



جامعة طيبة

EE 351 – Electromagnetic Fields

Lecture 8

Electric Flux Density and Gauss's Law

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This lecture covers following Sections of Sadiku's book Elements of Electromagnetics

Section 3.6, 4.4, 4.5



Electric flux density (\vec{D}) is an alternative to electric field intensity (E) as a way to quantify an electric field independent of the medium. It is defined as:

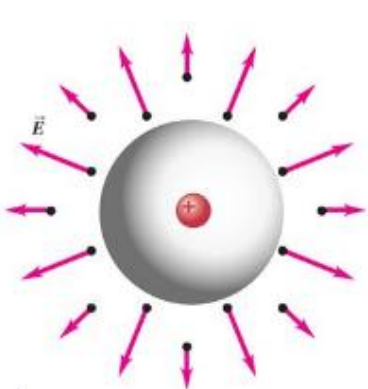
$$\vec{D} = \epsilon \vec{E}$$

E depends on medium of permittivity ϵ . But D is independent of medium. For example for a point charge the two quantities are given by:

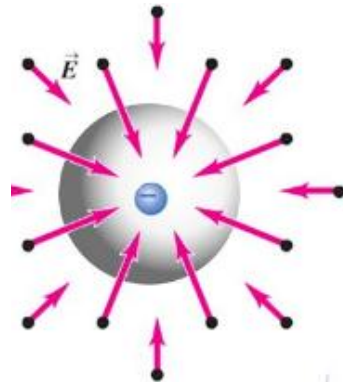
$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2} \hat{a}_r \quad \vec{D} = \frac{Q}{4\pi r^2} \hat{a}_r$$

Representation of Electric Flux Density

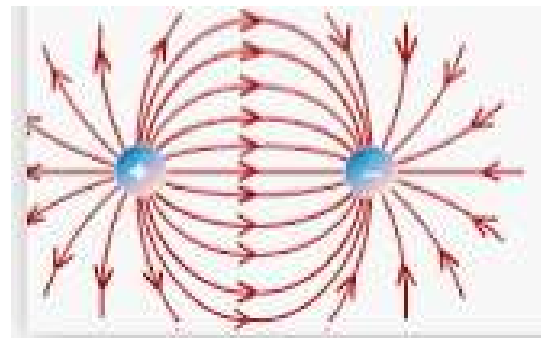
Flux density is proportional to the electric field lines emanating from charges. The flux lines start from positive charge and end at negative charge. More lines indicate more charge



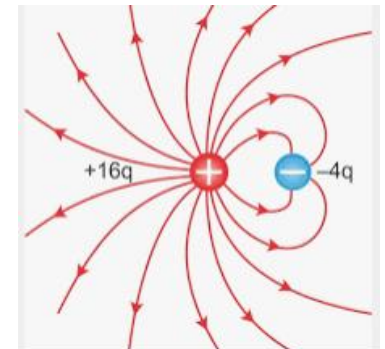
Positive charge



Negative charge



A Dipole of equal charges



A dipole of Large and small charges

Flux of a Vector (Section 3.3)



Flux of a vector field is defined as the surface integral of the vector through the surface

$$\psi = \int_S \vec{A} \cdot d\vec{s}$$

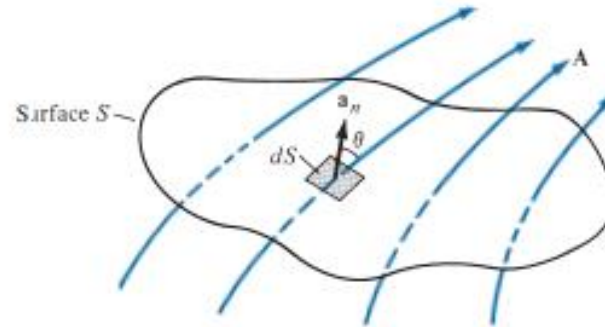


FIGURE 3.10 The flux of a vector field \mathbf{A} through surface S .

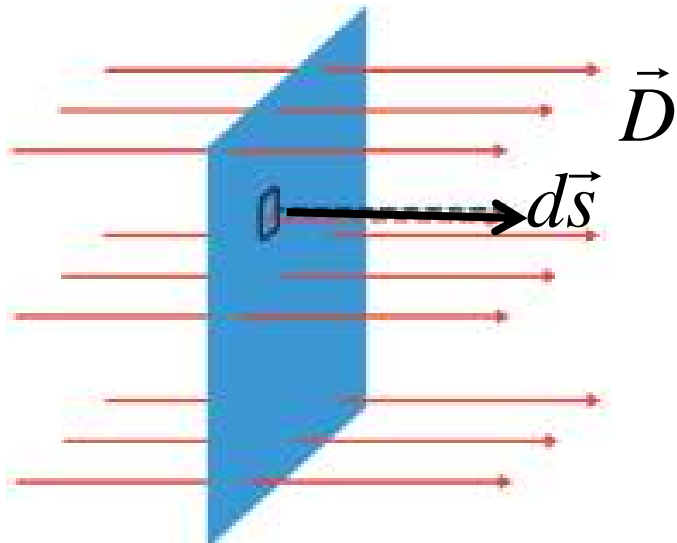
In Electromagnetics , we are interested in Flux of the Flux Density (\mathbf{D}), which is the total flux on the surface

$$\psi = \int_S \vec{D} \cdot d\vec{s}$$

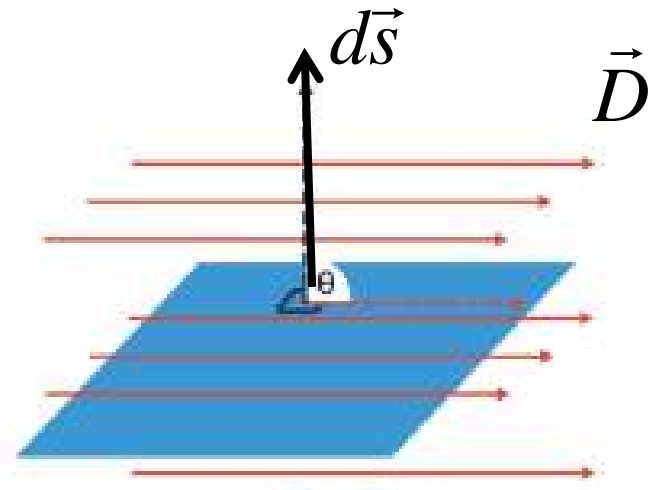


$$\psi = \int_S \vec{D} \cdot d\vec{s}$$

All flux passing through S (D and S in same direction)



