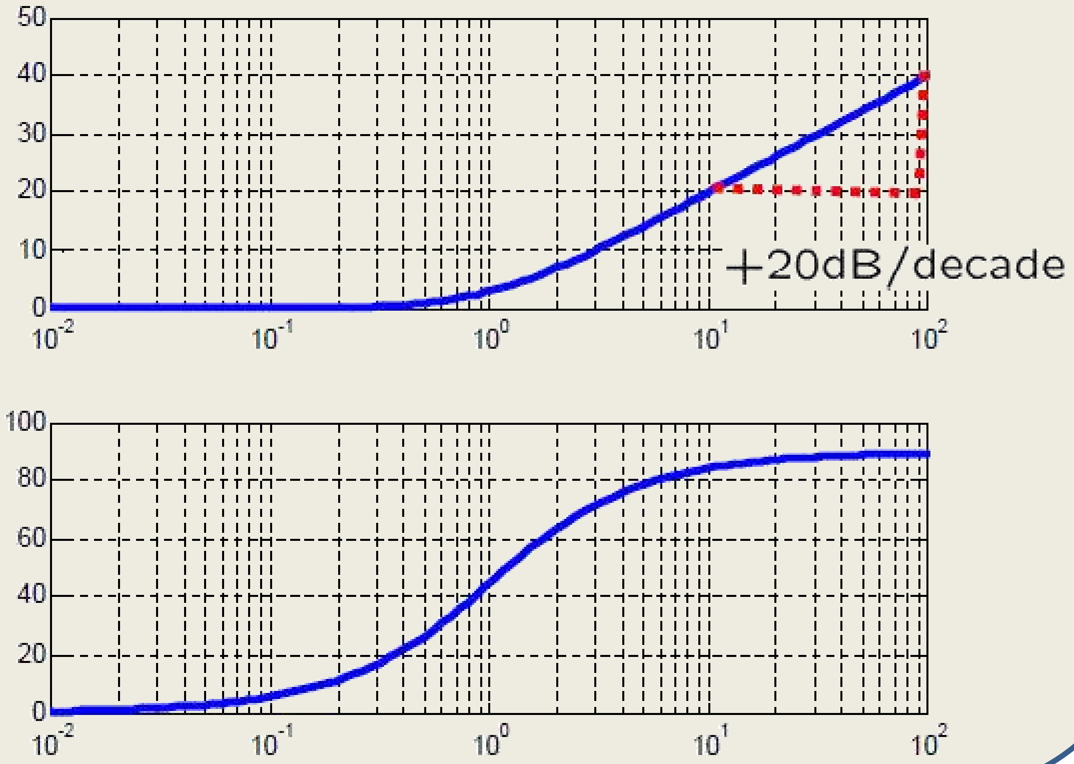
$$\approx \begin{cases} 1 & \text{if } 1 \gg \omega T \\ \frac{1}{j\omega T} & \text{if } 1 \ll \omega T \end{cases}$$



Bode plot of an inverse system

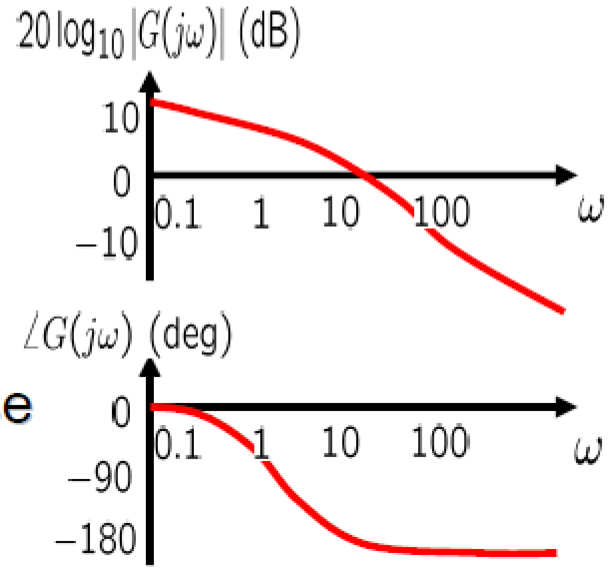
- TF $$G(s) = Ts + 1 = \left(\frac{1}{Ts+1}\right)^{-1}$$

Mirror image of the original Bode plot with respect to ω-axis.



Sketching Bode plot

- Basic functions
  - ✓ Constant gain
  - ✓ Differentiator and integrator
  - ✓ Double integrator
  - ✓ First order system and its inverse
  - Second order system
  - Time delay
- Product of basic functions
  1. Sketch Bode plot of each factor, and
  2. Add the Bode plots graphically.



