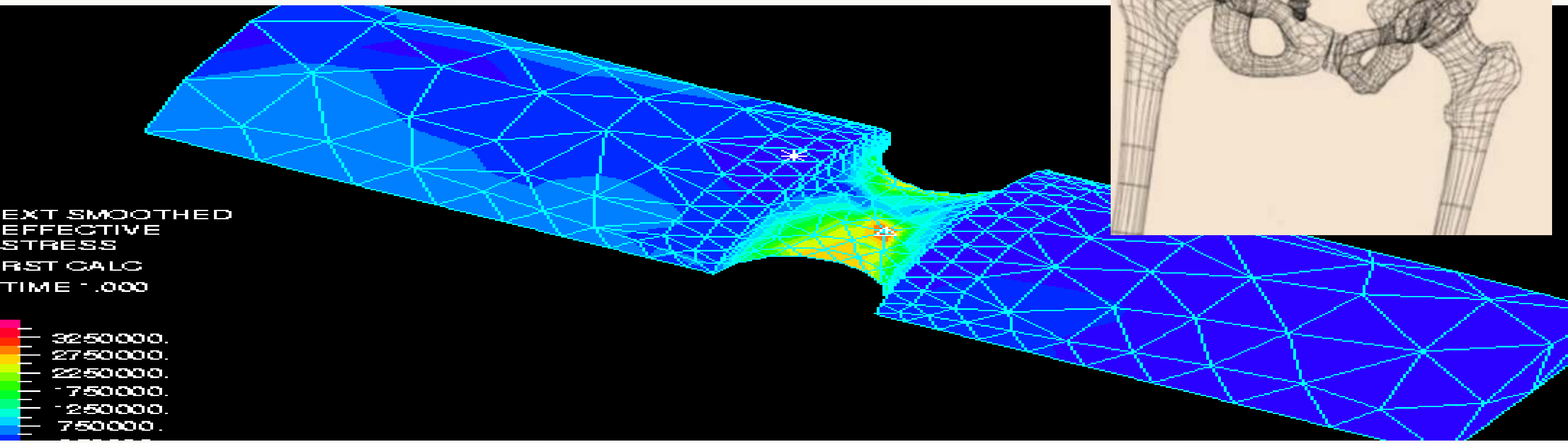
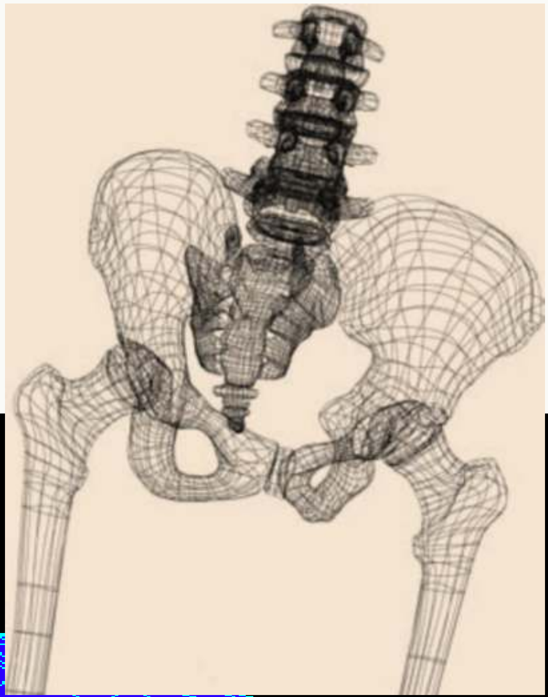
Fahad Bin Sultan University
جامعة فهد بن سلطان

# College of Engineering
# Department of Civil Engineering



## CIVE 517 : Finite Element Analysis

# Introduction to the Stiffness (Displacement) Method: Analysis of a system of springs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Fall, 2021

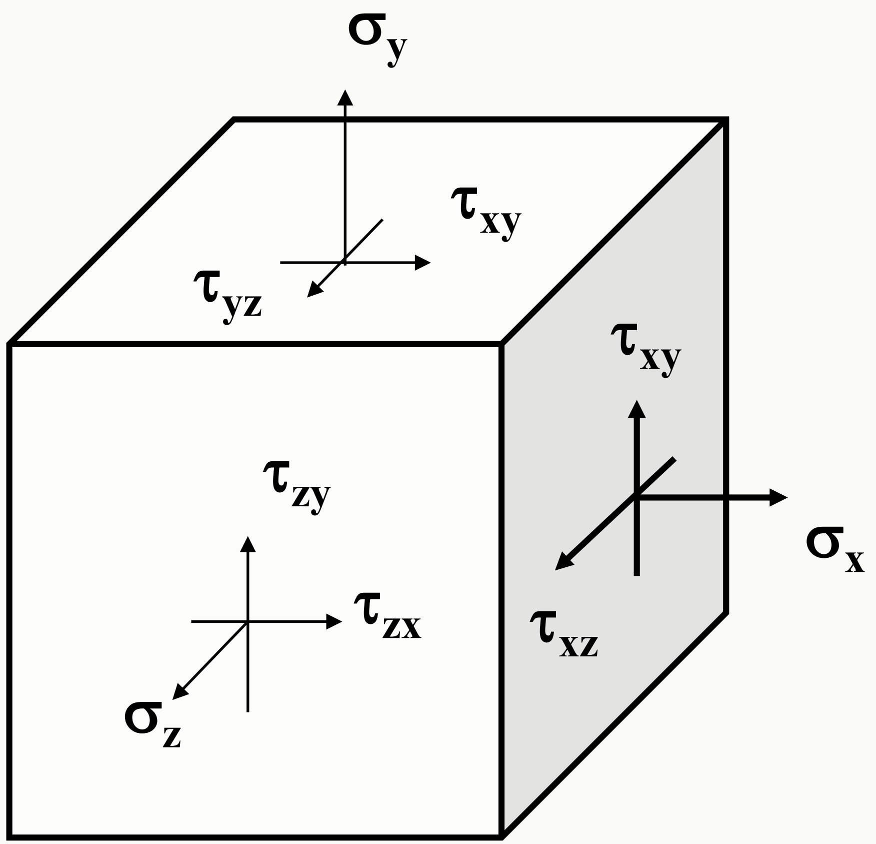
## FEM analysis scheme

**Step 1:** Divide the problem domain into non overlapping regions ("**elements**") connected to each other through special points ("**nodes**")

**Step 2:** Describe the behavior of each element

**Step 3:** Describe the behavior of the entire body by putting together the behavior of each of the elements (this is a process known as "**assembly**")

# Three-Dimensional Stress Analysis




Three-dimensional stress on an element.

# Equations of Equilibrium

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X_b = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + Y_b = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z_b = 0$$

# Strain Displacement

(u,v,w) are the x, y and z components of displacement.

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \qquad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

# Stress-Strain Relationships

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

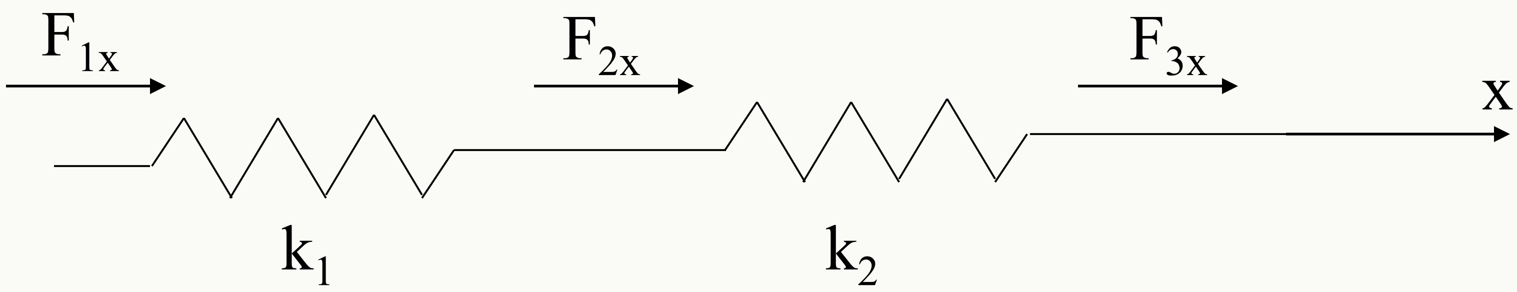
# 3D Stress-Strain Matrix

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix}$$

Note: $G = \dfrac{E}{2(1+\nu)}$

# Step 1 - Select the Element Type

1. Discretize Body into Four-Noded Tetrahedral Elements.
2. Three degrees-of-freedom per node.
3. These are x , y, and z displacements.
4. $u_i$ - x displacement at $i^{th}$ node.
5. $v_i$ - y displacement at $i^{th}$ node.
6. $w_i$ - z displacement at $i^{th}$ node.