No flux passing through S (D and S are perpendicular)



$$\text{Total area} = \int_S d\vec{s} = S$$

$$\psi = \int_S \vec{D} \cdot d\vec{s} = DS \cos 90 = 0$$

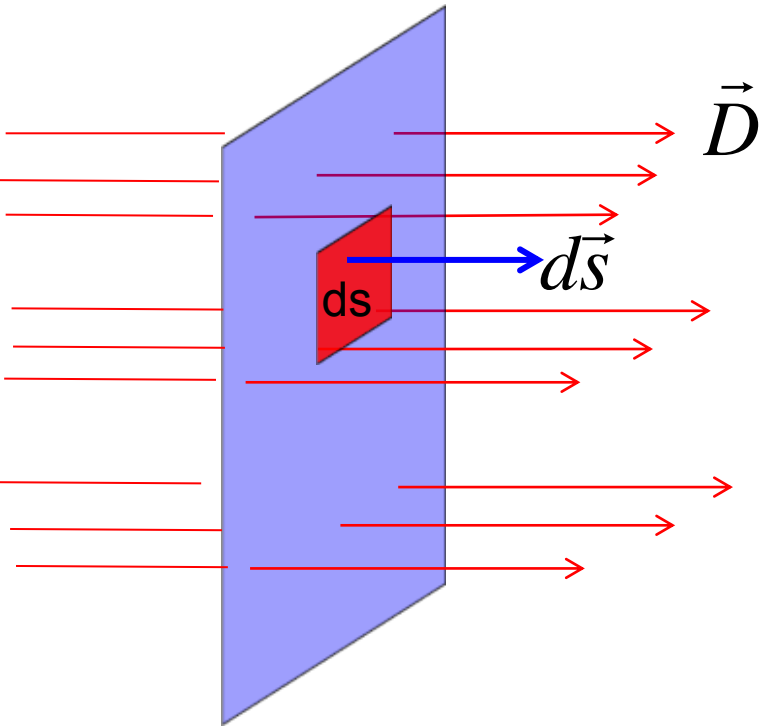
D is constant so we can bring out of Integral

$$\psi = \int_S \vec{D} \cdot d\vec{s} = DS$$

Electric Flux

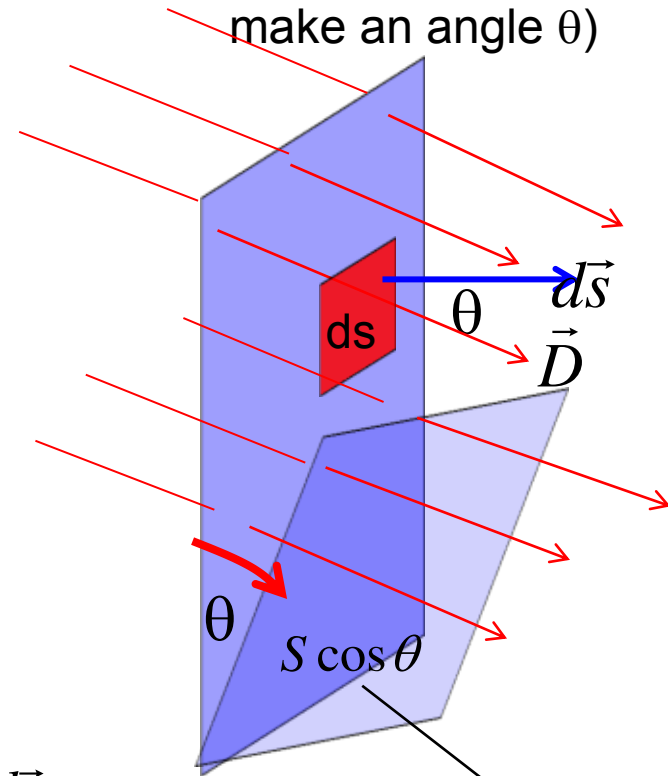


All flux passing through S (\vec{D} and S are in same direction)



$$\psi = \int_S \vec{D} \cdot d\vec{s} = DS$$

General Case: Some flux passing through S (\vec{D} and S make an angle θ)



$$\begin{aligned}\psi &= \int_S \vec{D} \cdot d\vec{s} \\ \Rightarrow \psi &= D \int_S ds \cos \theta \\ \Rightarrow \psi &= DS \cos \theta\end{aligned}$$

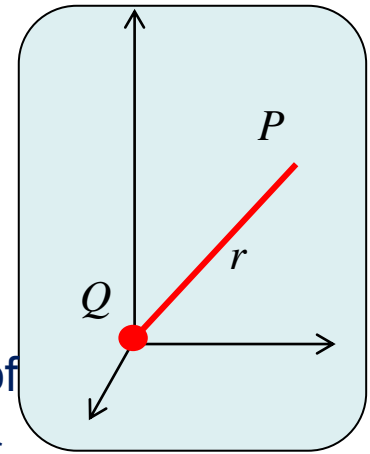
All the flux will pass through a surface that is perpendicular to the direction of the Flux density

Total Flux of a Point Charge

Electric Field due to a point charge a point P which is located at a distance 'r' from the charge

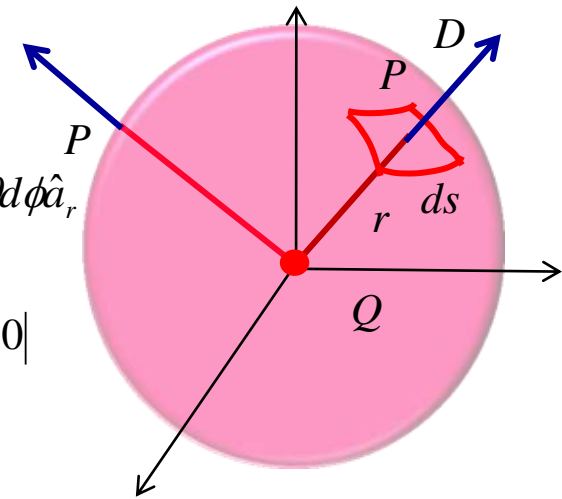
$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2} \hat{a}_r$$

Electric Flux density due to the point charge the point $\vec{D} = \frac{Q}{4\pi r^2} \hat{a}_r$



This flux density will emanate in all directions. At surface of a sphere of radius 'r', the magnitude of all D vectors will be equal pointing in the \hat{a}_r direction. Total flux crossing the spherical surface will be:

$$\begin{aligned} \psi &= \int_S \vec{D} \cdot d\vec{s} & d\vec{s} &= r^2 \sin \theta d\theta d\phi \hat{a}_r \\ \Rightarrow \psi &= \int_0^\pi \int_0^{2\pi} \frac{Q}{4\pi r^2} \hat{a}_r \cdot r^2 \sin \theta d\theta d\phi \hat{a}_r \Rightarrow \psi = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{Q}{4\pi} \hat{a}_r \cdot \hat{a}_r \sin \theta d\theta d\phi \\ & & & \Rightarrow \psi = \frac{Q}{4\pi} \left| -\cos \theta \right|_0^\pi \left| \phi \right|_0^{2\pi} = \frac{Q}{4\pi} \left| 1 + 1 \right| \left| 2\pi - 0 \right| \\ & & & \Rightarrow \psi = Q \end{aligned}$$



