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Digital Signal Processing

DSP

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Lecture 6

The Discrete Fourier Transform

The spectrum of a sampled function is given by

$$X_s(\Omega) = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\Omega T}$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

where $-\pi < \omega < \pi$ or $0 < \omega < 2\pi$.

Since it *impossible* to carry-out the summation from n=-∞ to ∞, let us consider a *truncated* version of x[n]:

$$\tilde{x}[n] = \begin{cases} x[n] & (0 \leq n < N-1), \\ 0 & (\text{elsewhere}). \end{cases}$$

The corresponding Fourier transform is

$$\tilde{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \tilde{x}[n]e^{-j\omega n}$$

$$= \sum_{n=0}^{N-1} x[n]e^{-j\omega n}.$$

Now let us choose ***N*** *frequency points* from $\omega = 0$ to 2π:

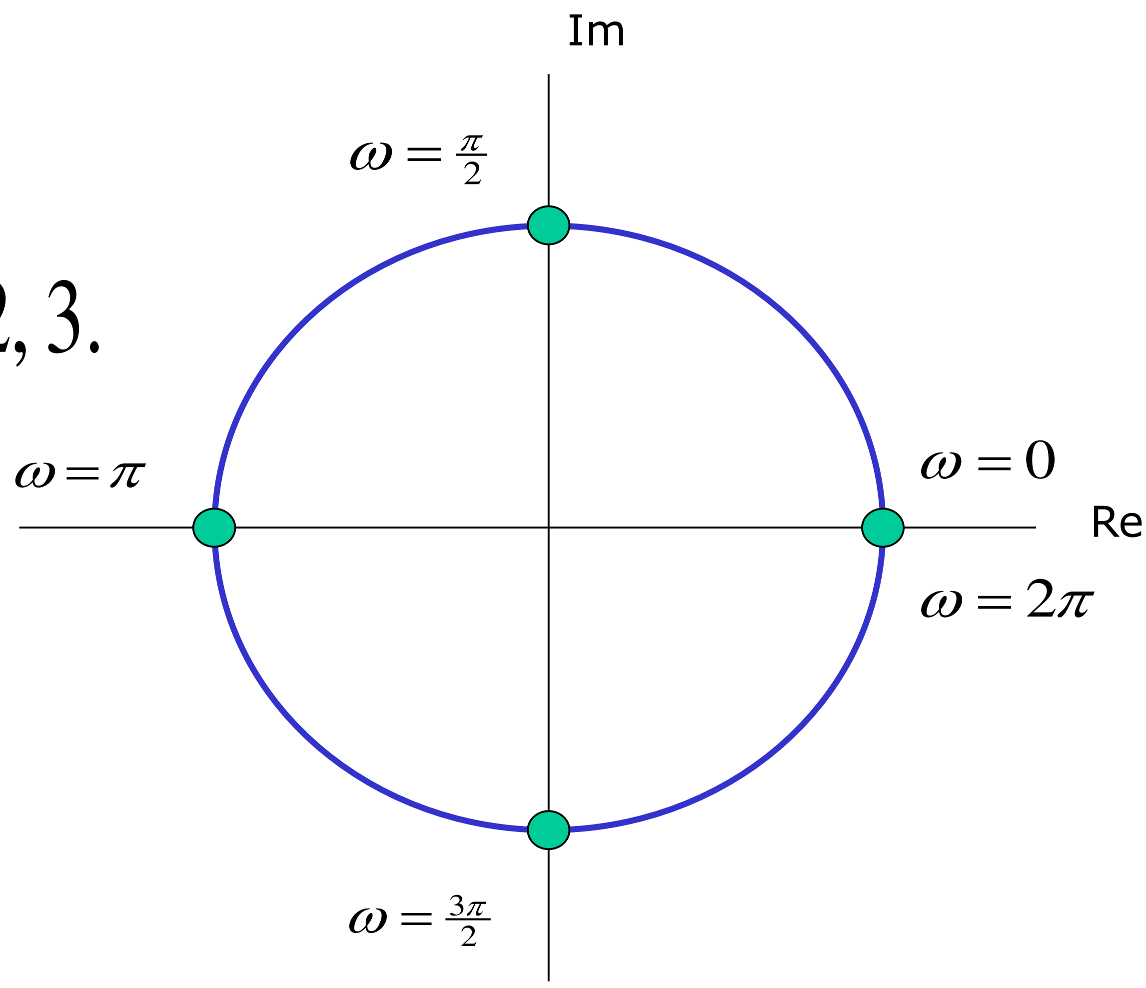
$$\omega = 2\pi \frac{k}{N} \qquad k = 0, 1, \ldots, N-1.$$

When plugged into $e^{j\omega}$, these values correspond to points along the unit circle.