Bode plot of a 2nd order system

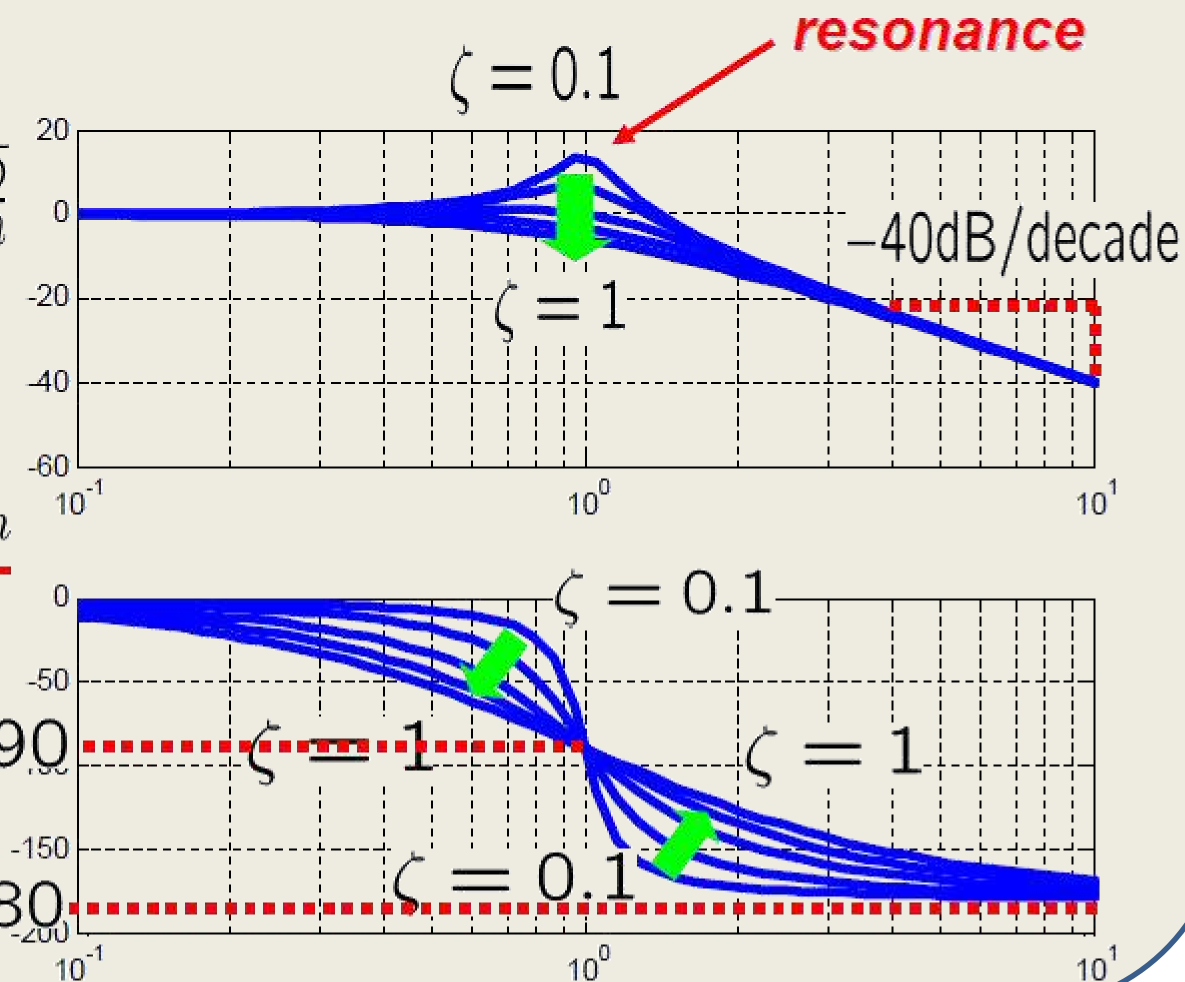
$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

Resonant freq.