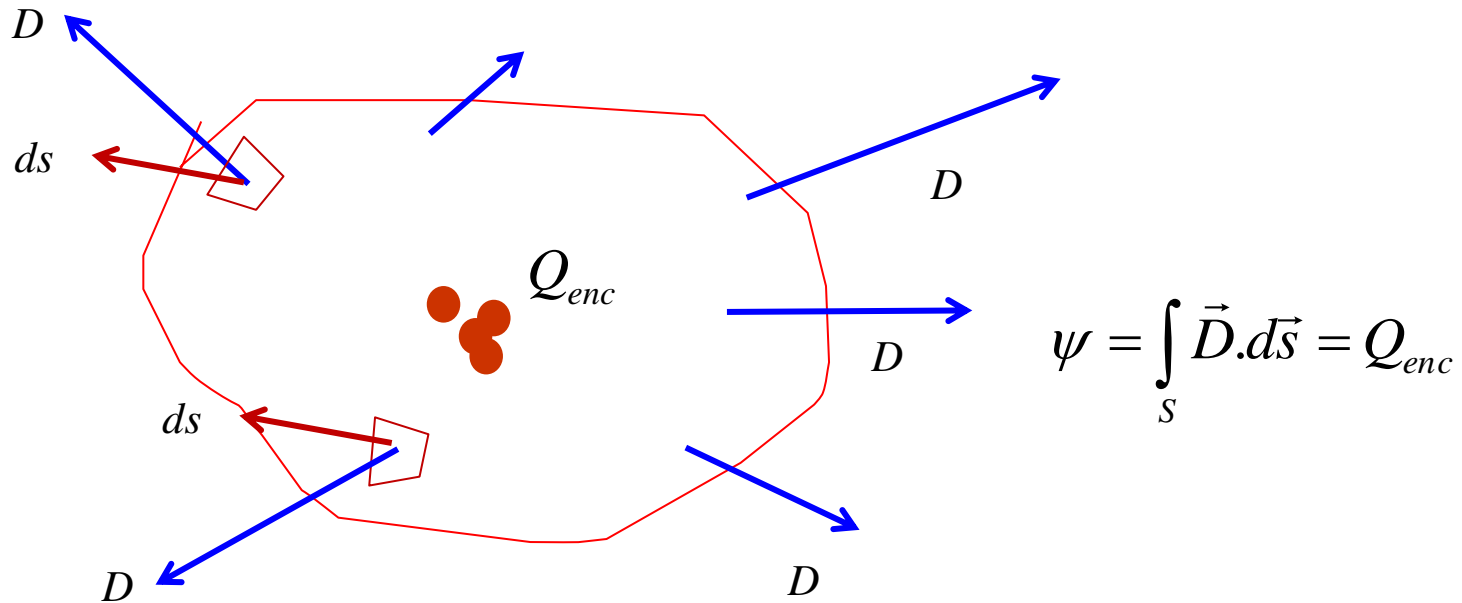
Very Important result: Total flux passing through a closed surface is equal to charge inside the surface (Gauss's Law)

Gauss's Law

The total electric flux through any *closed surface* is equal to the total charge enclosed by that closed surface

$$\psi = \int_S \vec{D} \cdot d\vec{s} = Q_{enc}$$

The surface can be any arbitrary shape. It does not have to be spherical

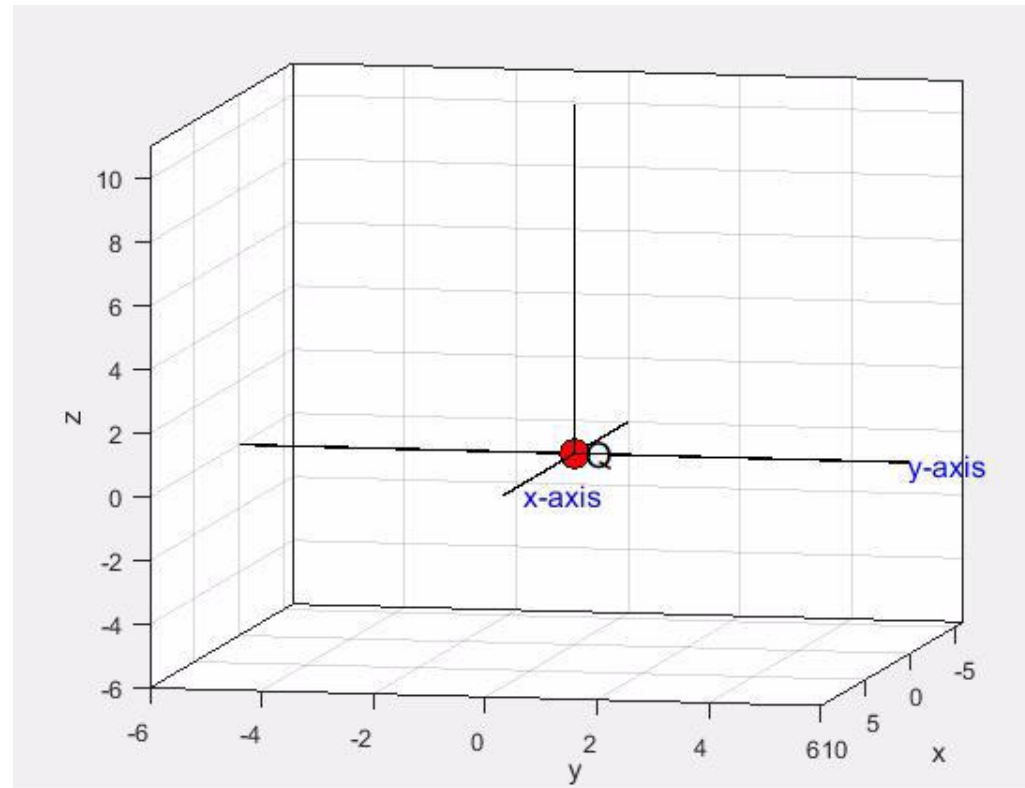
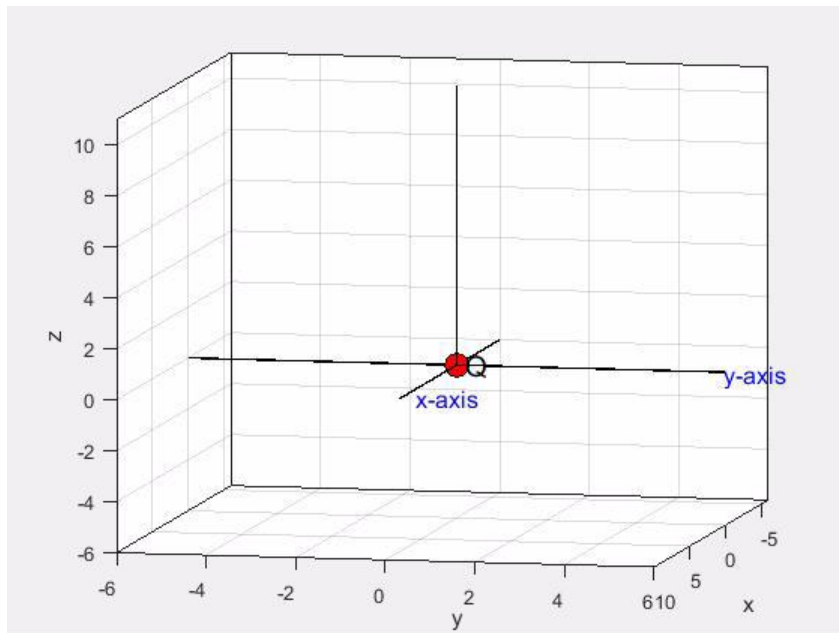


