ME 8043114 Automatic Control

Lecture 6: Root Locus

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Course roadmap

Modeling

- Laplace transform
- Transfer function
- Models for systems
  - electrical
  - mechanical
  - electromechanical
- Linearization, delay

Analysis

- Time response
  - Transient
  - Steady state
- Frequency response
  - • Bode plot
- Stability
  - Routh-Hurwitz
  - • Nyquist

Design

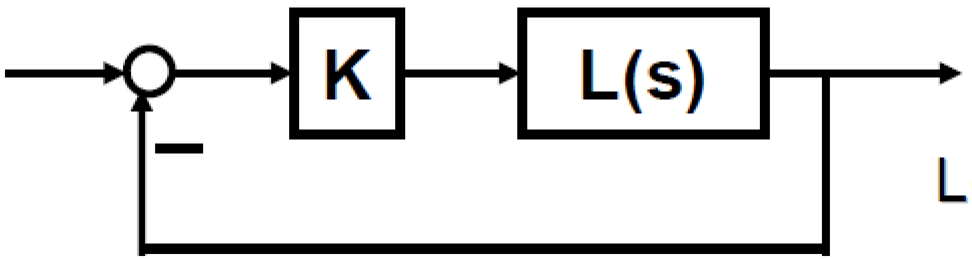
- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

Lecture Outline

- Root Locus
- How to use the root locus

What is Root Locus?

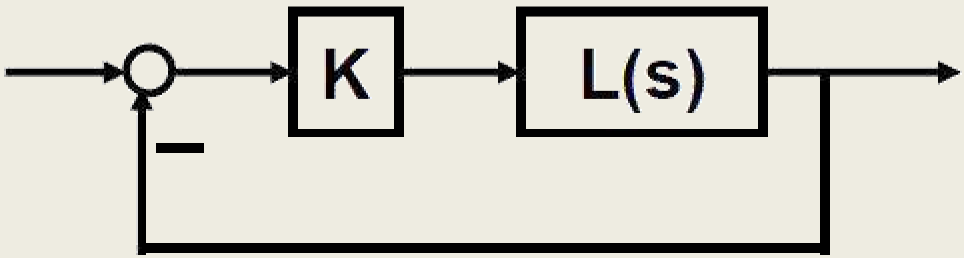
- *Pole location* of the feedback system characterizes stability and transient properties.
- Consider a feedback system that has one parameter (gain) K>0 to be designed.



L(s): open-loop TF

- *Root locus* graphically shows how poles of CL system varies as K varies from 0 to infinity.

A simple example



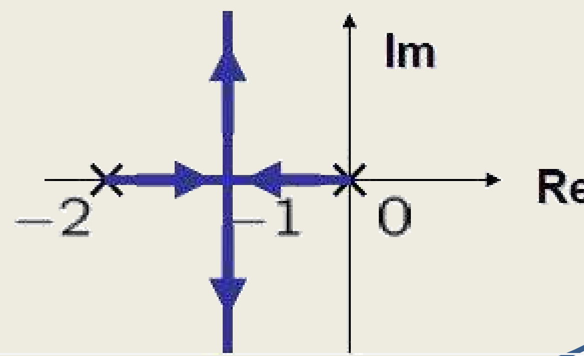
$$L(s) = \frac{1}{s(s+2)}$$

- Characteristic eq. $1 + K\frac{1}{s(s+2)} = 0$

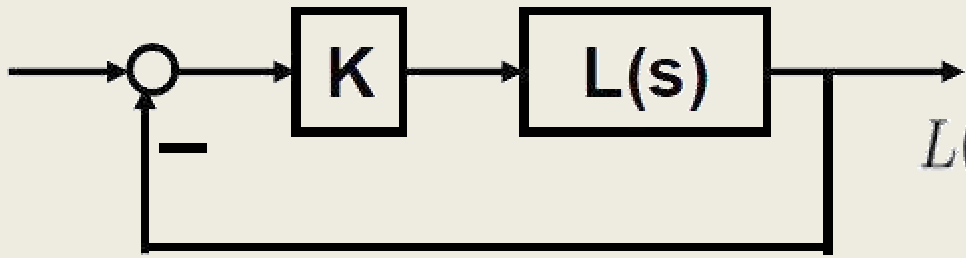
Closed-loop poles

$$\Rightarrow s^2 + 2s + K = 0 \Rightarrow s = -1 \pm \sqrt{1-K}$$

- K=0: s=0,-2
- K=1: s=-1,-1
- K>1: complex numbers



A more complicated example



$$L(s) = \frac{s+1}{s(s+2)(s+3)}$$

- Characteristic eq. $1 + K\dfrac{s+1}{s(s+2)(s+3)} = 0$

  $\Rightarrow s(s+2)(s+3) + K(s+1) = 0 \Rightarrow s = ???$

- It is hard to solve this analytically for each K.
- Is there some way to sketch roughly root locus by hand? (In Matlab, use command "*rlocus.m*".)

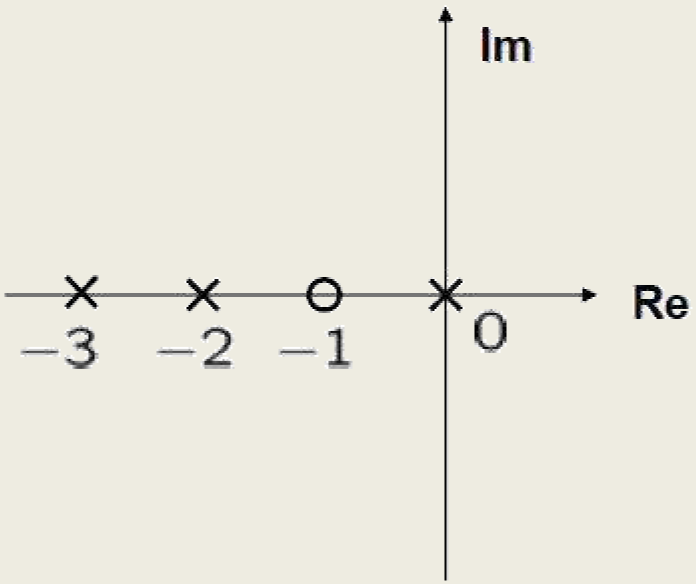
Root locus sketching algorithm

- Step 0: Mark open-loop poles and zeros
- Step 1: On the real axis
- Step 2: Asymptotes
- Step 3: Breakaway points
- Step 4: Angles of departures and arrivals