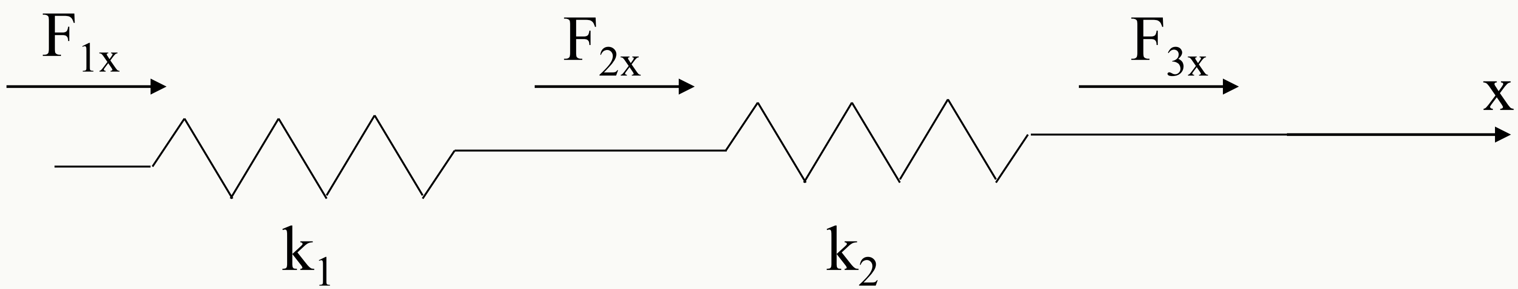

**Problem**

Analyze the behavior of the system composed of the two springs loaded by external forces as shown above

**Given**

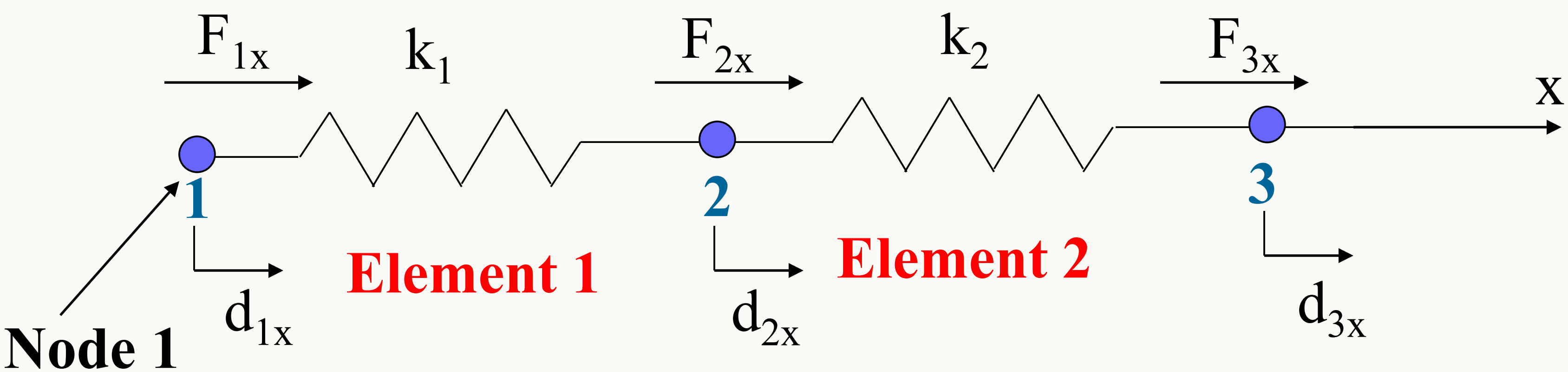
$F_{1x}$ , $F_{2x}$ , $F_{3x}$ are external loads. Positive directions of the forces are along the positive x-axis

$k_1$ and $k_2$ are the stiffnesses of the two springs

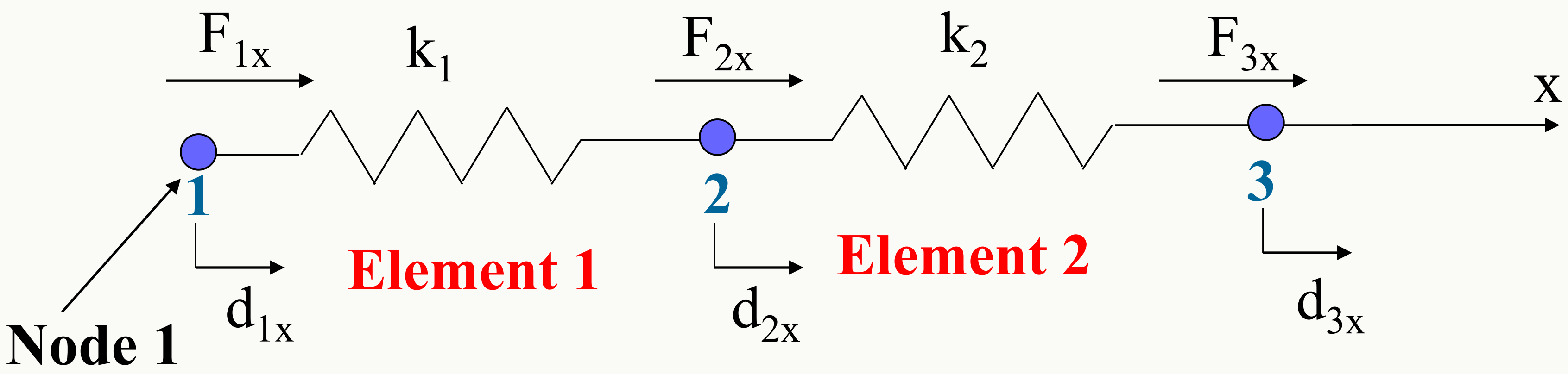



**Solution**

**Step 1:** In order to analyze the system we break it up into smaller parts, i.e., "elements" connected to each other through "nodes"