Gauss's Law: Video Demonstration

The total electric flux through any *closed surface* is equal to the total charge enclosed by that closed surface

Point Charge Enclosed in a cylindrical Gaussian Surface

Point Charge Enclosed in a spherical Gaussian Surface



Application of Gauss's Law to find Electric Field

Coulomb's Law Method of calculating Electric fields:

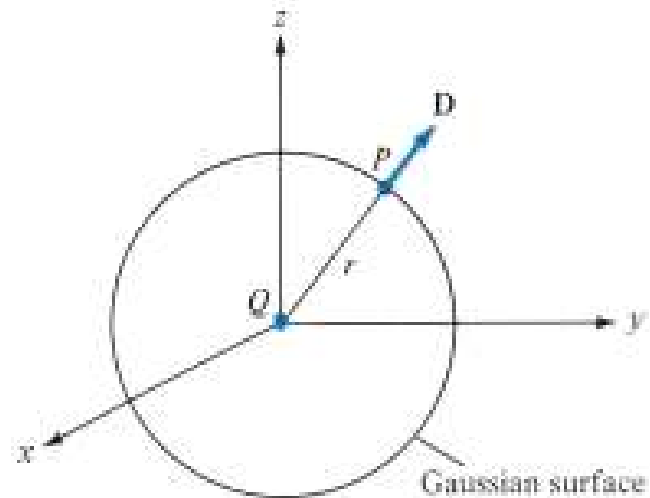
1. determination of the differential vector field
2. Symmetry and cancellation
3. Integration of all the components

Gauss's law method:

1. Enclose the charge into a Gaussian surface
2. Find the direction of field by symmetry
3. Integrate both sides of the Gauss's Law:

$$\Rightarrow \iint_S \vec{D} \cdot d\vec{S} = Q_{enc}$$

4. Find Electric field $\Rightarrow \vec{E} = \frac{\vec{D}}{\epsilon}$



FIC

Gaussian Surface around a point charge

Application of Gauss's Law to find Electric Field: Point Charge

Gauss's law method

$$\Rightarrow \iint_S \vec{D} \cdot d\vec{S} = Q_{enc}$$

1. Enclose the charge in Gaussian surface which is a spherical surface
2. Direction of D is \hat{a}_r So we can evaluate the surface integral

$$\begin{aligned} LHS \Rightarrow \iint_S \vec{D} \cdot d\vec{S} &= \iint_S D_r \hat{a}_r \cdot r^2 \sin \theta d\theta d\phi \hat{a}_r = r^2 D_r \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \sin \theta d\theta d\phi \\ &= r^2 D_r \left[-\cos \theta \right]_0^{\pi} \left[\phi \right]_0^{2\pi} = 4\pi r^2 D_r \end{aligned}$$

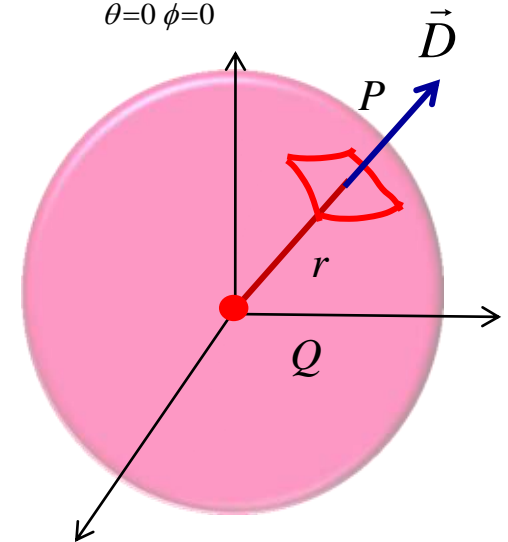
$$LHS = RHS$$

$$\Rightarrow 4\pi r^2 D_r = Q$$

$$\Rightarrow D_r = \frac{Q}{4\pi r^2}$$

$$\Rightarrow \vec{D} = \frac{Q}{4\pi r^2} \hat{a}_r$$

$$\Rightarrow \vec{E} = \frac{Q}{4\pi r \epsilon_0} \hat{a}_r$$



Application of Gauss's Law to find Electric Field: Line Charge

From the Coulomb's law method the Electric field due to Infinite line charge is given by:

$$\Rightarrow \vec{E} = \frac{\rho_L}{2\pi\epsilon_0\rho} \hat{a}_\rho$$

Let us apply the Gaussian surface method

Step 1: Enclose the charge into a proper symmetrical surface

In this case the symmetrical surface is a cylinder extending from $z=-h/2$ to $z=+h/2$

Step 2: Determine direction of field on the Gaussian surface by symmetry

Two opposite segments along z-direction will cancel the z- component and the ρ components are added. So the flux density has only ρ component. $\vec{D} = D_\rho \hat{a}_\rho$