Root locus: Step 0

- *Root locus is symmetric w.r.t. the real axis.*
- *The number of branches = order of L(s)*
- *Mark poles of L with "x" and zeros of L with "o".*

$$L(s) = \frac{s+1}{s(s+2)(s+3)}$$



Root locus sketching algorithm

- Step 0: Mark open-loop poles and zeros
- Step 1: On the real axis
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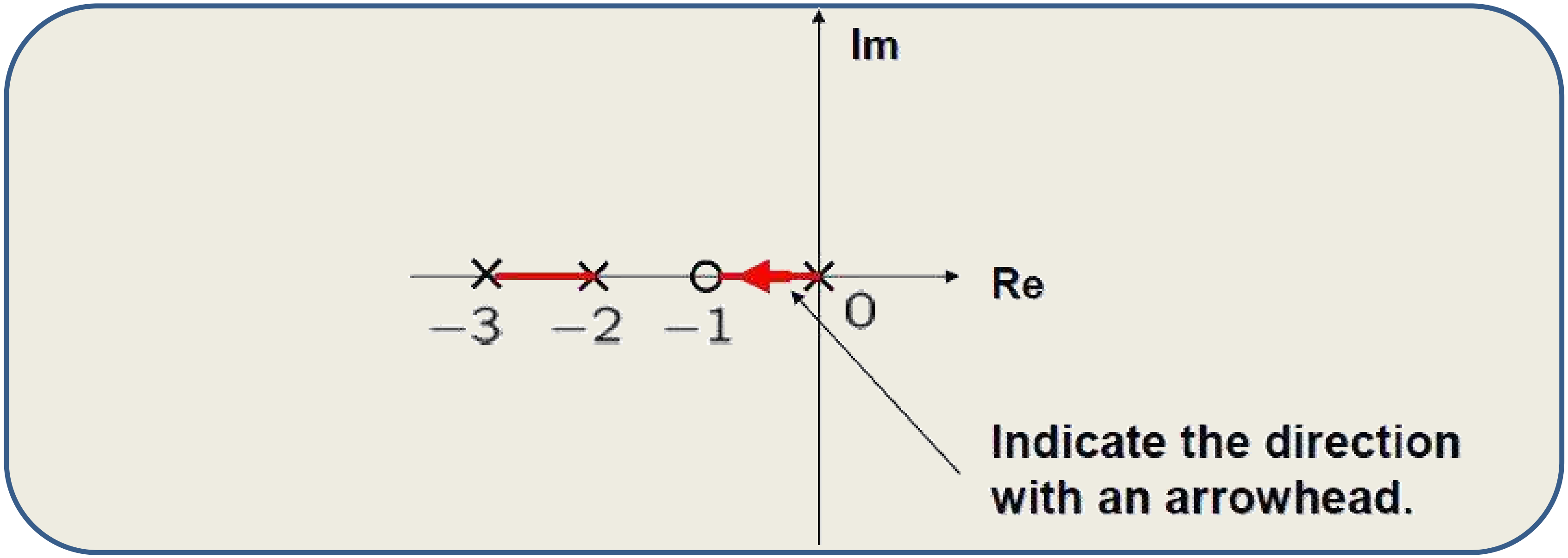
- Step 3: Breakaway points
- Step 4: Angles of departures and arrivals

Root locus: Step 1 (On the real axis)

- *RL includes all points on real axis to the left of an odd number of real poles/zeros.*
- *RL originates from the poles of L and terminates at the zeros of L, including infinity zeros.*



Root locus sketching algorithm

- Step 0: Mark open-loop poles and zeros
- Step 1: On the real axis
- Step 2: Asymptotes
- Step 3: Breakaway points
- Step 4: Angles of departures and arrivals

Root locus: Step 2 (Asymptotes)

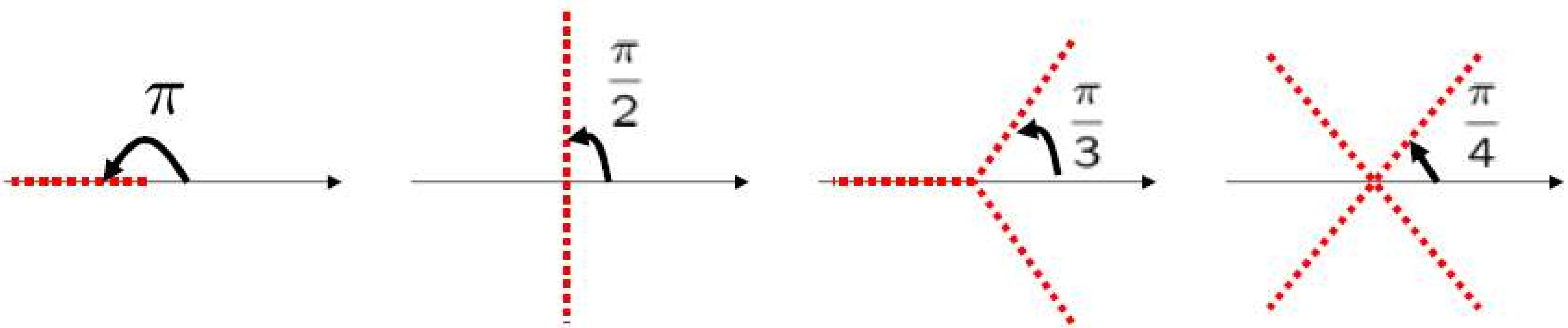
- *Number of asymptotes = relative degree (r) of L:*

$$r := \underbrace{n}_{\deg(\text{den})} - \underbrace{m}_{\deg(\text{num})}$$

- *Angles of asymptotes are*

$$\frac{\pi}{r} \times (2k+1),\ k = 0, 1, \ldots$$

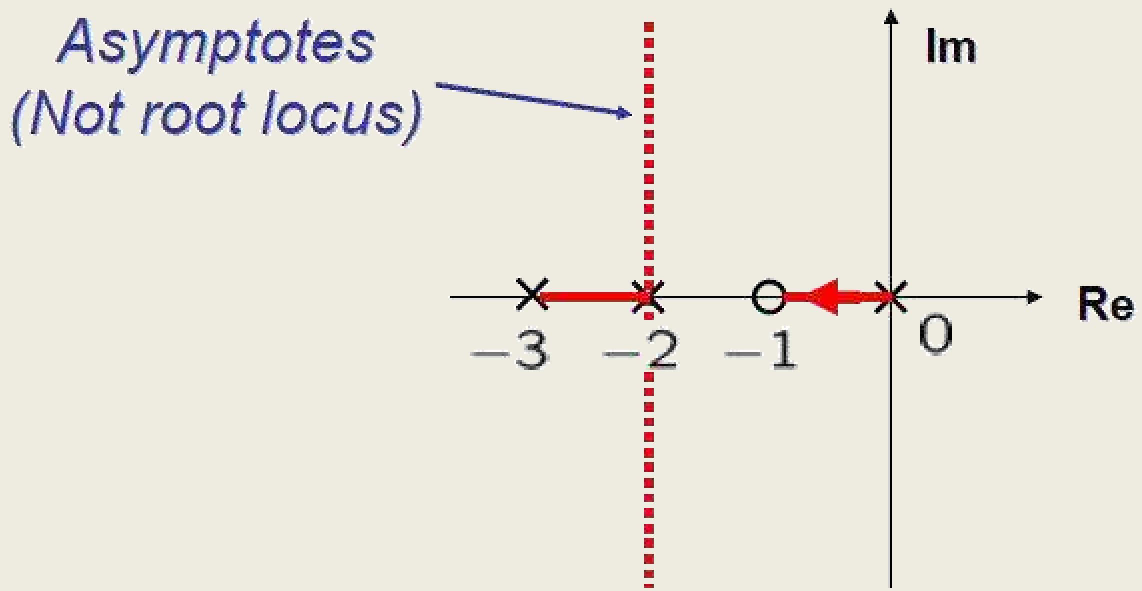
$r = 1$ $\quad r = 2$ $\quad r = 3$ $\quad r = 4$