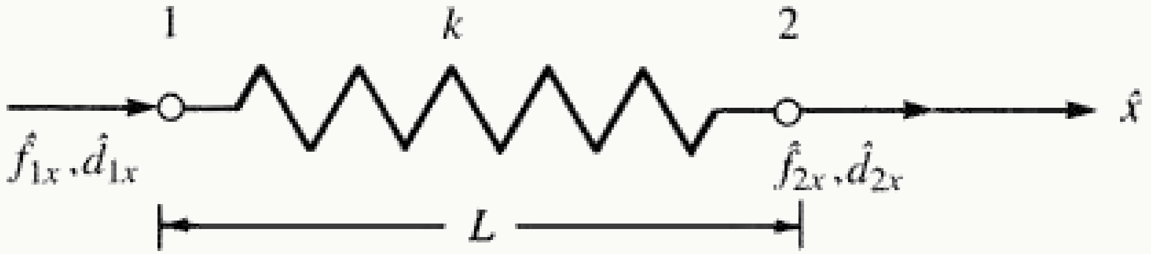



**Unknowns:** nodal displacements $d_{1x}$, $d_{2x}$, $d_{3x}$,




**Solution**

**Step 2:** Analyze the behavior of a single element (spring)

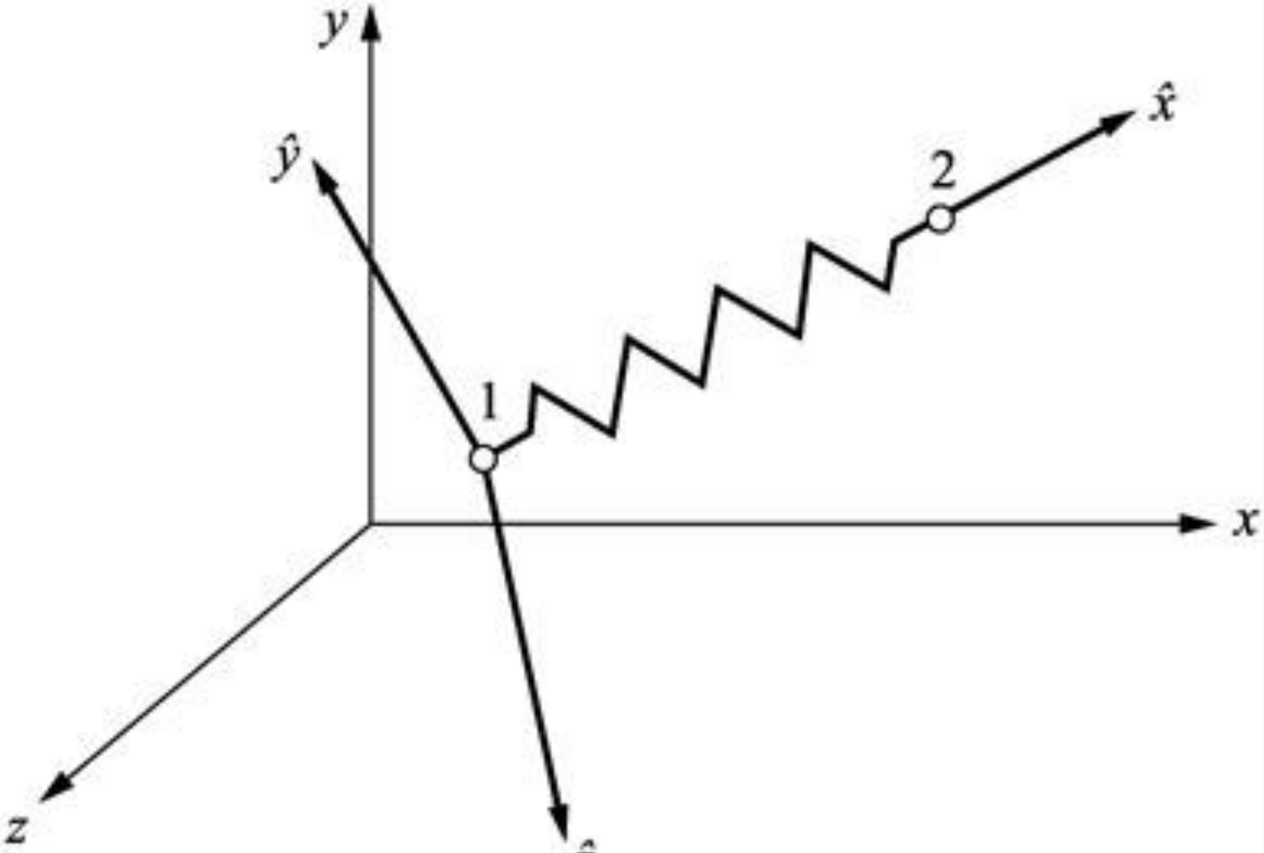



Two nodes: 1, 2

Nodal displacements: $\hat{d}_{1x}$ $\hat{d}_{2x}$

Nodal forces: $\hat{f}_{1x}$ $\hat{f}_{2x}$

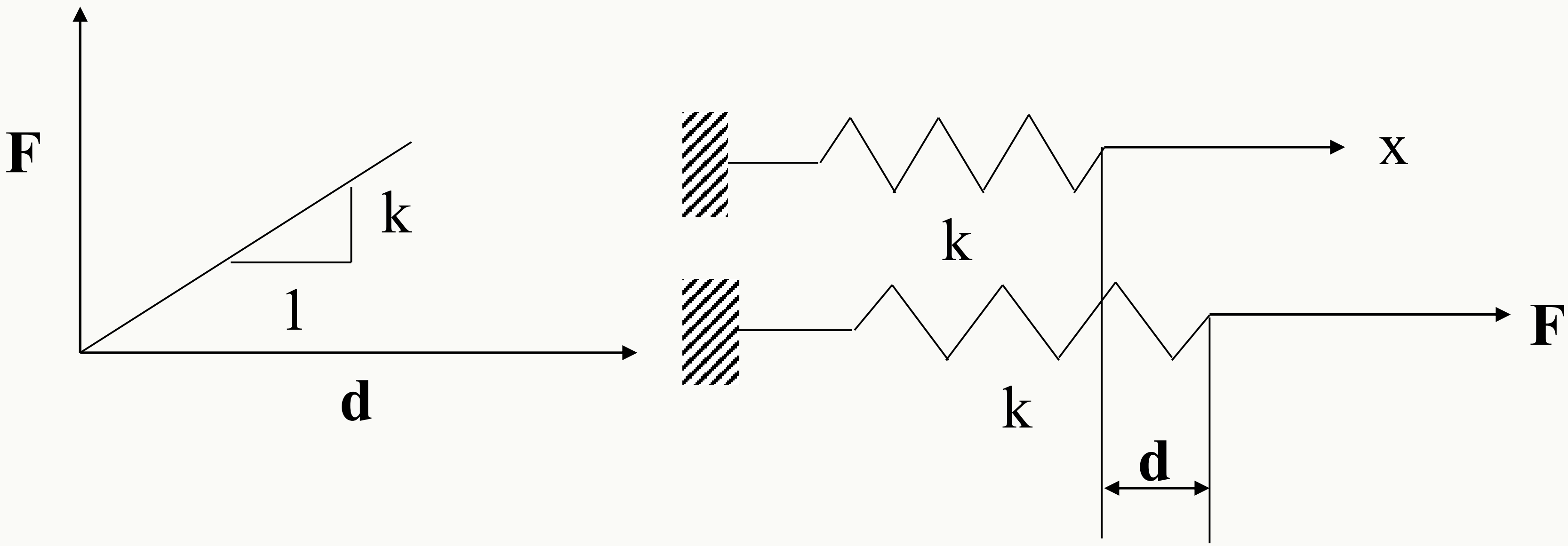
Spring constant: k





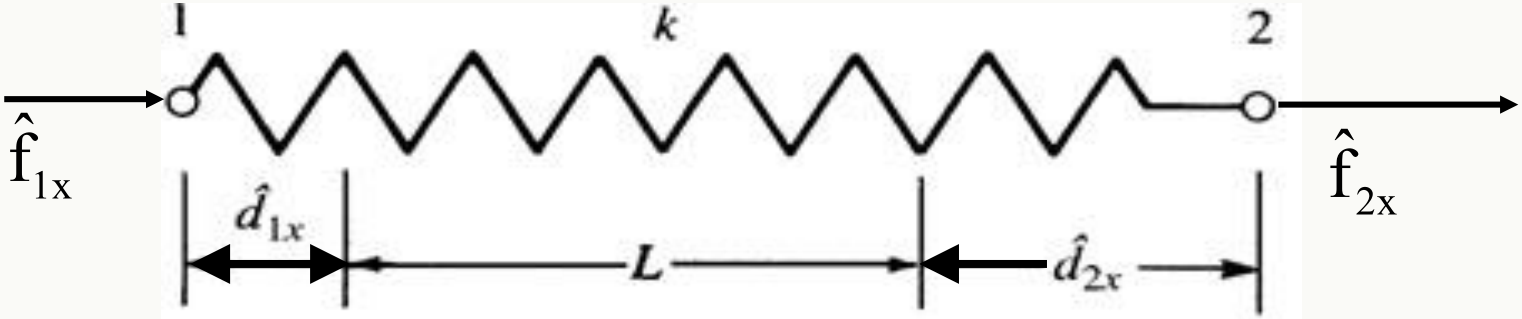
Local $(\hat{x}, \hat{y}, \hat{z})$ and global (x,y,z) coordinate systems

# Behavior of a linear spring (recap)




**F** = Force in the spring
**d** = deflection of the spring
**k** = “stiffness” of the spring

**Hooke’s Law**
**F = kd**




Hooke's law for our spring element

$$\hat{f}_{2x} = k(\hat{d}_{2x} - \hat{d}_{1x}) \qquad \textbf{Eq (1)}$$

Force equilibrium for our spring element (recap free body diagrams)

$$\hat{f}_{1x} + \hat{f}_{2x} = 0$$

$$\Rightarrow \hat{f}_{1x} = -\hat{f}_{2x} = -k(\hat{d}_{2x} - \hat{d}_{1x}) \qquad \textbf{Eq (2)}$$

Collect Eq (1) and (2) in matrix form

$$\underbrace{\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix}}_{\underline{\hat{f}}} = \underbrace{\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}}_{\underline{\hat{k}}} \underbrace{\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}}_{\underline{\hat{d}}}$$

$$\boxed{\underline{\hat{f}} = \underline{\hat{k}}\,\underline{\hat{d}}}$$

Element force vector

Element **stiffness matrix**

Element nodal displacement vector

**Note**

1. The element stiffness matrix is "**symmetric**", i.e. $\hat{\underline{k}}^{T} = \hat{\underline{k}}$
2. The element stiffness matrix is **singular**, i.e.,

$$\det(\hat{\underline{k}}) = k^2 - k^2 = 0$$

The consequence is that the matrix is NOT invertible. It is not possible to invert it to obtain the displacements. Why?