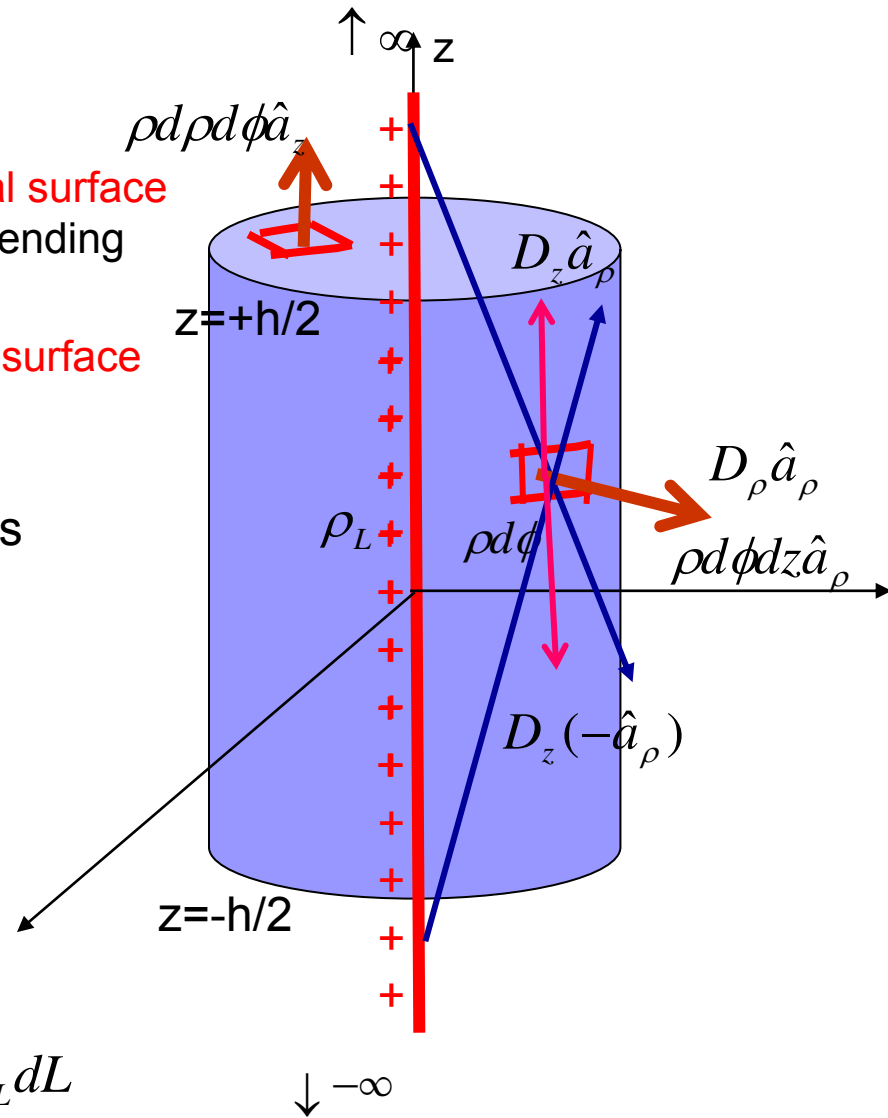
Step 3: apply Gauss's Law

$$\int_s \vec{D} \cdot d\vec{s} = Q_{enc}$$

There are three surfaces. Top, bottom and cylindrical

$$LHS = \int_s \vec{D} \cdot d\vec{s}$$

$$RHS = Q_{enc} = \int_L \rho_L dL$$



Application of Gauss's Law to find Electric Field: Line Charge

$$LHS = \int_s \vec{D} \cdot d\vec{s}$$

$$\Rightarrow LHS = \int_{Top} \vec{D} \cdot d\vec{s} + \int_{Bottom} \vec{D} \cdot d\vec{s} + \int_{cylinder} \vec{D} \cdot d\vec{s}$$

$$\Rightarrow LHS = \int_{Top} \cancel{\vec{D}_\rho \hat{a}_\rho} \cdot \rho d\rho d\phi \hat{a}_z + \int_{Bottom} \cancel{\vec{D}_\rho \hat{a}_\rho} \cdot \rho d\rho d\phi (-\hat{a}_z) + \int_{Bottom} \vec{D}_\rho \hat{a}_\rho \cdot \rho d\phi dz \hat{a}_\rho$$

$$\Rightarrow LHS = \rho D_\rho \int_{\phi=0}^{2\pi} \int_{z=-h/2}^{+h/2} d\phi dz \Rightarrow LHS = 2\pi D_\rho \rho h$$

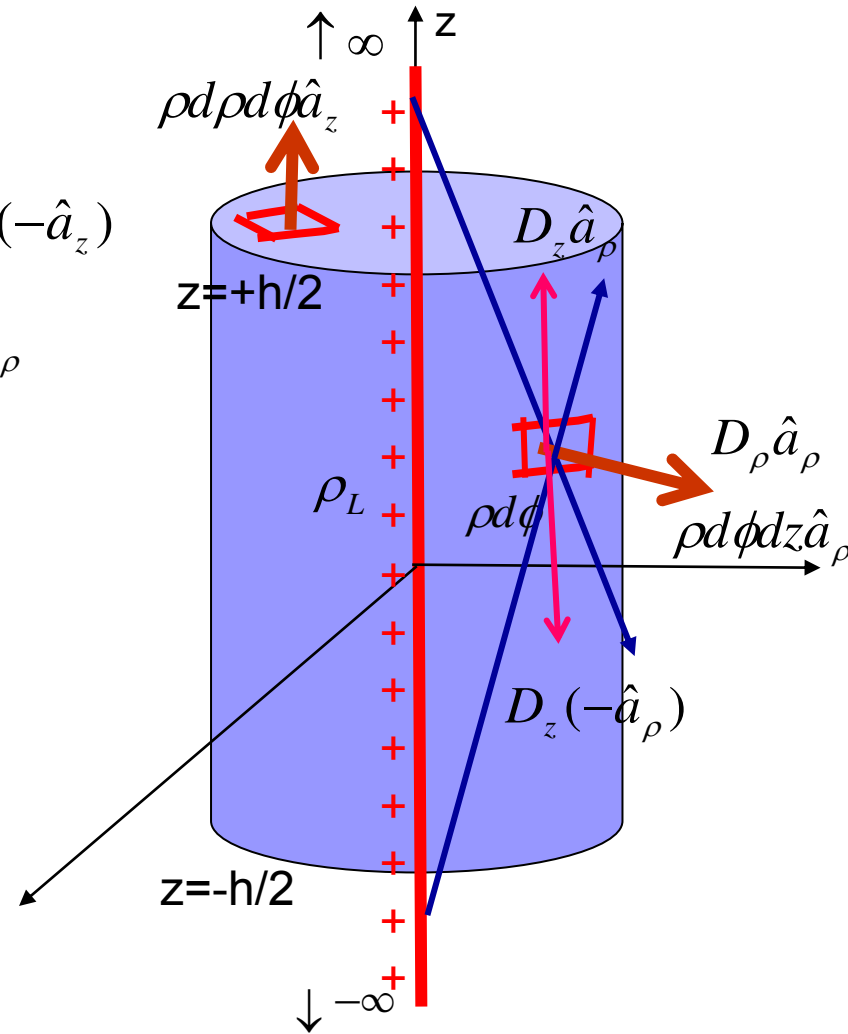
$$RHS = Q_{enc} \Rightarrow RHS = \int_S \rho_L dL = \int_{z=-h/2}^{+h/2} \rho_L dz$$

$$\Rightarrow RHS = \rho_L h$$

$$LHS = RHS$$

$$\Rightarrow 2\pi D_\rho \rho h = \rho_L h$$

$$\Rightarrow D_\rho = \frac{\rho_L}{2\pi\rho} \Rightarrow \vec{D} = \frac{\rho_L}{2\pi\rho} \hat{a}_\rho \Rightarrow \vec{E} = \frac{\rho_L}{2\pi\epsilon_0\rho} \hat{a}_\rho$$