Root locus: Step 2 (Asymptotes)

- *Intersections of asymptotes* $\dfrac{\sum \text{pole} - \sum \text{zero}}{r}$

$$L(s) = \frac{s+1}{s(s+2)(s+3)} \Rightarrow \frac{\sum \text{pole} - \sum \text{zero}}{r} = \frac{(0+(-2)+(-3))-(-1)}{2} = -2$$



Root locus sketching algorithm

- Step 0: Mark open-loop poles and zeros
- Step 1: On the real axis
- Step 2: Asymptotes
- Step 3: Breakaway points
- Step 4: Angles of departures and arrivals

Root locus: Step 3

- Breakaway points are among roots of $\boxed{\dfrac{dL(s)}{ds} = 0}$

Points where two or more branches meet and break away.

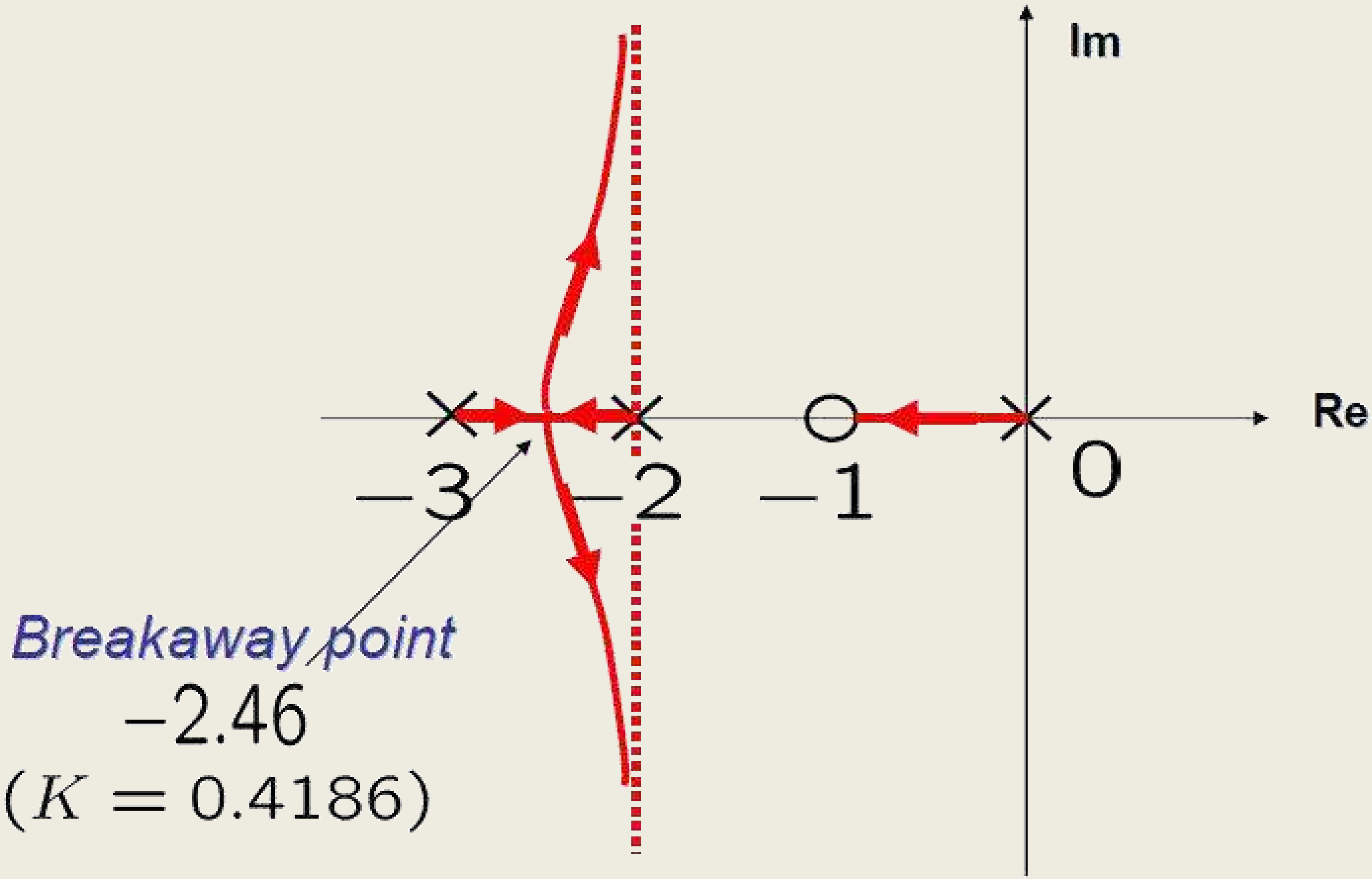
$$L(s) = \frac{s+1}{s(s+2)(s+3)} \Rightarrow \frac{dL(s)}{ds} = -2\frac{s^3+4s^2+5s+3}{(*)} = 0$$