$$\omega_n\sqrt{1-2\zeta^2}\approx\omega_n$$

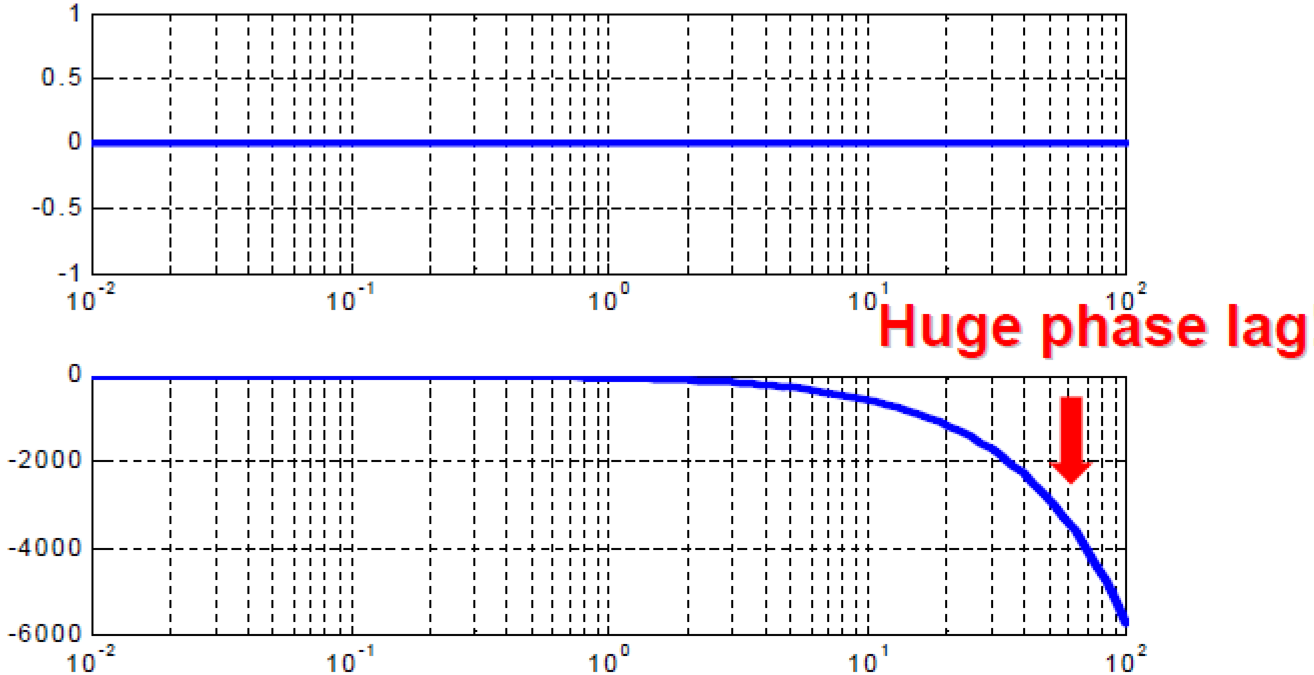
Peak gain

$$\frac{1}{2\zeta\sqrt{1-\zeta^2}}\approx\frac{1}{2\zeta}$$



Bode plot of a time delay

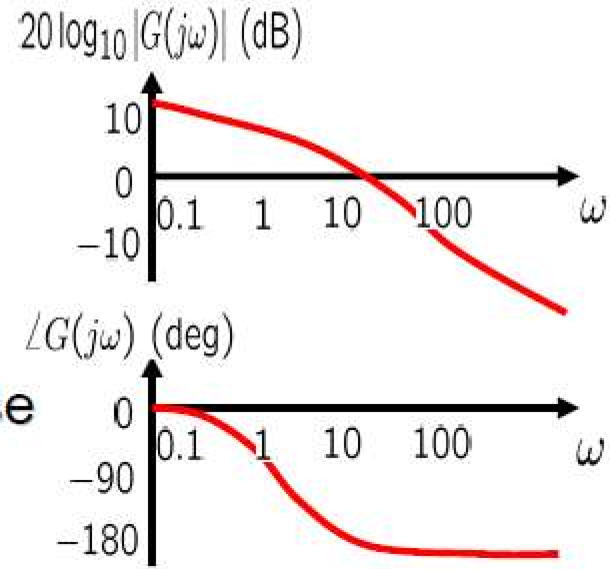
$$G(s) = e^{-Ts} \Rightarrow |G(j\omega)| = 1, \forall\omega,\ \angle G(j\omega) = -\omega T(\text{rad})$$



The phase lag causes instability of the closed-loop system, and thus, the difficulty in control.

Sketching Bode plot

- Basic functions
  - ✓ Constant gain
  - ✓ Differentiator and integrator
  - ✓ Double integrator
  - ✓ First order system and its inverse
  - ✓ Second order system
  - ✓ Time delay



- Product of basic functions
  1. Sketch Bode plot of each factor, and
  2. Add the Bode plots graphically.

Main advantage of Bode plot!

An advantage of Bode plot

- Bode plot of a series connection $G_1(s)G_2(s)$ is the addition of each Bode plot of G_1 and G_2.
  - Gain

$$20 \log_{10} |G_1(j\omega)G_2(j\omega)| = 20 \log_{10} |G_1(j\omega)| + 20 \log_{10} |G_2(j\omega)|$$

  - Phase

$$\angle G_1(j\omega)G_2(j\omega) = \angle G_1(j\omega) + \angle G_2(j\omega)$$

- We use this property to design C(s) so that G(s)C(s) has a "desired" shape of Bode plot.