The spring is not constrained in space and hence it can attain multiple positions in space for the same nodal forces

e.g.,

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} = \begin{Bmatrix} -2 \\ 2 \end{Bmatrix}$$

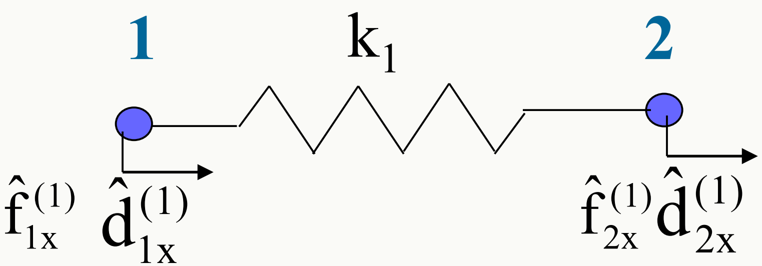
$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{Bmatrix} 3 \\ 4 \end{Bmatrix} = \begin{Bmatrix} -2 \\ 2 \end{Bmatrix}$$

**Solution**

**Step 3:** Now that we have been able to describe the behavior of each spring element, lets try to obtain the behavior of the original structure by assembly

Split the original structure into component elements

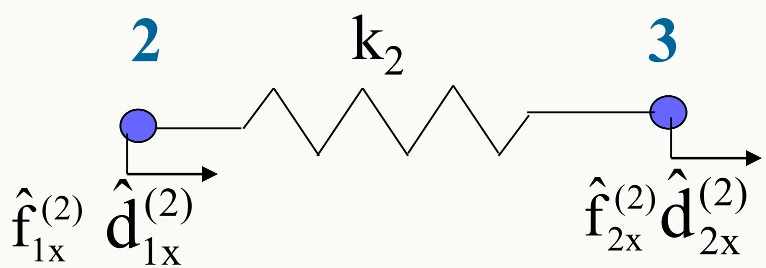
**Element 1**




$$\underbrace{\begin{Bmatrix}\hat{f}_{1x}^{(1)}\\ \hat{f}_{2x}^{(1)}\end{Bmatrix}}_{\underline{\hat{f}}^{(1)}} = \underbrace{\begin{bmatrix}k_1 & -k_1\\ -k_1 & k_1\end{bmatrix}}_{\underline{\hat{k}}^{(1)}} \underbrace{\begin{Bmatrix}\hat{d}_{1x}^{(1)}\\ \hat{d}_{2x}^{(1)}\end{Bmatrix}}_{\underline{\hat{d}}^{(1)}}$$

**Eq (3)**

**Element 2**

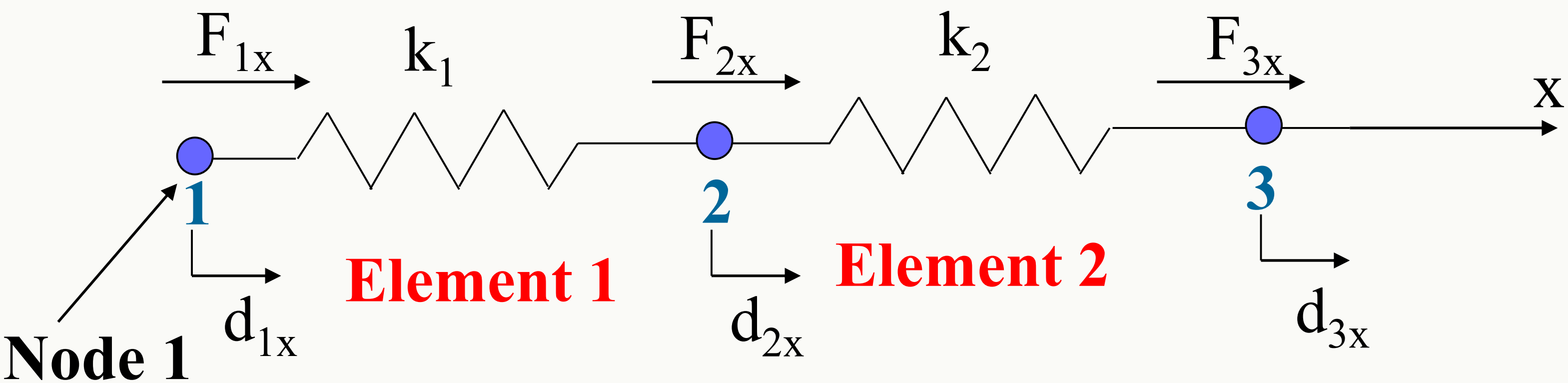



$$\underbrace{\begin{Bmatrix}\hat{f}_{1x}^{(2)}\\ \hat{f}_{2x}^{(2)}\end{Bmatrix}}_{\underline{\hat{f}}^{(2)}} = \underbrace{\begin{bmatrix}k_2 & -k_2\\ -k_2 & k_2\end{bmatrix}}_{\underline{\hat{k}}^{(2)}} \underbrace{\begin{Bmatrix}\hat{d}_{1x}^{(2)}\\ \hat{d}_{2x}^{(2)}\end{Bmatrix}}_{\underline{\hat{d}}^{(2)}}$$

**Eq (4)**

To assemble these two results into a single description of the response of the entire structure we need to link between the **local** and **global** variables.

**Question 1:** How do we relate the **local** (element) **displacements** back to the **global (**structure) displacements?




$$\boxed{\begin{aligned} \hat{d}_{1x}^{(1)} &= d_{1x} \\ \hat{d}_{2x}^{(1)} &= \hat{d}_{1x}^{(2)} = d_{2x} \\ \hat{d}_{2x}^{(2)} &= d_{3x} \end{aligned}}$$

**Eq (5)**

Hence, equations (3) and (4) may be rewritten as

$$\begin{Bmatrix} \hat{f}_{1x}^{(1)} \\ \hat{f}_{2x}^{(1)} \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix} \qquad \begin{Bmatrix} \hat{f}_{1x}^{(2)} \\ \hat{f}_{2x}^{(2)} \end{Bmatrix} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} d_{2x} \\ d_{3x} \end{Bmatrix}$$

Or, we may **expand** the matrices and vectors to obtain

$$\underbrace{\begin{Bmatrix} \hat{f}_{1x}^{(1)} \\ \hat{f}_{2x}^{(1)} \\ 0 \end{Bmatrix}}_{\underline{\hat{f}}^{(1)e}} = \underbrace{\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\underline{\hat{k}}^{(1)e}} \underbrace{\begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \end{Bmatrix}}_{\underline{d}} \qquad \underbrace{\begin{Bmatrix} 0 \\ \hat{f}_{1x}^{(2)} \\ \hat{f}_{2x}^{(2)} \end{Bmatrix}}_{\underline{\hat{f}}^{(2)e}} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}}_{\underline{\hat{k}}^{(2)e}} \underbrace{\begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \end{Bmatrix}}_{\underline{d}}$$

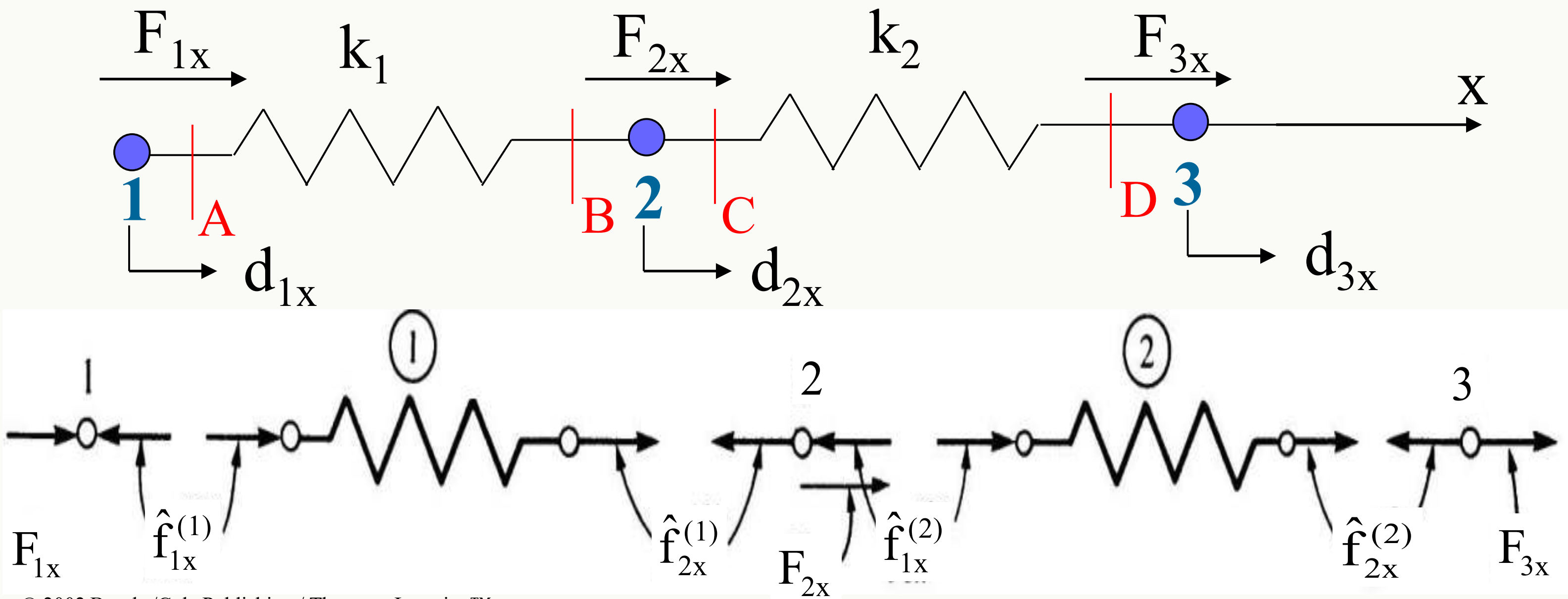
**Eq (6)** **Eq (7)**

$\underline{\hat{k}}^{(1)e}$ Expanded element stiffness matrix of element 1 (local)