$$\Rightarrow s = -2.4656,\ -0.7672 \pm 0.7926i$$

For each candidate s, check the positivity of $K = -\dfrac{1}{L(s)}$

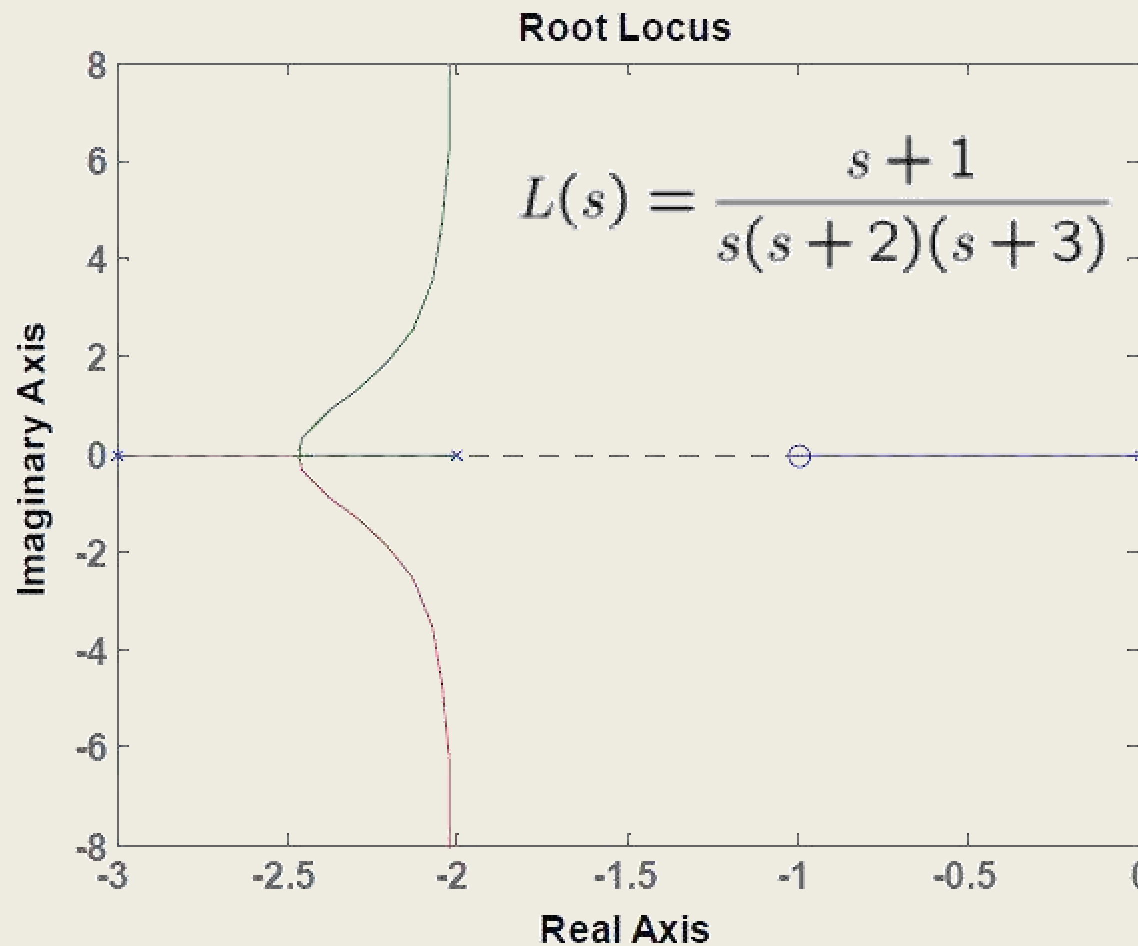
$$\Rightarrow K = 0.4186,\ 1.7907 \mp 4.2772i$$

Root locus: Step 3

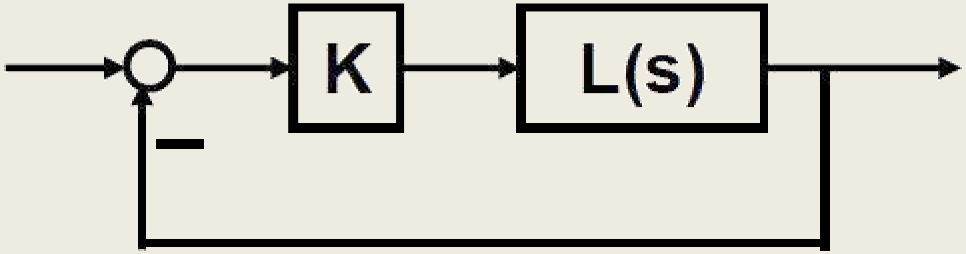


Matlab command "rlocus.m"

```
num=[1 1];
den=[1 5 6 0];
sys=tf(num,den);
rlocus(sys)
```