Short proofs

- Use polar representation

$$G_1(j\omega) = |G_1(j\omega)|e^{j\angle G_1(j\omega)} \quad G_2(j\omega) = |G_2(j\omega)|e^{j\angle G_2(j\omega)}$$

Then,

$$\begin{aligned} G_1(j\omega)G_2(j\omega) &= |G_1(j\omega)||G_2(j\omega)|e^{j\angle G_1(j\omega)}e^{j\angle G_2(j\omega)} \\ &= |G_1(j\omega)||G_2(j\omega)|e^{j\{\angle G_1(j\omega)+\angle G_2(j\omega)\}} \end{aligned}$$

Therefore,

$$20\log_{10}|G_1(j\omega)G_2(j\omega)| = 20\log_{10}|G_1(j\omega)|\cdot|G_2(j\omega)| = 20\log_{10}|G_1(j\omega)| + 20\log_{10}|G_2(j\omega)|$$

$$\angle G_1(j\omega)G_2(j\omega) = \angle G_1(j\omega) + \angle G_2(j\omega)$$

Example 1

- Sketch the Bode plot of a transfer function

$$G(s) = \frac{10}{s}$$

1. Decompose G(s) into a product form:

$$G(s) = 10 \cdot \frac{1}{s}$$

2. Sketch a Bode plot for each component on the same graph.
3. Add them all on both gain and phase plots.

Example 1 (cont'd)