As an example, if N = 4, we have

$$\omega = 2\pi \frac{k}{4} \quad k = 0, 1, 2, 3.$$

$$\omega = 0, \tfrac{\pi}{2}, \pi, \tfrac{3\pi}{2}.$$



When we insert $\omega = 2\pi \frac{k}{N}.$ into

$$\tilde{X}(e^{j\omega}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$

we get

$$X[k] = \tilde{X}(e^{j\omega})\Big|_{\omega = 2\pi \frac{k}{N}} = \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{nk}{N}}.$$

This function of ***k*** is the definition of the discrete Fourier transform.

$$\boxed{X[k] = \mathrm{DFT}\{x[n]\} \equiv \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{nk}{N}}.}$$

Example: Find the discrete Fourier transform of x[n]=1, for N=4.

Solution: as before, we have only four time samples and four frequency samples. The values of the time samples are 1 1 1 1 (for *n*=0, 1, 2, 3). Inserting these values into the DFT definition, we have

$$X[k] = \mathrm{DFT}\{1\} = \sum_{n=0}^{3} (1) e^{-j2\pi\frac{nk}{4}}$$

$$= (1)e^{-j2\pi\frac{0k}{4}} + (1)e^{-j2\pi\frac{1k}{4}} + (1)e^{-j2\pi\frac{2k}{4}} + (1)e^{-j2\pi\frac{3k}{4}}$$

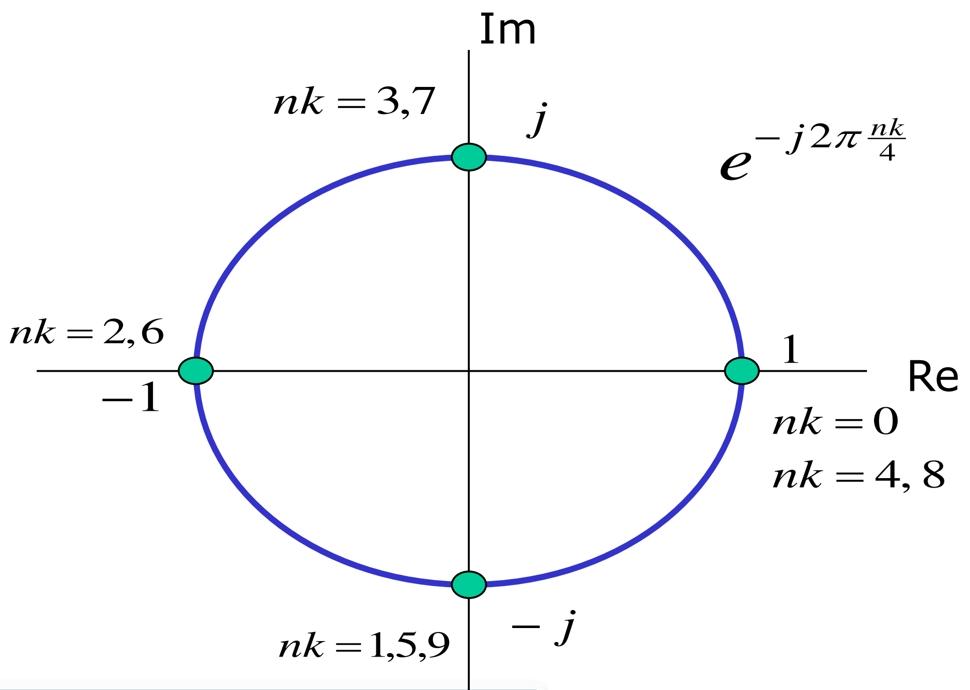
$$= 1 + e^{-j2\pi\frac{k}{4}} + e^{-j2\pi\frac{2k}{4}} + e^{-j2\pi\frac{3k}{4}}.$$

This result *is* dependent upon *k*.

$$X(0) = 1 + e^{-j2\pi\frac{0}{4}} + e^{-j2\pi\frac{2(0)}{4}} + e^{-j2\pi\frac{3(0)}{4}}$$

$$= 1 + 1 + 1 + 1 = 4.$$

To perform the rest of the calculations, it is good to have our circle of value of $e^{-j2\pi nk/4}$



$$X(1) = 1 + e^{-j2\pi\frac{1}{4}} + e^{-j2\pi\frac{2(1)}{4}} + e^{-j2\pi\frac{3(1)}{4}}$$
$$= 1 - j + (-1) + j = 0.$$

$$X(2) = 1 + e^{-j2\pi\frac{2}{4}} + e^{-j2\pi\frac{2(2)}{4}} + e^{-j2\pi\frac{3(2)}{4}}$$
$$= 1 + (-1) + 1 + (-1) = 0.$$

$$X(3) = 1 + e^{-j2\pi\frac{3}{4}} + e^{-j2\pi\frac{2(3)}{4}} + e^{-j2\pi\frac{3(3)}{4}}$$
$$= 1 + j + (-1) - j = 0.$$