$\underline{\hat{f}}^{(1)e}$ Expanded nodal force vector for element 1 (local)

$\underline{d}$ Nodal load vector for the entire structure (global)

**Question 2:** How do we relate the **local** (element) **nodal forces** back to the **global (**structure) forces? Draw 5 FBDs






$$\text{At node 1:} \quad F_{1x} - \hat{f}_{1x}^{(1)} = 0$$

$$\text{At node 2:} \quad F_{2x} - \hat{f}_{2x}^{(1)} - \hat{f}_{1x}^{(2)} = 0$$

$$\text{At node 3:} \quad F_{3x} - \hat{f}_{2x}^{(2)} = 0$$

In vector form, the nodal force vector (global)

$$\underline{F} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix} = \begin{Bmatrix} \hat{f}_{1x}^{(1)} \\ \hat{f}_{2x}^{(1)} + \hat{f}_{1x}^{(2)} \\ \hat{f}_{2x}^{(2)} \end{Bmatrix}$$

Recall that the expanded element force vectors were

$$\underline{\hat{f}}^{(1)e} = \begin{Bmatrix} \hat{f}_{1x}^{(1)} \\ \hat{f}_{2x}^{(1)} \\ 0 \end{Bmatrix} \text{ and } \underline{\hat{f}}^{(2)e} = \begin{Bmatrix} 0 \\ \hat{f}_{1x}^{(2)} \\ \hat{f}_{2x}^{(2)} \end{Bmatrix}$$

Hence, the global force vector is simply the sum of the **expanded** element nodal force vectors

$$\underline{F} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix} = \underline{\hat{f}}^{(1)e} + \underline{\hat{f}}^{(2)e}$$

But we know the expressions for the expanded local force vectors from Eqs (6) and (7)

$$\underline{\hat{f}}^{(1)e} = \underline{\hat{k}}^{(1)e}\underline{d} \quad \text{and} \quad \underline{\hat{f}}^{(2)e} = \underline{\hat{k}}^{(2)e}\underline{d}$$

Hence

$$\underline{F} = \underline{\hat{f}}^{(1)e} + \underline{\hat{f}}^{(2)e} = \underline{\hat{k}}^{(1)e}\underline{d} + \underline{\hat{k}}^{(2)e}\underline{d} = \left(\underline{\hat{k}}^{(1)e} + \underline{\hat{k}}^{(2)e}\right)\underline{d}$$

$$\boxed{\underline{F} = \underline{K}\,\underline{d}}$$

$\underline{F} =$ Global nodal force vector

$\underline{d} =$ Global nodal displacement vector

$\underline{K} =$ Global stiffness matrix

$=$ sum of expanded element stiffness matrices

For our original structure with two springs, the **global stiffness matrix** is

$$\underline{K} = \underbrace{\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\underline{\hat{k}}^{(1)e}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}}_{\underline{\hat{k}}^{(2)e}}$$

$$= \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}$$

**NOTE**

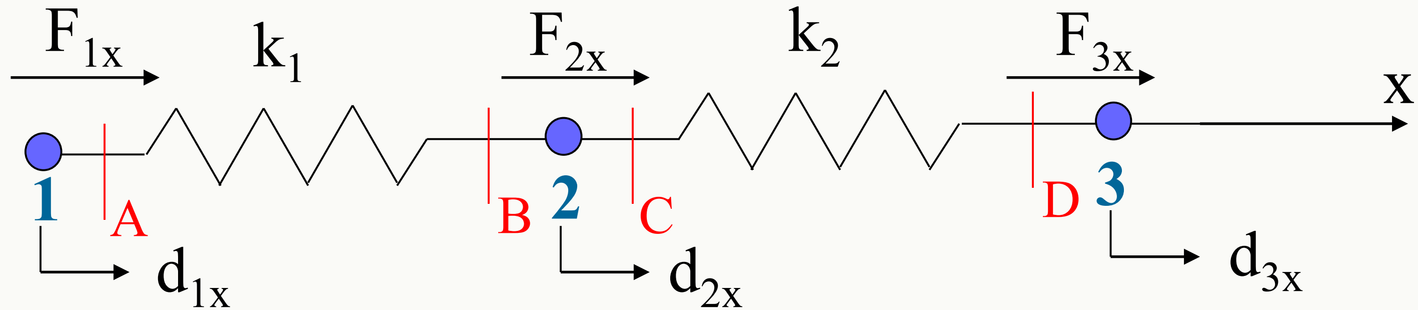
1. The global stiffness matrix is **symmetric**
2. The global stiffness matrix is **singular**

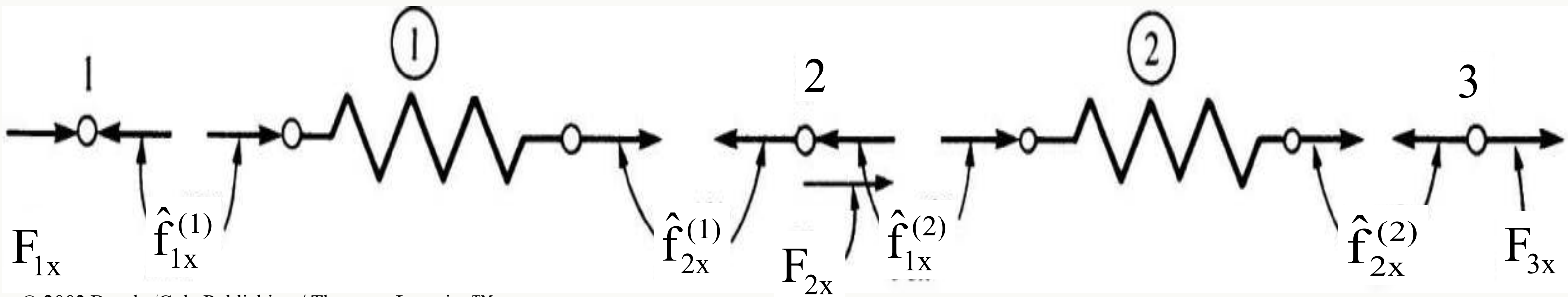
The system equations $\boxed{\underline{F} = \underline{K}\,\underline{d}}$ imply

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \end{Bmatrix}$$

$$\begin{aligned} & F_{1x} = k_1 d_{1x} - k_1 d_{2x} \\ \Rightarrow\ & F_{2x} = -k_1 d_{1x} + (k_1 + k_2) d_{2x} - k_2 d_{3x} \\ & F_{3x} = -k_2 d_{2x} + k_2 d_{3x} \end{aligned}$$

These are the 3 equilibrium equations at the 3 nodes.