Application of Gauss's Law to find Electric Field: Surface Charge

$$LHS = \int_s \vec{D} \cdot d\vec{s}$$

$$\Rightarrow LHS = \int_{Top} \vec{D} \cdot d\vec{s} + \int_{Bottom} \vec{D} \cdot d\vec{s} + \int_{Front} \vec{D} \cdot d\vec{s} + \int_{Back} \vec{D} \cdot d\vec{s} + \int_{Right} \vec{D} \cdot d\vec{s} + \int_{Left} \vec{D} \cdot d\vec{s}$$

All the $D \cdot ds$ are zero except the top and bottom surfaces because the field is in z-direction on all the surfaces.

$$\Rightarrow LHS = \underbrace{\int_{x=-X/2}^{X/2} \int_{y=-Y/2}^{+Y/2} D_z \hat{a}_z \cdot dxdy \hat{a}_z}_{\text{Top}} + \underbrace{\int_{x=-X/2}^{X/2} \int_{y=-Y/2}^{+Y/2} D_z (-\hat{a}_z) \cdot dxdy (-\hat{a}_z)}_{\text{Bottom}}$$

$$\Rightarrow LHS = D_z XY + D_z XY = 2D_z XY$$

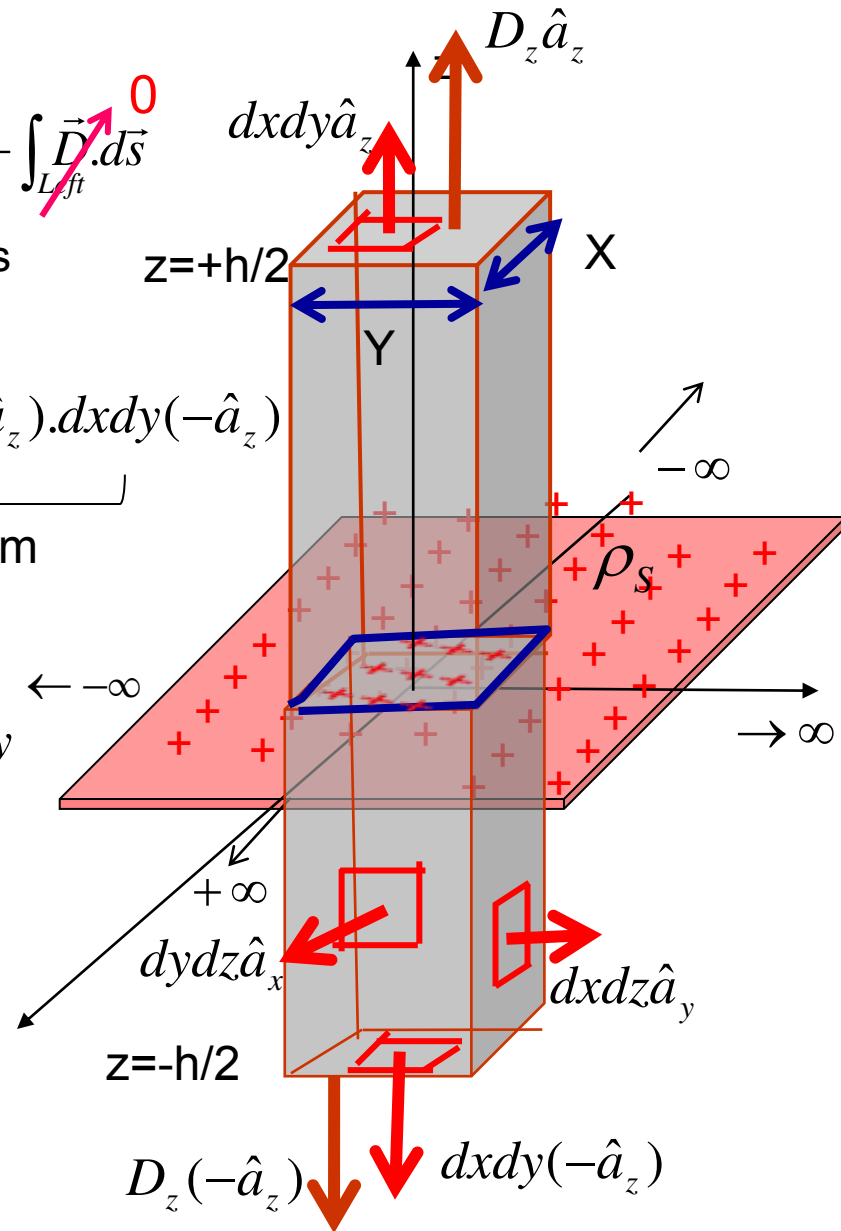
$$RHS = Q_{enc} \Rightarrow RHS = \int_s \rho_s dS = \int_{x=-X/2}^{+X/2} \int_{y=-Y/2}^{Y/2} \rho_s dxdy$$

$$\Rightarrow RHS = \rho_s XY$$

$$LHS = RHS$$

$$\Rightarrow 2D_z XY = \rho_s XY \Rightarrow D_z = \frac{\rho_s}{2}$$

$$\Rightarrow D = \frac{\rho_s}{2} \hat{a}_z \Rightarrow E = \frac{\rho_s}{2\epsilon_0} \hat{a}_z$$



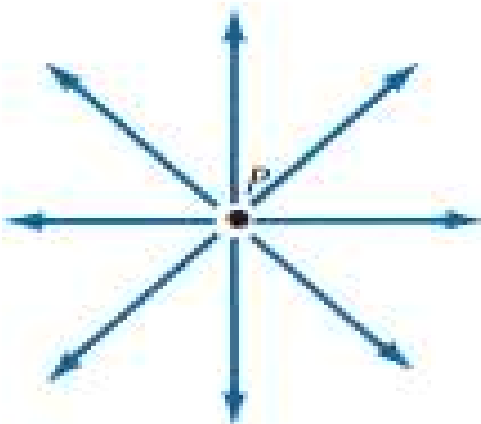
Divergence of a Vector Field (Section 3.6)