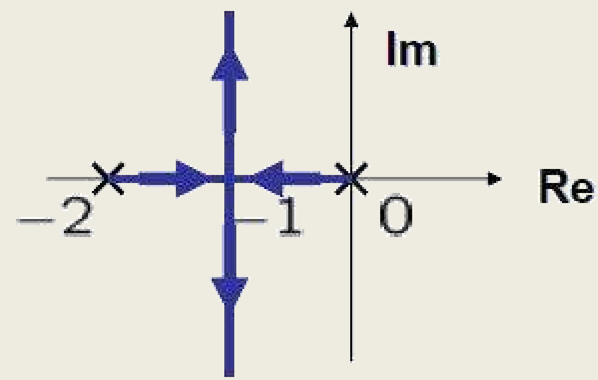


A simple example: revisited



$$L(s) = \frac{1}{s(s+2)}$$

- Asymptotes
  - Relative degree 2
  - Intersection $\dfrac{0+(-2)}{2} = -1$



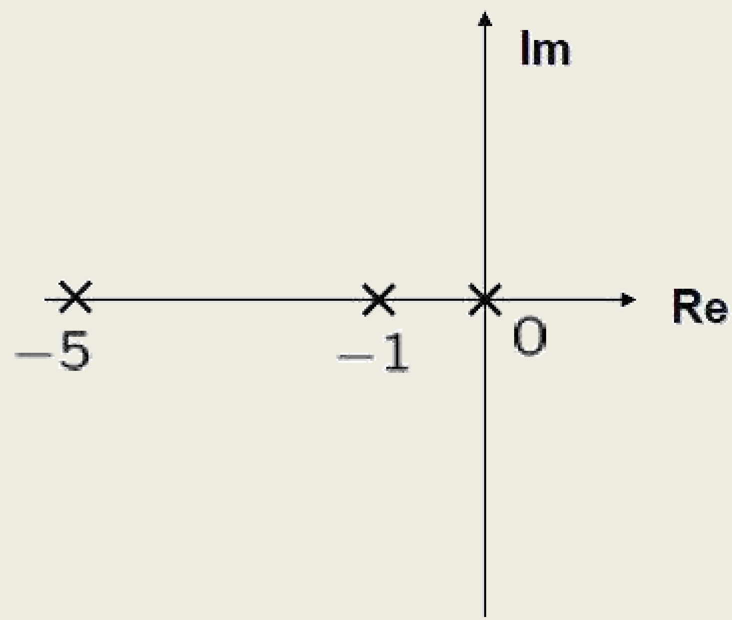
- Breakaway point

$$L'(s) = \frac{-(2s+2)}{*} = 0 \;\Rightarrow\; s = -1$$

Root locus: Step 0

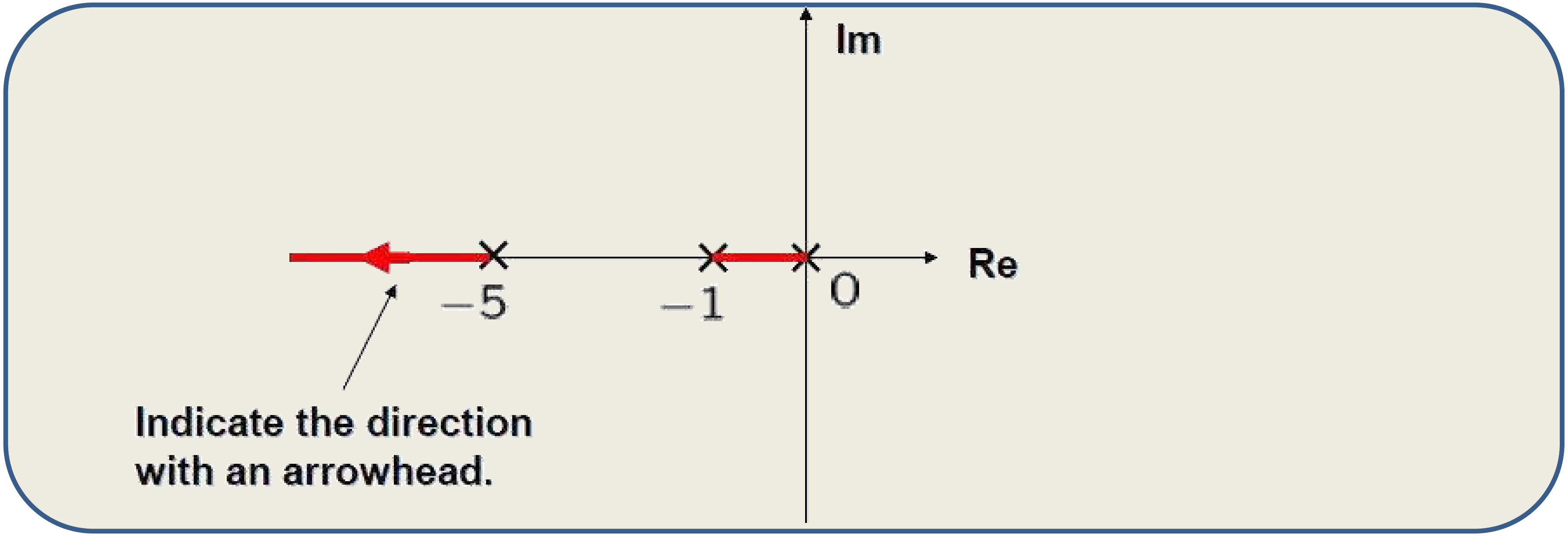
- *Root locus is symmetric w.r.t. the real axis.*
- *The number of branches = order of L(s)*
- *Mark poles of L with "x" and zeros of L with "o".*

$$L(s) = \frac{1}{s(s+1)(s+5)}$$



Root locus: Step 1 (Real axis)

- *RL includes all points on real axis to the left of an odd number of real poles/zeros.*
- *RL originates from the poles of L and terminates at the zeros of L, including infinity zeros.*



Root locus: Step 2 (Asymptotes)

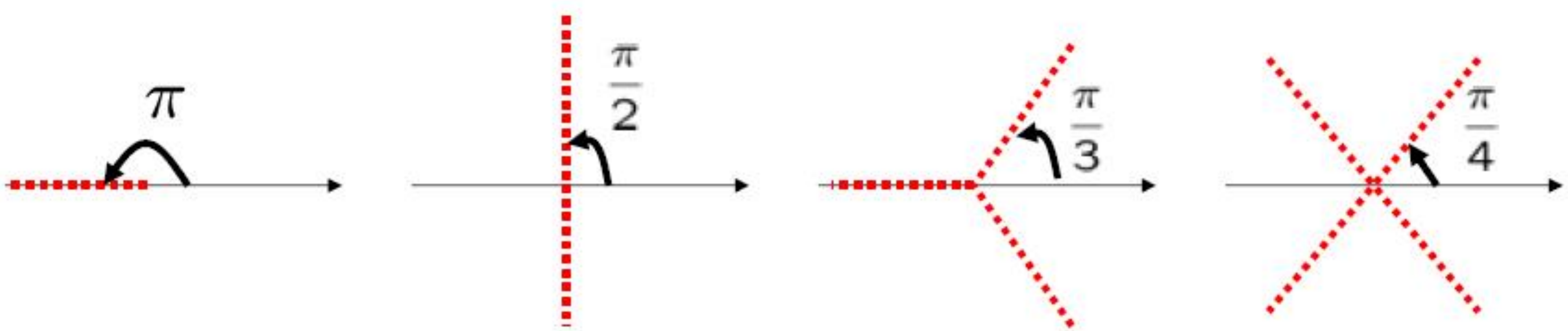
- *Number of asymptotes = relative degree (r) of L:*

$$r := \deg(\text{den}) - \deg(\text{num})$$

- *Angles of asymptotes are*

$$\frac{\pi}{r} \times (2k+1), \ \ k = 0, 1, \ldots$$

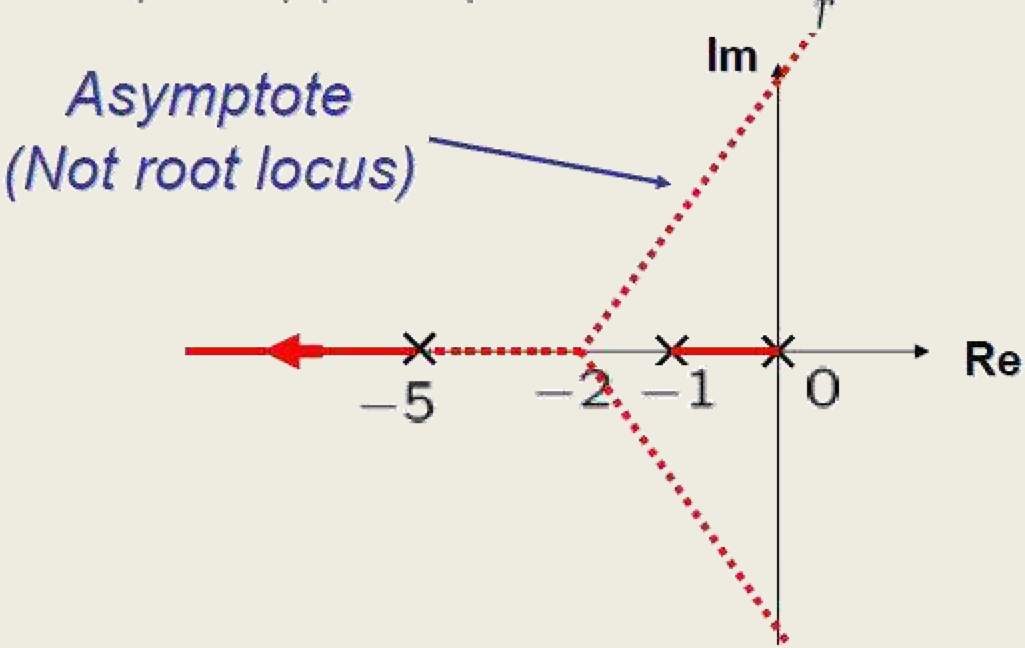
$r = 1$ $\quad$ $r = 2$ $\quad$ $r = 3$ $\quad$ $r = 4$