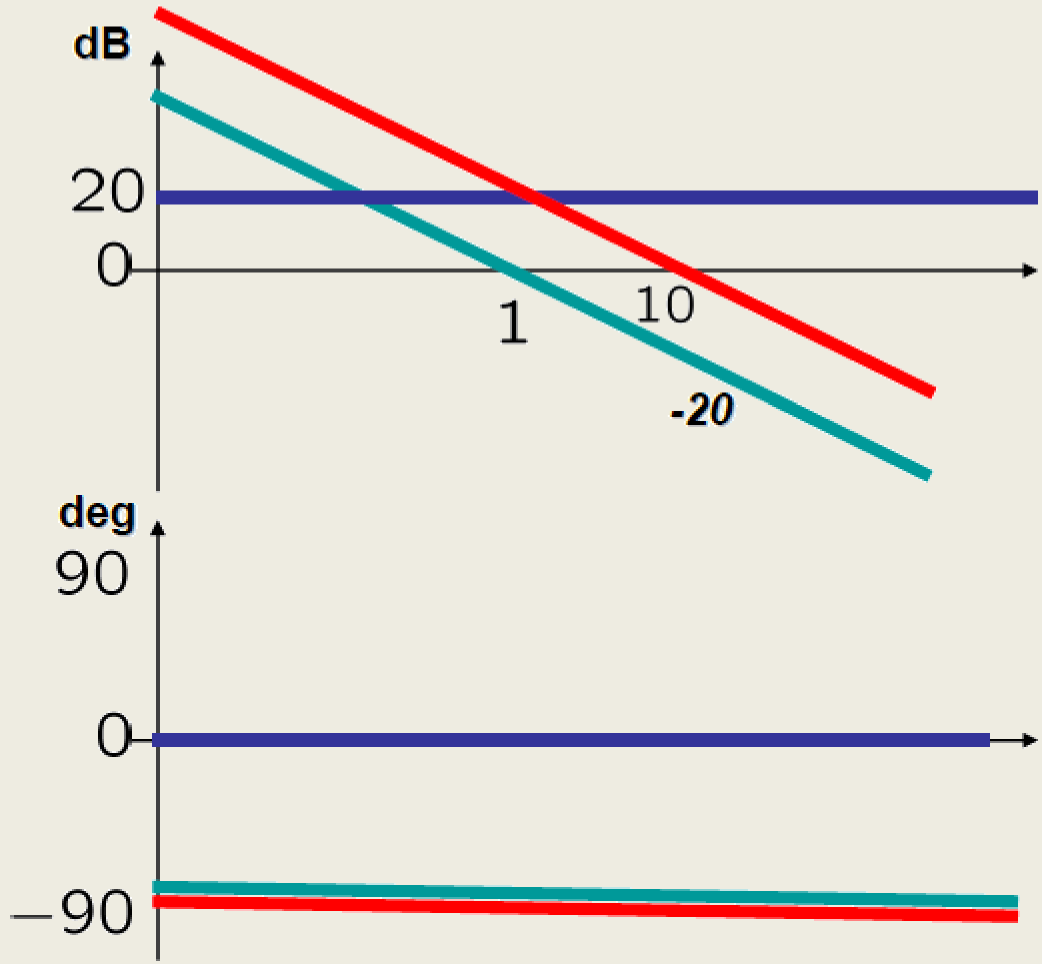
$$G(s) = 10$$

$$\times$$

$$G(s) = \frac{1}{s}$$

$$\Downarrow$$

$$G(s) = \frac{10}{s}$$



Example 2

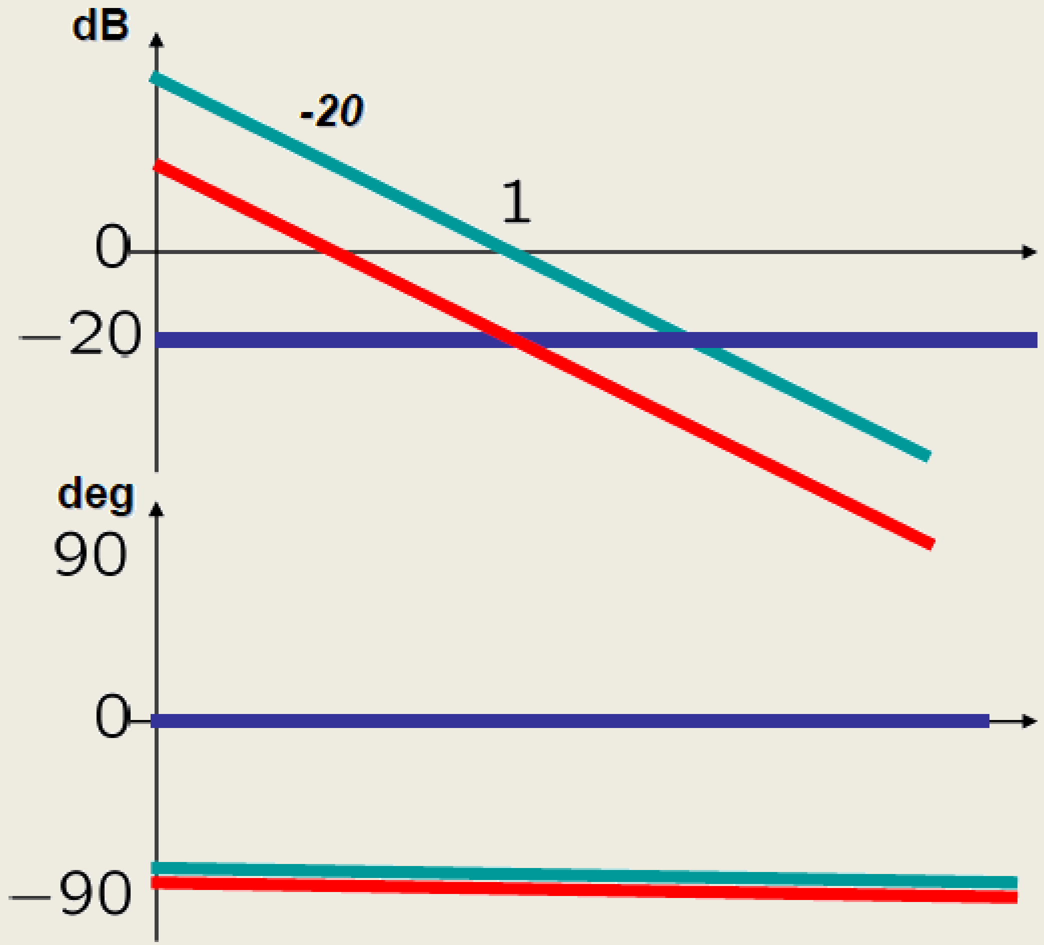
$$G(s) = 0.1$$

$$\times$$

$$G(s) = \frac{1}{s}$$

$$\Downarrow$$

$$G(s) = \frac{0.1}{s}$$



Example 3

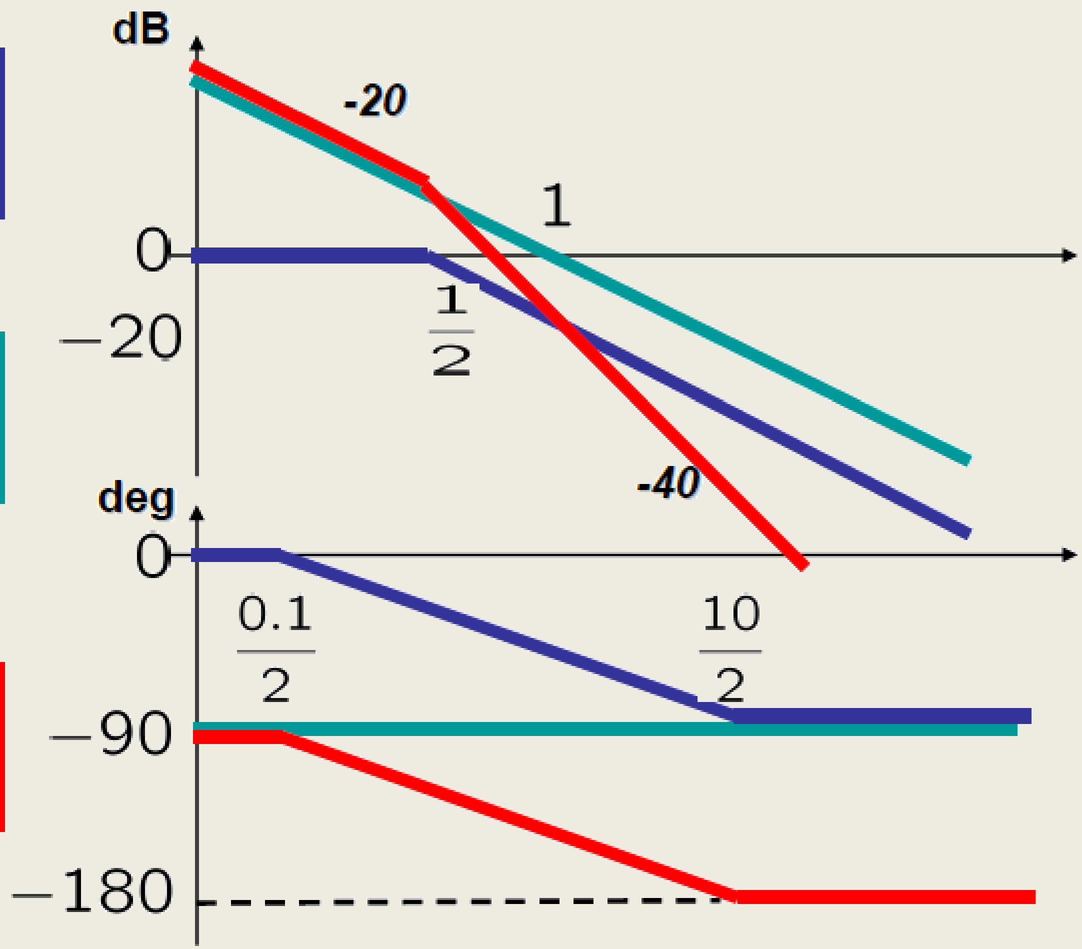
$$G(s) = \frac{1}{2s+1}$$

$\times$

$$G(s) = \frac{1}{s}$$

$\Downarrow$

$$G(s) = \frac{1}{s(2s+1)}$$



Example 4

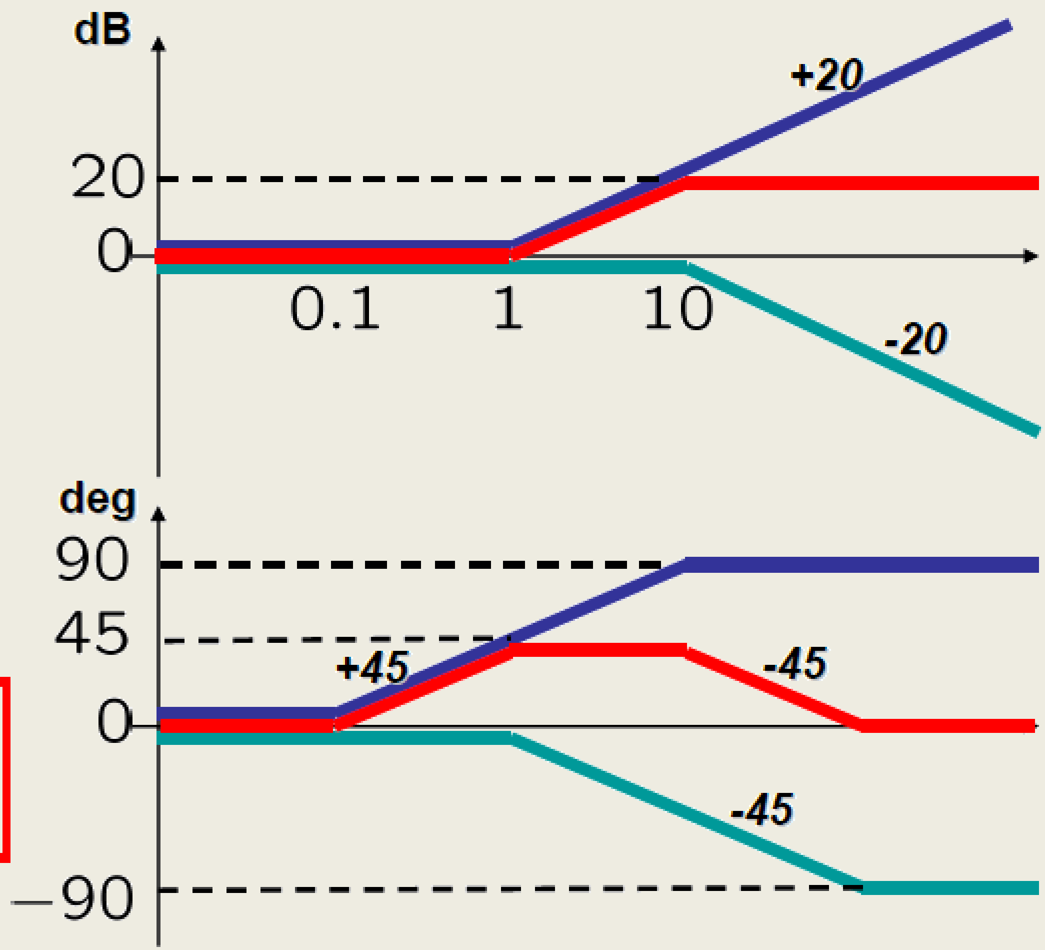
$$G(s) = s + 1$$