- Divergence is a vector operation which provides a relation between surface integral of a vector field and the total flux leaving an enclosed volume
- Divergence operator acts on a vector field and the result is a scalar quantity. In Cartesian coordinates, divergence of a vector is given by

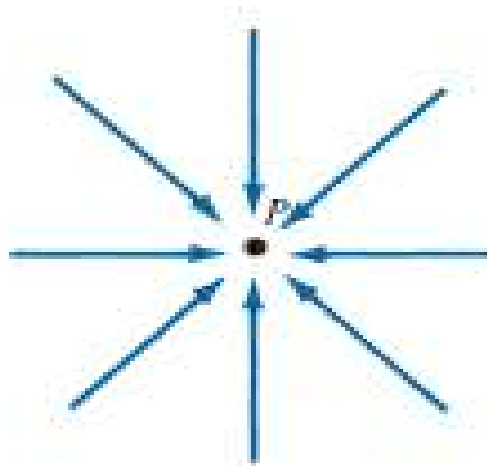
$$\operatorname{div}\vec{A} = \nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Physical meaning of divergence

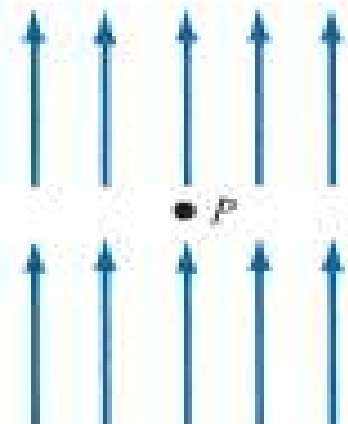
It gives an indication of flux leaving a differential volume



Positive Divergence
(Flux leaving a differential volume)



Negative Divergence
(Flux entering a differential volume)



Zero Divergence
(Flux entering = Flux leaving)

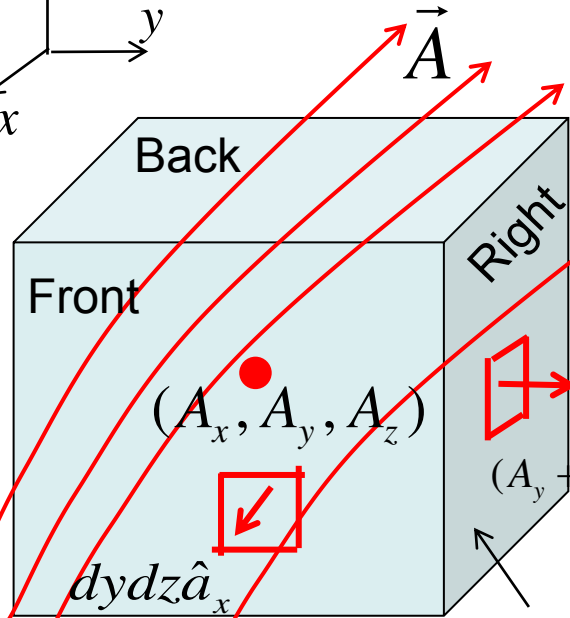
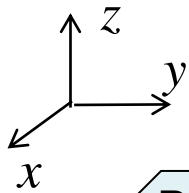
Mathematical Representation of the Physical Meaning of Divergence

- Divergence is the total flux per unit volume which leaves a differential volume. Mathematically

$$\text{div} \vec{A} = \lim_{\Delta v \rightarrow 0} \frac{\oint_S \vec{A} \cdot d\vec{s}}{\Delta v}$$

In Cartesian $\Delta v = dx dy dz \Rightarrow \text{div} \vec{A} = \frac{\oint_S \vec{A} \cdot d\vec{s}}{dx dy dz}$

$$\oint_S \vec{A} \cdot d\vec{s} = \oint_{\text{Front}} \vec{A} \cdot d\vec{s} \hat{a}_x + \oint_{\text{Back}} \vec{A} \cdot d\vec{s} (-\hat{a}_x) + \oint_{\text{Right}} \vec{A} \cdot d\vec{s} \hat{a}_y + \oint_{\text{Left}} \vec{A} \cdot d\vec{s} (-\hat{a}_y) + \oint_{\text{Top}} \vec{A} \cdot d\vec{s} \hat{a}_z + \oint_{\text{Bottom}} \vec{A} \cdot d\vec{s} (-\hat{a}_z)$$



At the center of the volume let the vector $\vec{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z$

Therefore, at front face $\vec{A}_{\text{front}} = \left(A_x + \frac{\partial A_x}{\partial x} \frac{dx}{2} \right) \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z$

$$\Rightarrow \oint_{\text{Front}} \vec{A} \cdot d\vec{s} = \left(A_x + \frac{\partial A_x}{\partial x} \frac{dx}{2} \right) \hat{a}_x \cdot dy dz \hat{a}_x$$

$$\Rightarrow \oint_{\text{Back}} \vec{A} \cdot d\vec{s} = \left(A_x - \frac{\partial A_x}{\partial x} \frac{dx}{2} \right) \hat{a}_x \cdot dx dz (-\hat{a}_x)$$

$$\left(A_x + \frac{\partial A_x}{\partial x} \frac{dx}{2}, A_y, A_z \right)$$

Differential volume dv

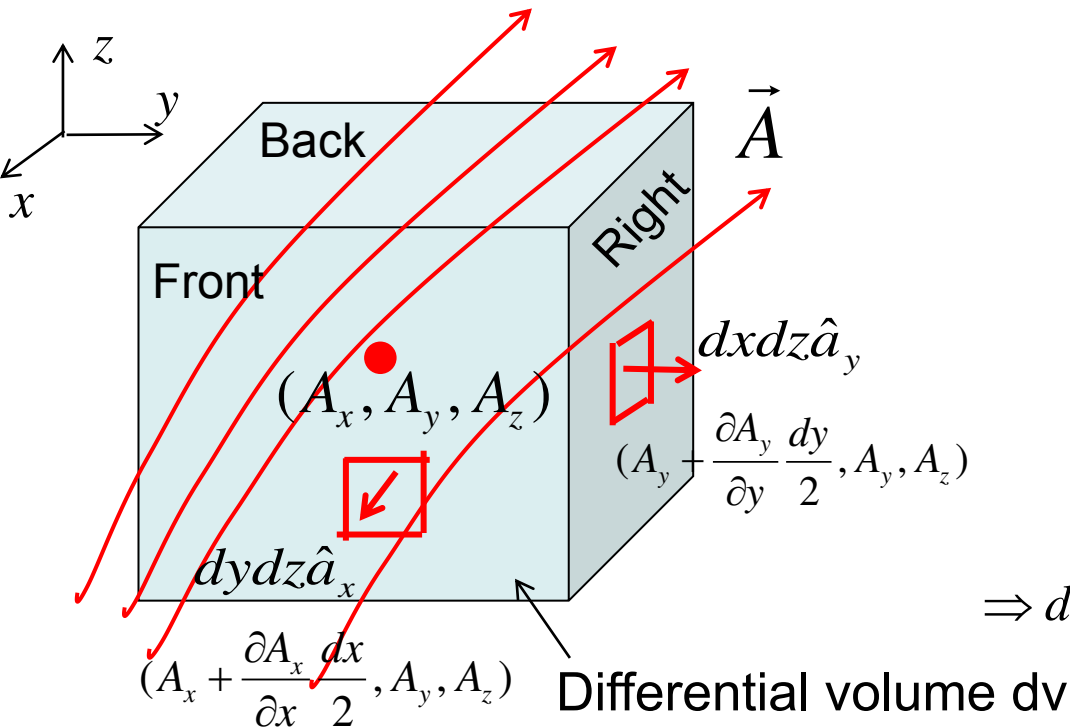
$$\Rightarrow \oint_{\text{Front+Back}} \vec{A} \cdot d\vec{s} = \frac{\partial A_x}{\partial x} dx dy dz$$