Root locus: Step 2 (Asymptotes)

- *Intersections of asymptotes* $\dfrac{\sum \text{pole} - \sum \text{zero}}{r}$

$$L(s) = \frac{1}{s(s+1)(s+5)} \Rightarrow \frac{\sum \text{pole} - \sum \text{zero}}{r} = \frac{0+(-1)+(-5)}{3} = -2$$



Root locus: Step 3 (Breakaway)

- *Breakaway points are among roots of* $\boxed{\frac{dL(s)}{ds} = 0}$

$$L(s) = \frac{1}{s(s+1)(s+5)} \Rightarrow \frac{dL(s)}{ds} = -\frac{3s^2+12s+5}{(*)} = 0$$

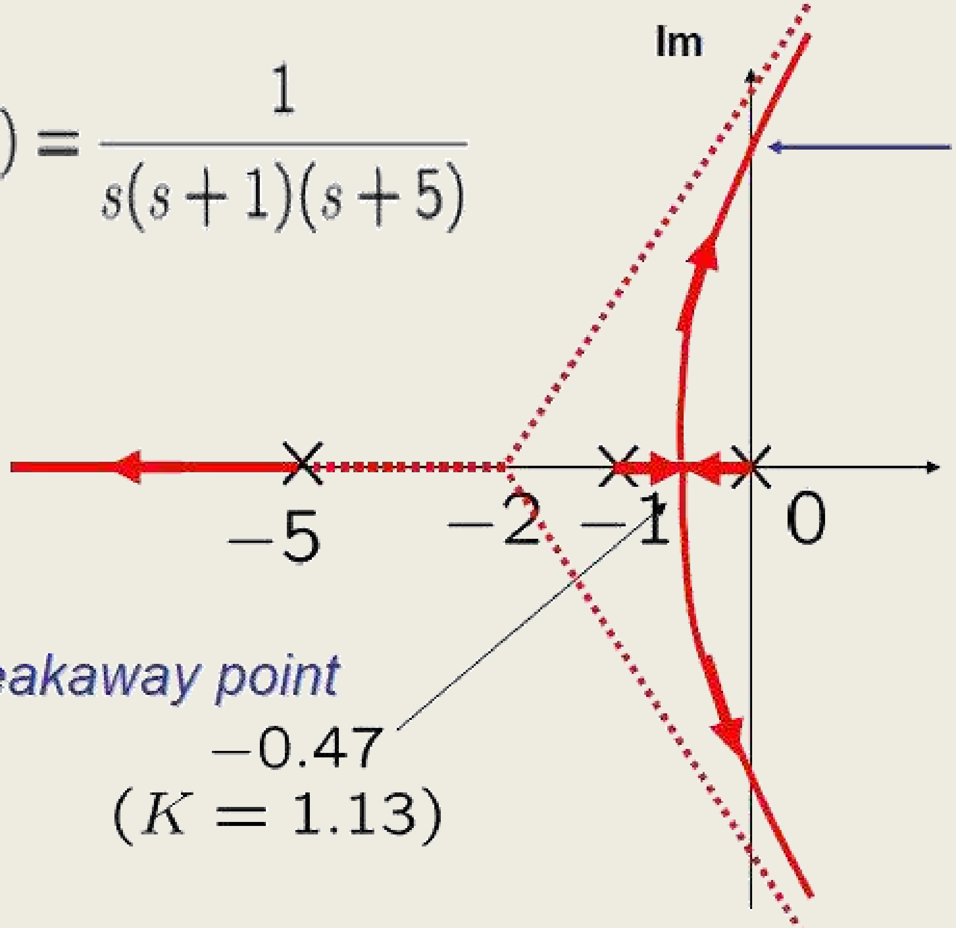
$$\Rightarrow s = -2 \pm \frac{\sqrt{21}}{3}$$

For each candidate s, check the positivity of $K = -\frac{1}{L(s)}$

$$\Rightarrow \begin{cases} s = -2 + \frac{\sqrt{21}}{3} \approx -0.47 & K \approx 1.13 \\ s = -2 - \frac{\sqrt{21}}{3} \approx -3.52 & K \approx -13.1 \end{cases}$$

Root locus: Step 3 (Breakaway)

$$L(s) = \frac{1}{s(s+1)(s+5)}$$



What is this value?
For what K?

Routh-Hurwitz!

Breakaway point
-0.47
$(K = 1.13)$

Finding K for critical stability

- Characteristic equation

$$1+\frac{K}{s(s+1)(s+5)}=0 \Leftrightarrow s^3+6s^2+5s+K=0$$

- Routh array

$$\begin{array}{c|cc} s^3 & 1 & 5 \\ s^2 & 6 & K \\ s^1 & \frac{30-K}{6} & \\ s^0 & K & \end{array}$$