At node 1: $F_{1x} - \hat{f}_{1x}^{(1)} = 0$

At node 2: $F_{2x} - \hat{f}_{2x}^{(1)} - \hat{f}_{1x}^{(2)} = 0$

At node 3: $F_{3x} - \hat{f}_{2x}^{(2)} = 0$

$$F_{1x} = k_1\left(d_{1x} - d_{2x}\right) = \hat{f}_{1x}^{(1)}$$

$$\begin{aligned} F_{2x} &= -k_1 d_{1x} + (k_1 + k_2)d_{2x} - k_2 d_{3x} \\ &= -k_1\left(d_{1x} - d_{2x}\right) + k_2\left(d_{2x} - d_{3x}\right) \\ &= \hat{f}_{2x}^{(1)} + \hat{f}_{1x}^{(2)} \end{aligned}$$

$$F_{3x} = -k_2\left(d_{2x} - d_{3x}\right) = \hat{f}_{2x}^{(2)}$$

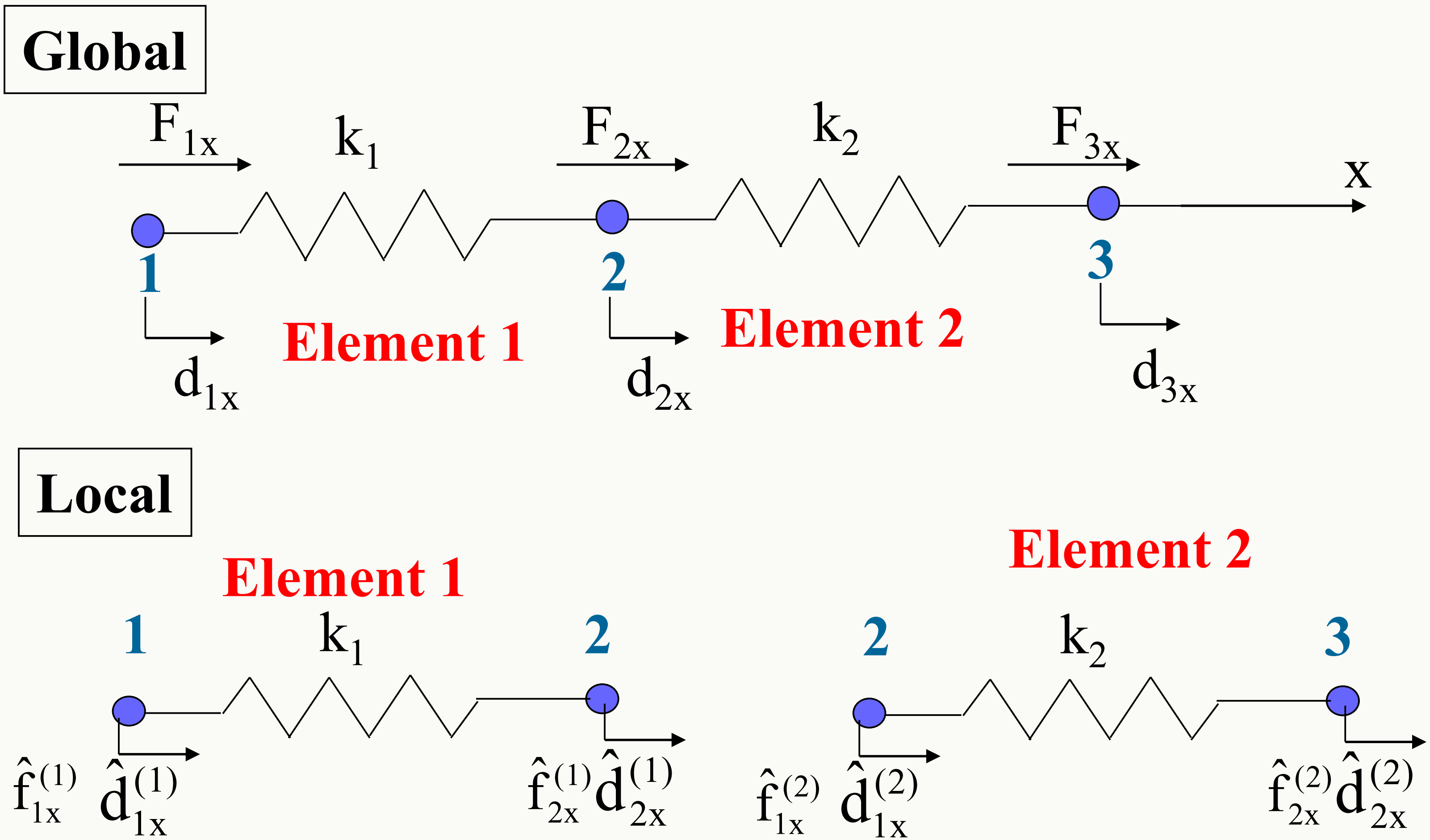
Notice that the sum of the forces equal zero, i.e., the structure is in static equilibrium.

$F_{1x} + F_{2x} + F_{3x} = 0$

Given the nodal forces, can we solve for the displacements?

To obtain unique values of the displacements, **at least one of the nodal displacements must be specified**.

# Direct assembly of the global stiffness matrix



**Node element connectivity chart :** Specifies the global node number corresponding to the local (element) node numbers

| ELEMENT | Node 1 | Node 2 |
|---|---|---|
| 1 | 1 | 2 |
| 2 | 2 | 3 |

Local node number

Global node number

## Stiffness matrix of element 1

$$\hat{\underline{k}}^{(1)} = \begin{array}{c} \begin{array}{cc} d_{1x} & d_{2x} \end{array} \\ \left[\begin{array}{c|c} k_1 & -k_1 \\ \hline -k_1 & k_1 \end{array}\right] \end{array} \begin{array}{c} d_{1x} \\ d_{2x} \end{array}$$

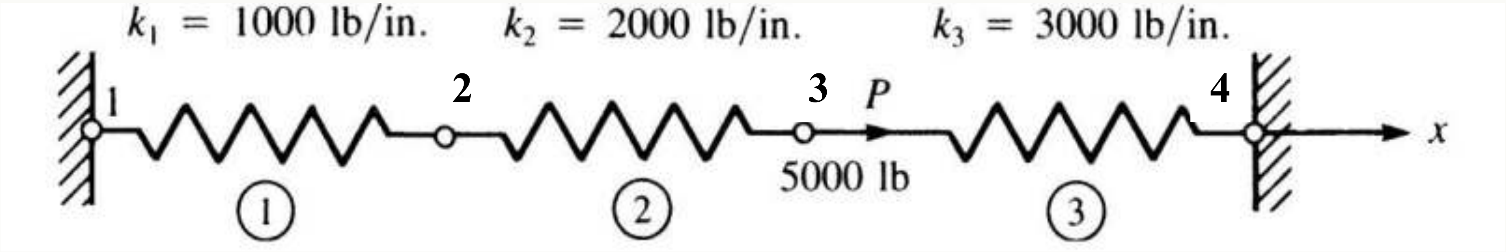
## Stiffness matrix of element 2

$$\hat{\underline{k}}^{(2)} = \begin{array}{c} \begin{array}{cc} d_{2x} & d_{3x} \end{array} \\ \left[\begin{array}{c|c} k_2 & -k_2 \\ \hline -k_2 & k_2 \end{array}\right] \end{array} \begin{array}{c} d_{2x} \\ d_{3x} \end{array}$$

## Global stiffness matrix

$$\underline{K} = \begin{array}{c} \begin{array}{ccc} d_{1x} & d_{2x} & d_{3x} \end{array} \\ \left[\begin{array}{c|c|c} k_1 & -k_1 & 0 \\ \hline -k_1 & k_1 + k_2 & -k_2 \\ \hline 0 & -k_2 & k_2 \end{array}\right] \end{array} \begin{array}{c} d_{1x} \\ d_{2x} \\ d_{3x} \end{array}$$

# Example 1

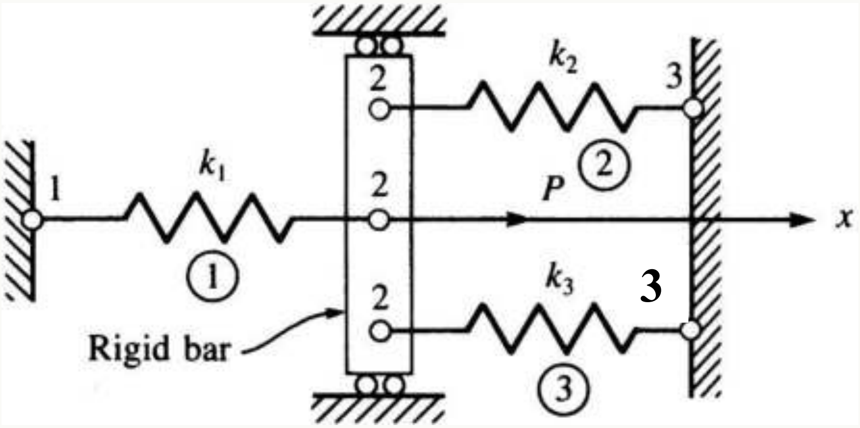





**Compute the global stiffness matrix of the assemblage of springs shown above**

$$\underline{K} = \begin{array}{c} \begin{array}{cccc} d_{1x} & d_{2x} & d_{3x} & d_{4x} \end{array} \\ \begin{bmatrix} 1000 & -1000 & 0 & 0 \\ -1000 & (1000+2000) & -2000 & 0 \\ 0 & -2000 & (2000+3000) & -3000 \\ 0 & 0 & -3000 & 3000 \end{bmatrix} \end{array} \begin{array}{c} d_{1x} \\ d_{2x} \\ d_{3x} \\ d_{4x} \end{array}$$

# Example 2






**Compute the global stiffness matrix of the assemblage of springs shown above**

$$\underline{K} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 + k_3 & -(k_2 + k_3) \\ 0 & -(k_2 + k_3) & (k_2 + k_3) \end{bmatrix}$$