The previous example suggests that the discrete Fourier transform can be calculated using a matrix equation:

$$X(0) = 1 + e^{-j2\pi\frac{0}{4}} + e^{-j2\pi\frac{2(0)}{4}} + e^{-j2\pi\frac{3(0)}{4}}.$$

$$X(1) = 1 + e^{-j2\pi\frac{1}{4}} + e^{-j2\pi\frac{2(1)}{4}} + e^{-j2\pi\frac{3(1)}{4}}.$$

$$X(2) = 1 + e^{-j2\pi\frac{2}{4}} + e^{-j2\pi\frac{2(2)}{4}} + e^{-j2\pi\frac{3(2)}{4}}.$$

$$X(3) = 1 + e^{-j2\pi\frac{3}{4}} + e^{-j2\pi\frac{2(3)}{4}} + e^{-j2\pi\frac{3(3)}{4}}.$$

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}.$$

For the previous examples we have $x[n]=1$

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Fourier transform of a constant is an impulse in frequency domain (just a D.C. component).

Suppose that we found the DFT of x[n] = {1 −1 1 −1}.

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \\ 0 \end{bmatrix}.$$

We get the Fourier transform of a sinewave at half the sampling frequency: *k=2* (for N = 4) corresponds to $\omega=\pi$, or $\Omega=\Omega_s/2$.

In general we have

$$\boxed{\frac{k}{N} = \frac{\Omega}{\Omega_s}.}$$

Example: Suppose we sample a 2 kHz sinewave at 8000 samples/second. If we perform a 1024-point DFT, where are the spikes in the transform?

Solution: $$\frac{k}{1024} = \frac{2}{8} \Rightarrow k = 256.$$

There will *also* be a spike at k=1024-256=768.

Example: Suppose we sample a sinewave at 16000 samples/second. After performing a 2048-point DFT, we have frequency spikes at k=128 and k = 1920. Find the frequency of the sinewave.

Solution: $$\frac{128}{2048} = \frac{f}{16000} \Rightarrow f = 1000\text{Hz}.$$

Inverse of DFT

$$x[n] = \mathrm{IDFT}\{X[K]\} = (1/N)\sum_{K=0}^{N-1} X[K]e^{j2\pi\frac{nk}{N}}.$$

Using MATLAB to Calculate DFT

Example:

- Assume N=4
- x[n]=[1,2,3,4]
- n=0,…,3
- Find X[k]; k=0,…,3

```
N=4;
x=[1,2,3,4]
for k1=1:N
    X(k1)=0;
    k=k1-1;
    for n1=1:N;
        n=n1-1;
        X(k1)=X(k1)+x(n1)*exp(-j*2*pi*k*n/N);
    end
end
x
X
```

```
X =

   10.0000            -2.0000 + 2.0000i   -2.0000 - 0.0000i   -2.0000 - 2.0000i
```

$$\boxed{X[k]=\sum_{n=0}^{N-1} x[n]e^{-j2\pi\frac{nk}{N}}.}$$

DFT Properties

Linearity	$\alpha\,x[n]+\beta\,y[n] \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} \alpha\,X[k]+\beta\,Y[k]$
Time Shifting	$x[n-n_0] \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X[k]e^{-j2\pi k n_0/N}$
Frequency Shifting	$x[n]e^{j2\pi k_0 n/N} \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X[k-k_0]$
Time Reversal	$x[-n]=x[N-n] \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X[-k]=X[N-k]$
Conjugation	$x^*[n] \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X^*[-k]=X^*[N-k]$
$\vdots$	$x^*[-n]=x^*[N-n] \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X^*[k]$
Time Scaling	$z[n]=\begin{cases} x[n/m] & , \; n/m \text{ an integer} \\ 0 & , \; \text{otherwise} \end{cases}$
$\vdots$	$N \to mN \;,\; Z[k]=(1/m)X[k]$