$$\times$$

$$G(s) = \frac{1}{0.1s + 1}$$

$$\Downarrow$$

$$G(s) = \frac{10(s+1)}{(s+10)}$$



Example 5

$$G(s) = 2$$

$$\times$$

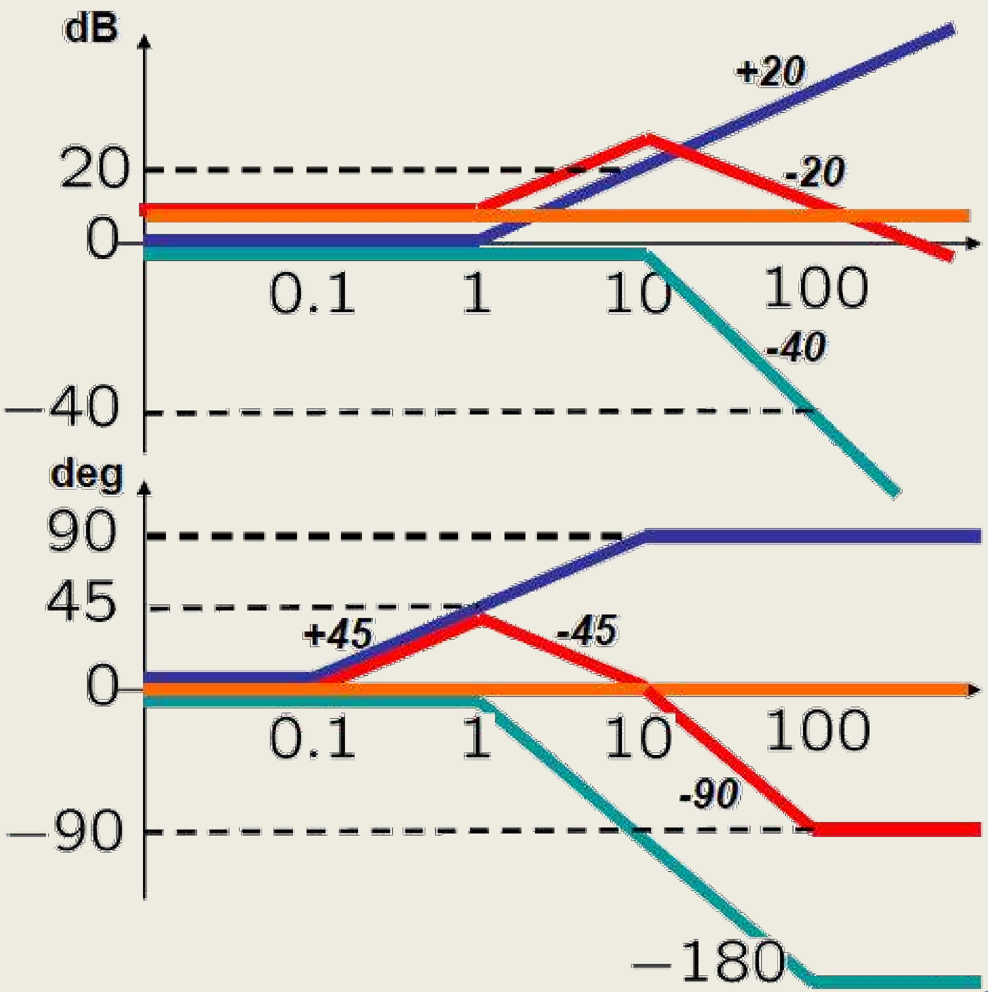
$$G(s) = s + 1$$

$$\times$$

$$G(s) = \frac{1}{(0.1s+1)^2}$$

$$\Downarrow$$

$$G(s) = \frac{200(s+1)}{(s+10)^2}$$



Remark

- Use Matlab "bode.m" to obtain precise shape.
- ALWAYS check the correctness of
  - Low frequency gain (DC gain) $G(0)$
  - High frequency gain $G(\infty)$
- Example

$$G(s) = \frac{10(s+1)}{s+5}$$

Summary

- Frequency Responce
  - Steady state response to a sinusoidal input
  - For a linear system, sinusoidal input generates sinusoidal output with same frequency but different amplitude and phase.
  - Bode plot is a graphical representation of frequency response function. (“bode.m”)
- Sketches of Bode Plot
  - Basic functions
  - Products of basic functions
- Sketching Bode Plot is …
  - To get a rough idea of the characteristics of a system.
  - To interpret the result obtained from computer.