Mathematical Representation of the Physical Meaning of Divergence

- Divergence is the total flux per unit volume which leaves a differential volume
- Mathematically

$$\text{div} \vec{A} = \lim_{\Delta v \rightarrow 0} \frac{\oint \vec{A} \cdot d\vec{s}}{\Delta v}$$

In Cartesian $\Delta v = dx dy dz \Rightarrow \text{div} \vec{A} = \frac{\oint \vec{A} \cdot d\vec{s}}{dx dy dz}$



$$\oint_{\text{Front+Back}} A \cdot ds = \frac{\partial A_x}{\partial x} dx dy dz$$

$$\oint_{\text{Left+Right}} A \cdot ds = \frac{\partial A_y}{\partial y} dx dy dz$$

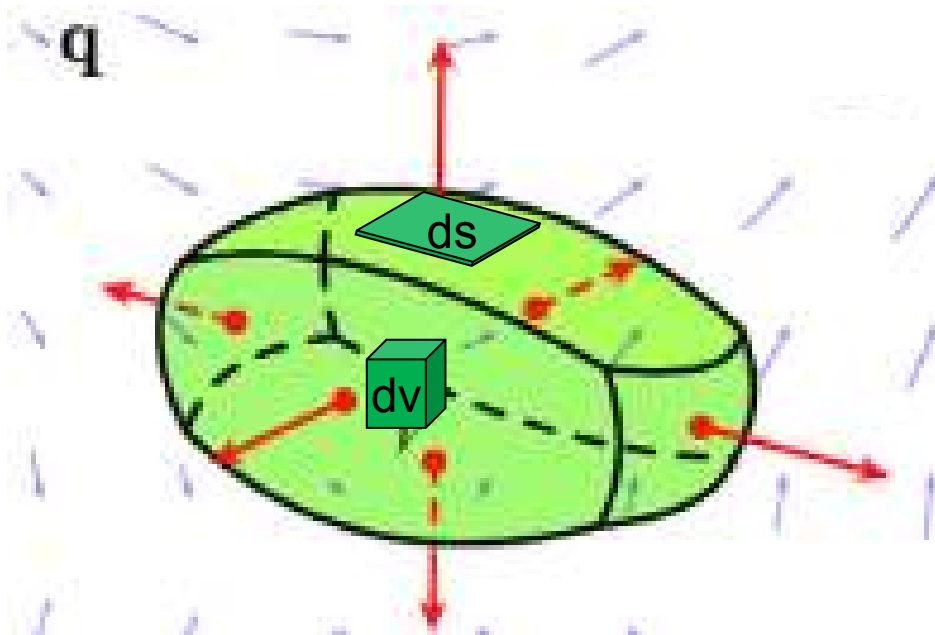
$$\oint_{\text{Top+Bottom}} A \cdot ds = \frac{\partial A_z}{\partial z} dx dy dz$$

$$\Rightarrow \text{div} \vec{A} = \frac{\oint \vec{A} \cdot d\vec{s}}{dx dy dz} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Divergence Theorem

Total outward flux of a vector field \mathbf{A} through a closed surface S is equal to the volume integral of the divergence of \mathbf{A}

$$\int_v \nabla \cdot \vec{A} \, dv = \oint_S \vec{A} \cdot d\vec{s}$$



Point Form of Gauss's Law

Integral form of Gauss's Law

$$\int_S \vec{D} \cdot d\vec{s} = Q_{enc} = \int_v \rho_v dv$$

From Divergence Theorem

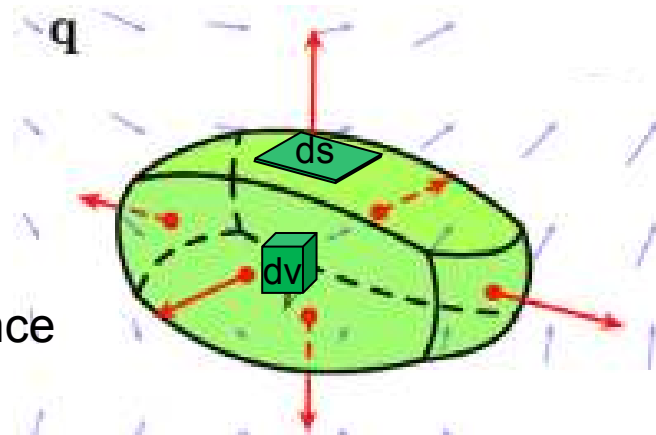
$$\int_v \nabla \cdot \vec{D} dv = \oint_S \vec{D} \cdot d\vec{s}$$

Replacing the surface integral with the total volume charge

$$\Rightarrow \int_v \nabla \cdot \vec{D} dv = \int_v \rho_v dv$$

$$\Rightarrow \nabla \cdot \vec{D} = \rho_v$$

This is the point form of Gauss's Law. Divergence of Flux density is equal to the volume charge density



Integral Form

$$\int_S \vec{D} \cdot d\vec{s} = \int_V \rho_v dv$$

Differential or Point form

$$\nabla \cdot \vec{D} = \rho_v$$

All Four Maxwell's Equations

TABLE 9.1 Generalized Forms of Maxwell's Equations

Differential Form	Integral Form	Remarks
$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v dv$	Gauss's Law
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$	Gauss's Law for Magnetic Fields
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_L \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S}$	Faraday's Law
$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S}$	Ampere's Law

*This is also referred to as Gauss's law for magnetic fields.

Divergence in Cylindrical and Spherical Systems

Cartesian

$$\operatorname{div} \vec{A} = \nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Cylindrical

$$\nabla \cdot \vec{A} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\rho) + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

Spherical

$$\nabla \cdot \vec{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (A_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi}$$