DFT Properties

Change of Period	$N \to qN,\ q$ a positive integer
⋮	$X_q[k] = \begin{cases} X[k/q] & , \ k/q \text{ an integer} \\ 0 & , \ \text{otherwise} \end{cases}$
Multiplication - Convolution Duality	$x[n]y[n] \xleftrightarrow[N]{\mathcal{DFT}} (1/N)Y[k] \circledast X[k]$
⋮	$x[n] \circledast y[n] \xleftrightarrow[N]{\mathcal{DFT}} Y[k]X[k]$
⋮	where $x[n] \circledast y[n] = \sum_{m=\langle N \rangle} x[m]y[n-m]$
Parseval's Theorem	$\sum_{n=\langle N \rangle} \lvert x[n] \rvert^2 = \frac{1}{N} \sum_{k=\langle N \rangle} \lvert X[k] \rvert^2$

Example of DFT

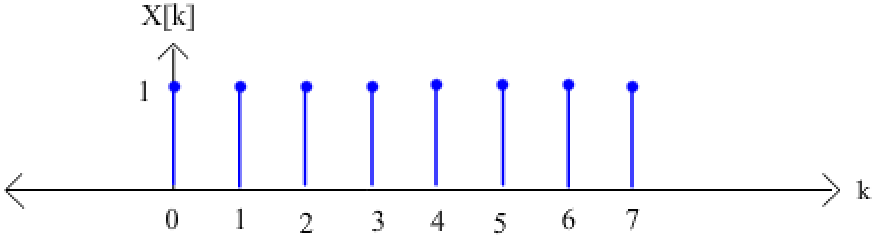
$$x[n] = \begin{cases} 1, & n = 0 \\ 0, & n = 1, \ldots, 7 \end{cases}$$

Find X[k]

$$W_N^{kn} = e^{-j\left(\frac{2\pi kn}{N}\right)}$$

- We know k=1,.., 7; N=8

$$X[k] = \sum_{n=0}^{7} x[n] W_N^{kn} = \sum_{n=0}^{7} \delta[n] W_N^{kn} = 1 \quad , \forall\, k$$



Example of DFT

Given $y[n] = \delta[n-2]$ and $N = 8$, find $Y[k]$.

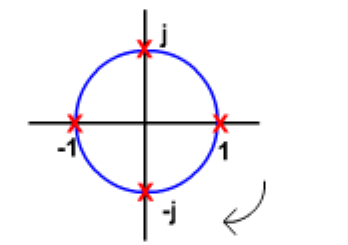
$$Y[k] = \sum_{n=0}^{7} \delta[n-2] W_N^{kn} = W_N^{2k} = e^{\frac{-j2\pi 2k}{N}} = e^{\frac{-j\pi k}{2}}$$

because N = 8

$$= (-j)^k$$

$$Y[k] = [1, -j, -1, j, 1, -j, -1, j]$$

$$x[n-n_0] \longleftrightarrow W_N^{n_0 k} X[k]$$



Polar plot for W_N^{2k}

Time shift Property of DFT

$$x[n-n_0]_{\text{mod } N} \leftrightarrow W_N^{kn_0} X[k]$$

Example of DFT

Given $x[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + \delta[n-3]$ and $N = 4$, find $X[k]$.

Summation for X[k]

$$X[k] = \sum^{3} x[n]W_4^{kn} = 1 + 2W_4^{k} + 3W_4^{2k} + W_4^{3k}$$

$$W_4 = e^{\frac{-j\pi}{2}}$$

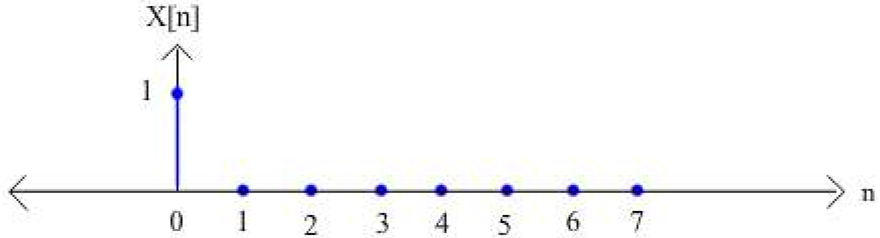
$$X[k] = 1 + 2e^{\frac{-j\pi k}{2}} + 3e^{-j\pi k} + e^{\frac{-j3\pi k}{2}}$$

Using the shift property!

Example of IDFT

Find the IDFT of $X[k] = 1, k = 0, 1, \ldots, 7.$

$$x[n] = \frac{1}{8}\sum_{n=0}^{7} W_N^{-kn} = \frac{1}{8} N\,\delta[n] = \delta[n]$$



Remember:

$$\boxed{x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-kn}, \; n = 0, 1, \ldots, N-1}$$