Stability condition

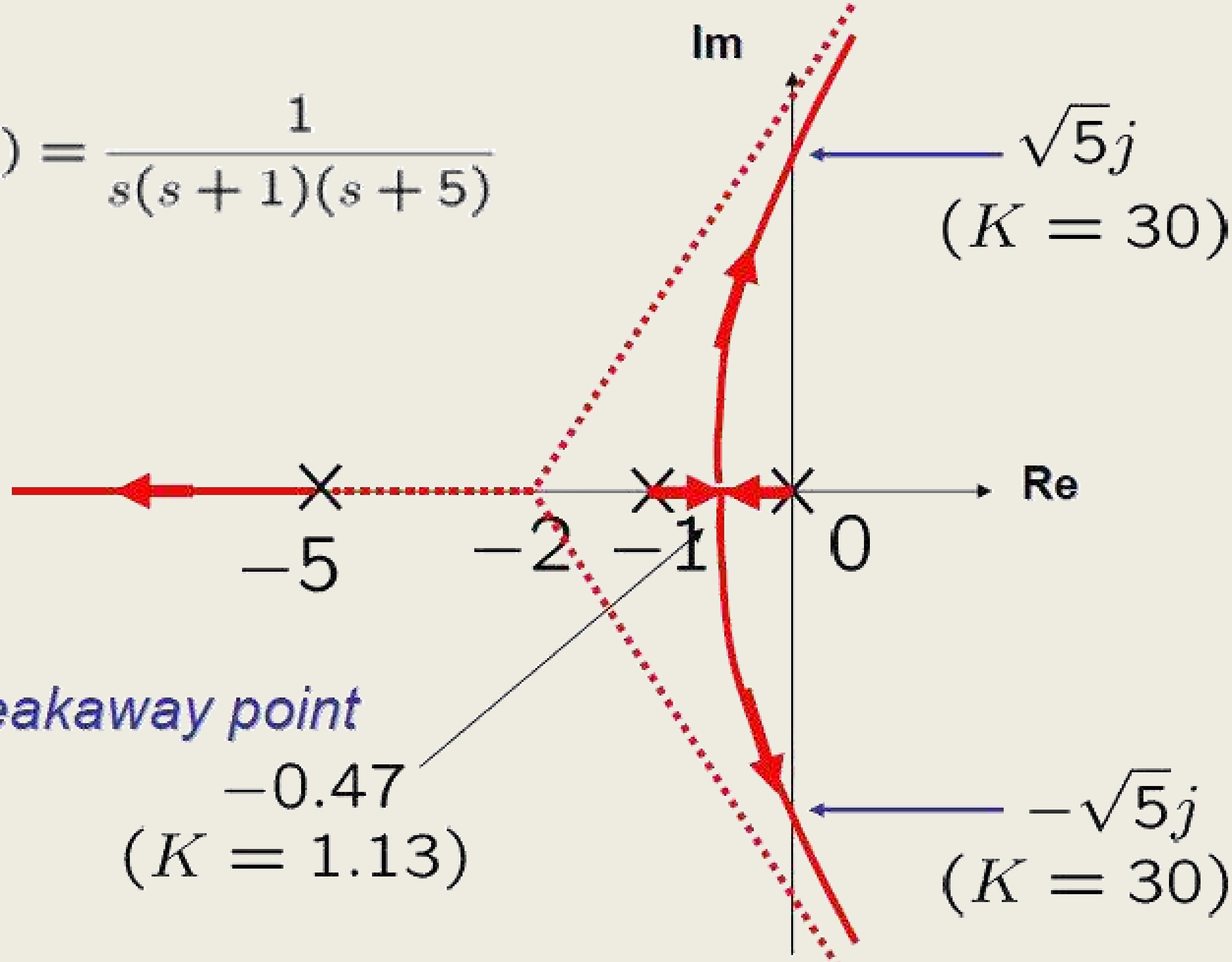
$0 < K < 30$

- When K=30

$$6s^2+30=0 \Rightarrow s=\pm\sqrt{5}j$$

Root locus

$$L(s) = \frac{1}{s(s+1)(s+5)}$$



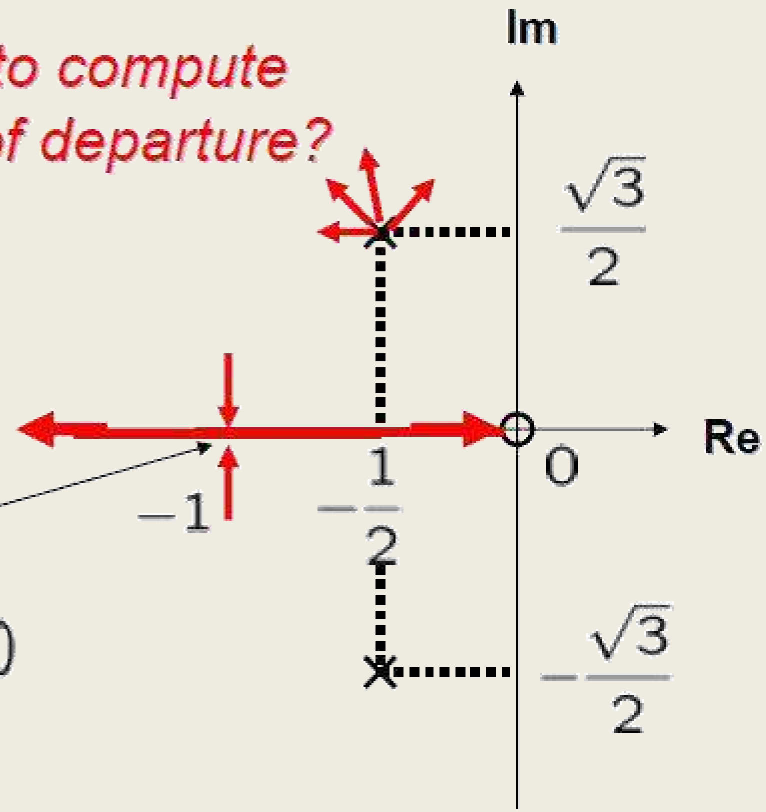
Example with complex poles

$$L(s) = \frac{s}{s^2 + s + 1}$$

After Steps 0,1,2,3, we obtain

zero	0
pole	$-\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$

How to compute angle of departure?



Breakaway point

$$s^2 + s + 1 - s(2s + 1) = 0$$

$$\Rightarrow s = \pm 1$$

Root locus sketching algorithm

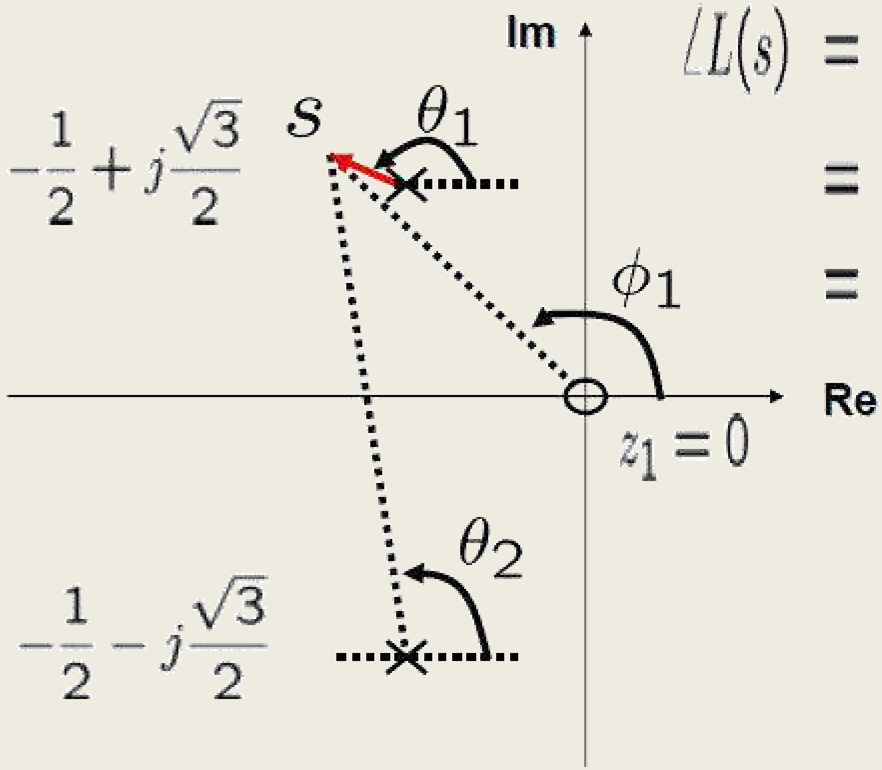
- Step 0: Mark open-loop poles and zeros
- Step 1: On the real axis
- Step 2: Asymptotes
- Step 3: Breakaway points
- Step 4: Angles of departures and arrivals