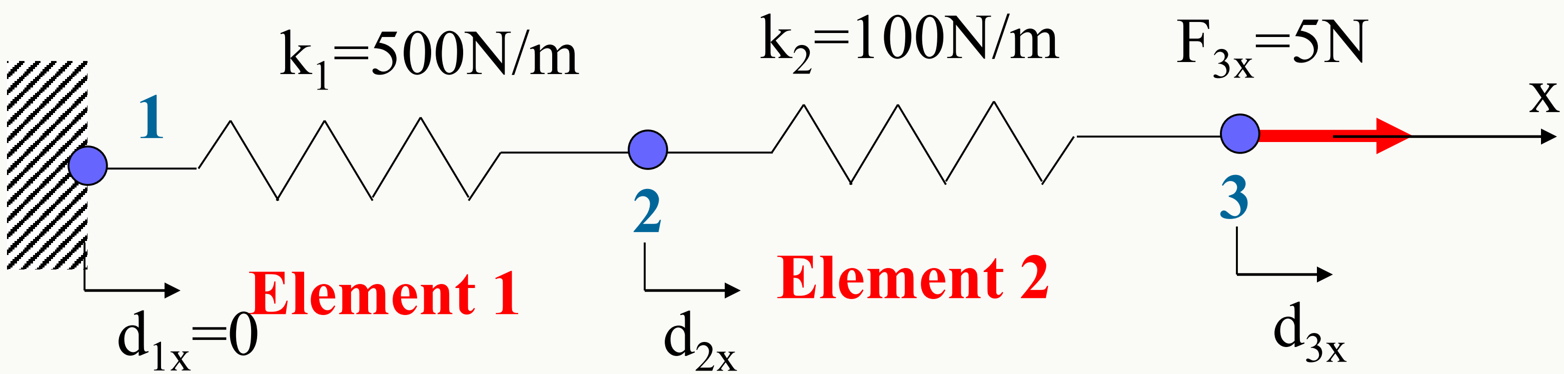
# Imposition of boundary conditions

Consider 2 cases

Case 1: **Homogeneous** boundary conditions (e.g., $d_{1x}=0$)

Case 2: **Nonhomogeneous** boundary conditions (e.g., one of the nodal displacements is known to be different from zero)

## Homogeneous boundary condition at node 1



**System equations**

0

$$\begin{bmatrix} 500 & -500 & 0 \\ -500 & 600 & -100 \\ 0 & -100 & 100 \end{bmatrix} \begin{bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \end{bmatrix} = \begin{bmatrix} F_{1x} \\ 0 \\ 5 \end{bmatrix}$$

Global Stiffness matrix

Nodal disp vector

Nodal load vector

Note that $F_{1x}$ is the wall reaction which is to be computed as part of the solution and hence is an unknown in the above equation

Writing out the equations explicitly

$$-500d_{2x} = F_{1x} \qquad \text{Eq(1)}$$

$$600d_{2x} - 100d_{3x} = 0 \qquad \text{Eq(2)}$$

$$-100d_{2x} + 100d_{3x} = 5 \qquad \text{Eq(3)}$$

**Eq(2) and (3) are used to find $d_{2x}$ and $d_{3x}$ by solving**

$$\begin{bmatrix} 600 & -100 \\ -100 & 100 \end{bmatrix} \begin{bmatrix} d_{2x} \\ d_{3x} \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} d_{2x} \\ d_{3x} \end{bmatrix} = \begin{bmatrix} 0.01\, m \\ 0.06\, m \end{bmatrix}$$

**NOTICE:** The matrix in the above equation may be obtained from the global stiffness matrix **by deleting the first row and column**

$$\begin{bmatrix} 500 & -500 & 0 \\ -500 & 600 & -100 \\ 0 & -100 & 100 \end{bmatrix} \qquad \begin{bmatrix} 600 & -100 \\ -100 & 100 \end{bmatrix}$$

Note use Eq(1) to compute $F_{1x} = -500 d_{2x} = -5N$

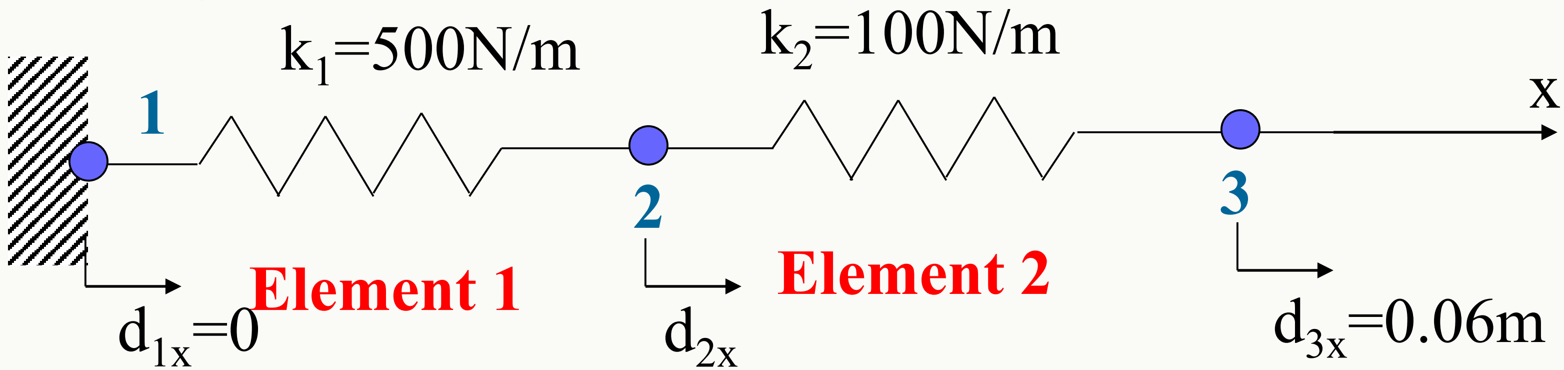
**NOTICE:**

1. Take care of **homogeneous** boundary conditions **by deleting the appropriate rows and columns** from the global stiffness matrix and solving the reduced set of equations for the unknown nodal displacements.

2. Both displacements and forces CANNOT be known at the same node. If the displacement at a node is known, the reaction force at that node is unknown (and vice versa)

# Imposition of boundary conditions…contd.

**Nonhomogeneous boundary condition: spring 2 is pulled at node 3 by 0.06 m)**



**System equations**

$$\begin{bmatrix} 500 & -500 & 0 \\ -500 & 600 & -100 \\ 0 & -100 & 100 \end{bmatrix} \begin{bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \end{bmatrix} = \begin{bmatrix} F_{1x} \\ 0 \\ F_{3x} \end{bmatrix}$$

$$d_{1x} \to 0 \qquad d_{3x} \to 0.06$$

Note that now $F_{1x}$ and $F_{3x}$ are not known.

Writing out the equations explicitly

$$-500 d_{2x} = F_{1x} \qquad \text{Eq(1)}$$

$$600 d_{2x} - 100(0.06) = 0 \qquad \text{Eq(2)}$$

$$-100 d_{2x} + 100(0.06) = F_{3x} \qquad \text{Eq(3)}$$

Now use only equation (2) to compute $d_{2x}$

$$600d_{2x} = 100(0.06)$$
$$\Rightarrow d_{2x} = 0.01m$$

Now use Eq(1) and (3) to compute $F_{1x}$ =-5N and $F_{3x}$=5N

**Recap of what we did**

**Step 1:** Divide the problem domain into non overlapping regions ("**elements**") connected to each other through special points ("**nodes**")

**Step 2:** Describe the behavior of each element ( $\underline{\hat{f}} = \underline{\hat{k}}\ \underline{\hat{d}}$ ) Element nodal displacement vector

**Step 3:** Describe the behavior of the entire body (by "**assembly**").

This consists of the following steps

1. Write the force-displacement relations of each spring in **expanded** form

$$\underline{\hat{f}}^{e} = \underline{\hat{k}^{e}}\ \underline{\hat{d}}$$

Global nodal displacement vector

## Recap of what we did…contd.

2. Relate the local forces of each element to the global forces at the nodes (use FBDs and force equilibrium).

$$\underline{F} = \sum \underline{\hat{f}}^{e}$$

Finally obtain

$$\boxed{\underline{F} = \underline{K}\ \underline{d}}$$

Where the **global stiffness matrix**

$$\underline{K} = \sum \underline{k}^{e}$$

**Recap of what we did…contd.**

Apply boundary conditions by partitioning the matrix and vectors

$$\begin{bmatrix} \underline{K}_{11} & \underline{K}_{12} \\ \underline{K}_{21} & \underline{K}_{22} \end{bmatrix} \begin{Bmatrix} \underline{d}_1 \\ \underline{d}_2 \end{Bmatrix} = \begin{Bmatrix} \underline{F}_1 \\ \underline{F}_2 \end{Bmatrix}$$