===

- Next
  - Nyquist stability criterion

Derivation of frequency response

$$Y(s) = G(s)R(s) = G(s)\frac{A\omega}{s^2+\omega^2} = \frac{k_1}{s+j\omega} + \frac{k_2}{s-j\omega} + C_g(s)$$

Term having denominator of G(s)

$$\begin{cases} k_1 = \lim_{s\to -j\omega}(s+j\omega)G(s)\frac{A\omega}{s^2+\omega^2} = G(-j\omega)\frac{A\omega}{-2j\omega} = -\frac{AG(-j\omega)}{2j} \\ k_2 = \lim_{s\to j\omega}(s-j\omega)G(s)\frac{A\omega}{s^2+\omega^2} = G(j\omega)\frac{A\omega}{2j\omega} = \frac{AG(j\omega)}{2j} \end{cases}$$

$$\Rightarrow \quad y(t) = k_1 e^{-j\omega t} + k_2 e^{j\omega t} + \mathcal{L}^{-1}\{C_g(s)\}$$

0 as t goes to infinity.

$$\Rightarrow \quad y_{ss}(t) = A|G(j\omega)|\underbrace{\frac{e^{j(\omega t+\angle G(j\omega))} - e^{-j(\omega t+\angle G(j\omega))}}{2j}}_{\sin(\omega t+\angle G(j\omega))}$$

Assignment

1 - Draw Bode plots of the following transfer functions:

1- $G(s) = 10$

2 - $G(s) = s^3$

3 - $G(s) = \frac{10}{s+10}$

4 - $G(s) = \frac{s}{5} + 1$

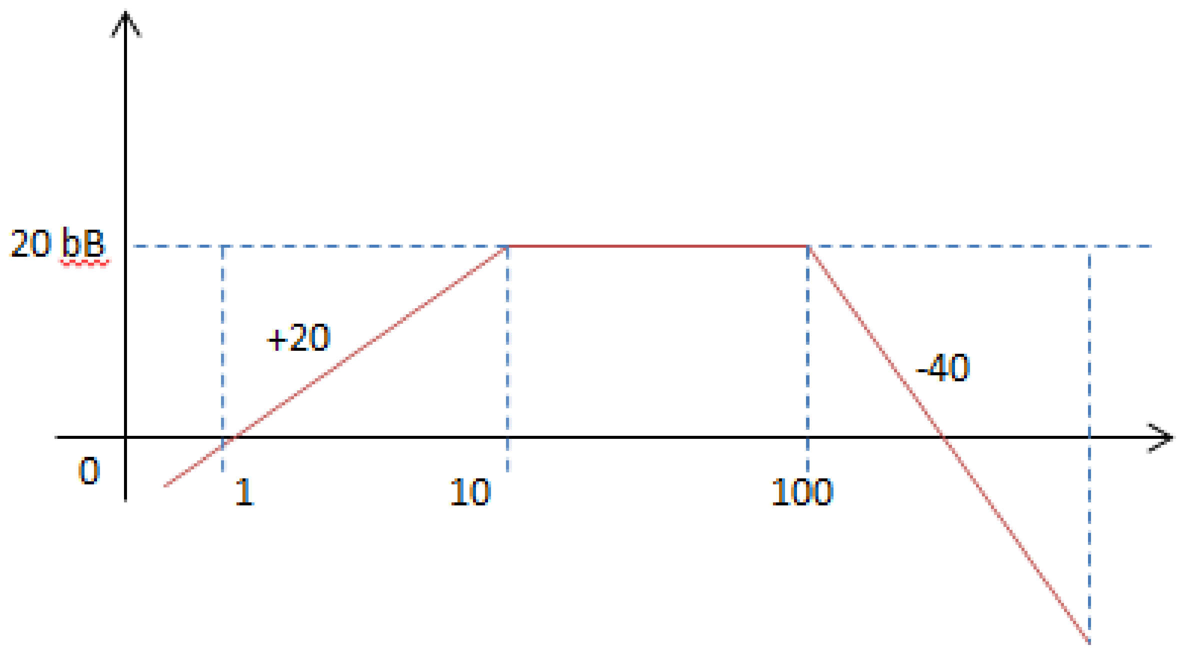
5 - $G(s) = \frac{4}{s^2+2s+4}$

6 - $G(s) = \frac{20}{s(s+1)^2}$

7 - $G(s) = \frac{s+2}{s^2}$

Assignment

2 - Find a transfer function having the gain plot below



End of Lecture 8