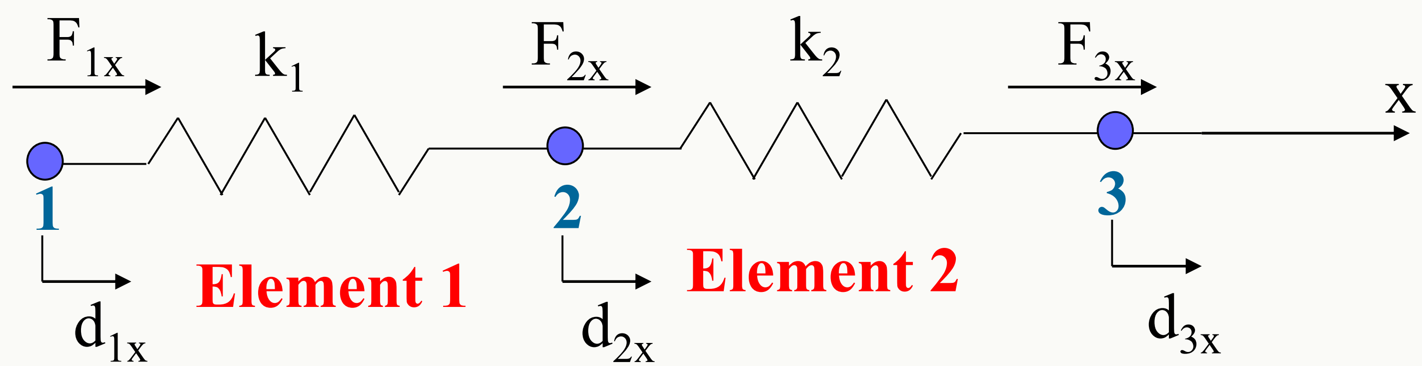
Solve for unknown nodal displacements

$$\underline{K}_{22}\underline{d}_2 = \underline{F}_2 - \underline{K}_{21}\underline{d}_1$$

Compute unknown nodal forces

$$\underline{F}_1 = \underline{K}_{11}\underline{d}_1 + \underline{K}_{12}\underline{d}_2$$

# Physical significance of the stiffness matrix




In general, we will have a stiffness matrix of the form (assume for now that we do not know $k_{11}$, $k_{12}$, etc)

$$\underline{K} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}$$

The finite element force-displacement relations:

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

## Physical significance of the stiffness matrix

The first equation is

$$k_{11}d_1 + k_{12}d_2 + k_{13}d_3 = F_1$$

**Force equilibrium equation at node 1**

**Columns of the global stiffness matrix**

What if $d_1=1$, $d_2=0$, $d_3=0$ ?

While nodes 2 and 3 are held fixed

$F_1 = k_{11}$ Force along node 1 due to unit displacement at node 1

$F_2 = k_{21}$ Force along node 2 due to unit displacement at node 1

$F_3 = k_{31}$ Force along node 3 due to unit displacement at node 1

Similarly we obtain the physical significance of the other entries of the global stiffness matrix

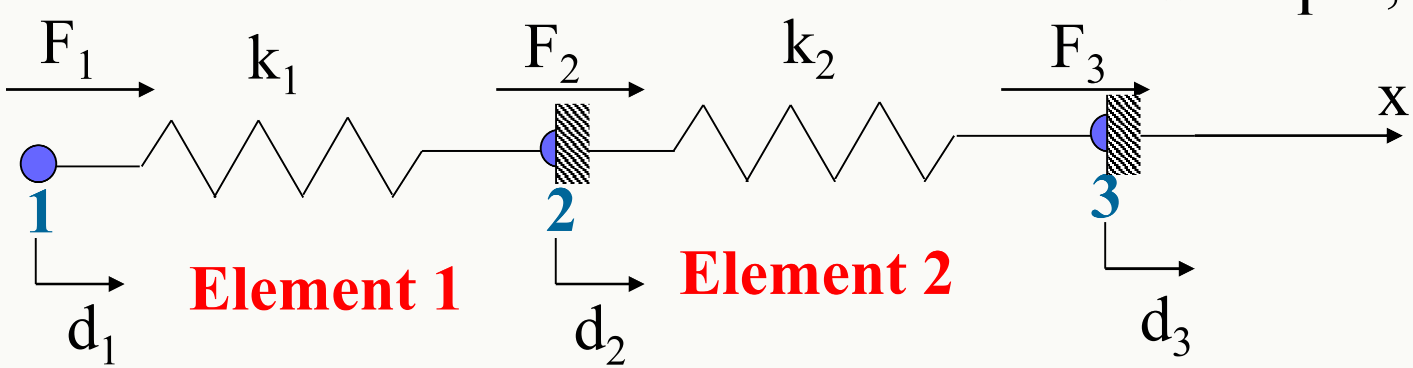
## Physical significance of the stiffness matrix

In general

$k_{ij}$ = Force at node 'i' due to **unit displacement** at node 'j' keeping **all the other nodes fixed**

This is an alternate route to generating the global stiffness matrix e.g., to determine the **first column of the stiffness matrix**

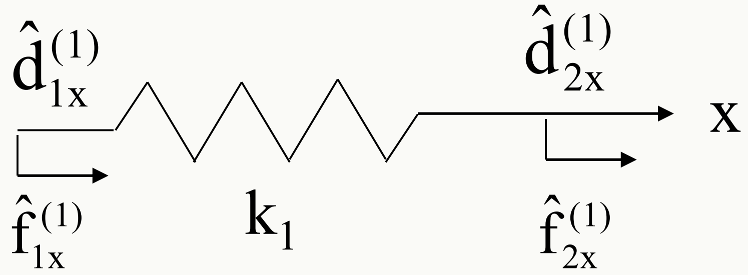
Set $\mathbf{d_1=1}$, $d_2=0$, $d_3=0$




Find $F_1=?$, $F_2=?$, $F_3=?$

## Physical significance of the stiffness matrix

For this special case, Element #2 does not have any contribution. Look at the free body diagram of Element #1

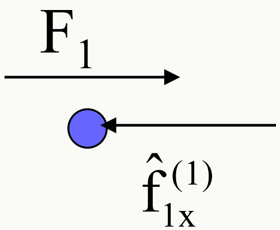



$$\hat{f}_{2x}^{(1)} = k_1(\hat{d}_{2x}^{(1)} - \hat{d}_{1x}^{(1)}) = k_1(0-1) = -k_1$$

$$\hat{f}_{1x}^{(1)} = -\hat{f}_{2x}^{(1)} = k_1$$

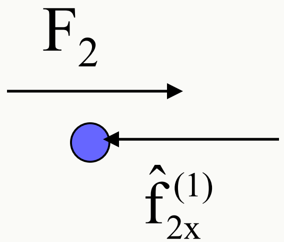
## Physical significance of the stiffness matrix

Force equilibrium at node 1




$$F_1 = \hat{f}_{1x}^{(1)} = k_1$$

Force equilibrium at node 2




$$F_2 = \hat{f}_{2x}^{(1)} = -k_1$$

Of course, $F_3=0$

$$F_1 = k_1 d_1 = k_1 = k_{11}$$

$$F_2 = -F_1 = -k_1 = k_{21}$$

$$F_3 = 0 = k_{31}$$

## Physical significance of the stiffness matrix

Hence the **first column** of the stiffness matrix is

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} k_1 \\ -k_1 \\ 0 \end{Bmatrix}$$

To obtain the **second column** of the stiffness matrix, calculate the nodal reactions at nodes 1, 2 and 3 when $d_1=0$, $\mathbf{d_2=1}$, $d_3=0$

Check that

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} -k_1 \\ k_1 + k_2 \\ -k_2 \end{Bmatrix}$$

## **Physical significance of the stiffness matrix**

To obtain the **third column** of the stiffness matrix, calculate the nodal reactions at nodes 1, 2 and 3 when $d_1=0$, $d_2=0$, $\mathbf{d_3=1}$

Check that

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -k_2 \\ k_2 \end{Bmatrix}$$

## Steps in solving a problem

**Step 1:** Write down the **node-element connectivity table** linking local and global displacements

**Step 2:** Write down the **stiffness matrix of each element**

**Step 3:** **Assemble** the element stiffness matrices to form the global stiffness matrix for the entire structure using the node element connectivity table

**Step 4:** Incorporate appropriate **boundary conditions**

**Step 5:** Solve resulting set of **reduced equations** for the **<u>unknown</u> displacements**

**Step 6:** Compute the **<u>unknown</u> nodal forces**