Root locus: Step 4
Angle of departure

- *Angle condition:* For a point "s" to be on RL,

$$p_1 = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$$

$$p_2 = -\frac{1}{2} - j\frac{\sqrt{3}}{2}$$



$$\angle L(s) = \angle \frac{s-z_1}{(s-p_1)(s-p_2)}$$
$$= \angle(s-z_1) - \angle(s-p_1) - \angle(s-p_2)$$
$$= \phi_1 - \theta_1 - \theta_2 = 180$$

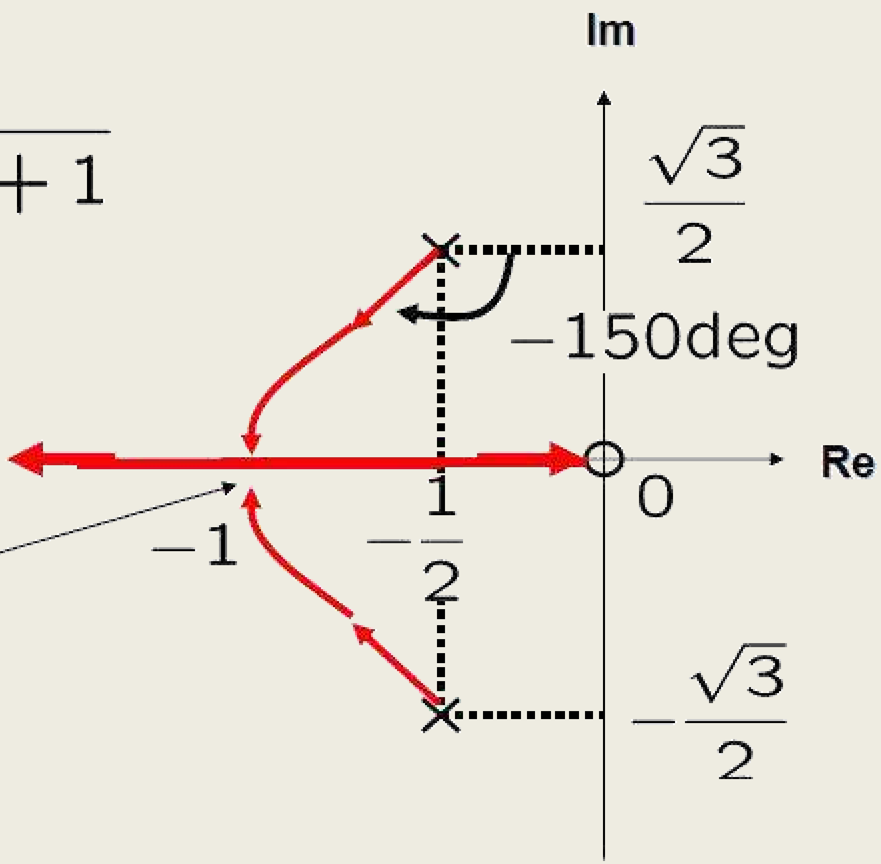
If s is close to p1

$$\phi_1 \approx 120, \ \ \theta_2 \approx 90$$

$$\Rightarrow \theta_1 = -150$$

Root locus

$$L(s) = \frac{s}{s^2 + s + 1}$$



Breakaway point

$$s = -1 \; (K = 1)$$

Course roadmap

Modeling

- ✓ Laplace transform
- ✓ Transfer function
- ✓ Models for systems
  - ✓ electrical
  - ✓ mechanical
  - ✓ electromechanical
- ✓ Linearization, delay

Analysis

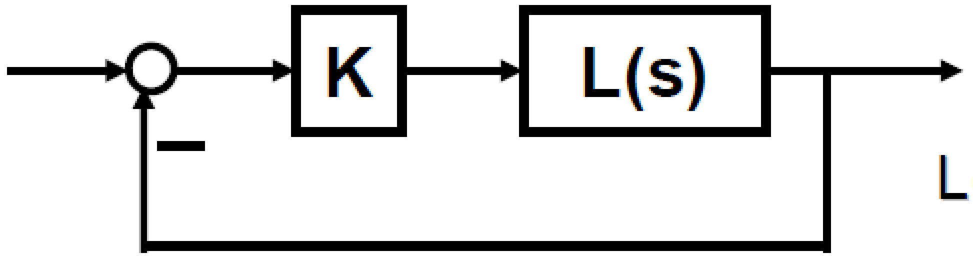
- ✓ Time response
  - ✓ Transient
  - ✓ Steady state
- Frequency response
  - Bode plot
- ✓ Stability
  - ✓ Routh-Hurwitz
  - Nyquist

Design

- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

What is Root Locus? (Review)

- *Pole location* of the feedback system characterizes stability and transient properties.
- Consider a feedback system that has one parameter (gain) K>0 to be designed.



L(s): open-loop TF

- *Root locus* graphically shows how poles of CL system varies as K varies from 0 to infinity.

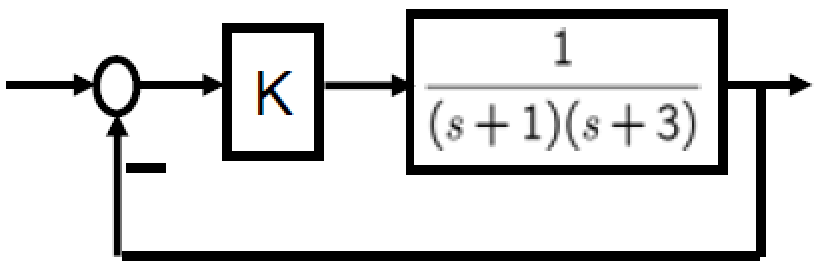
- How to use the root locus

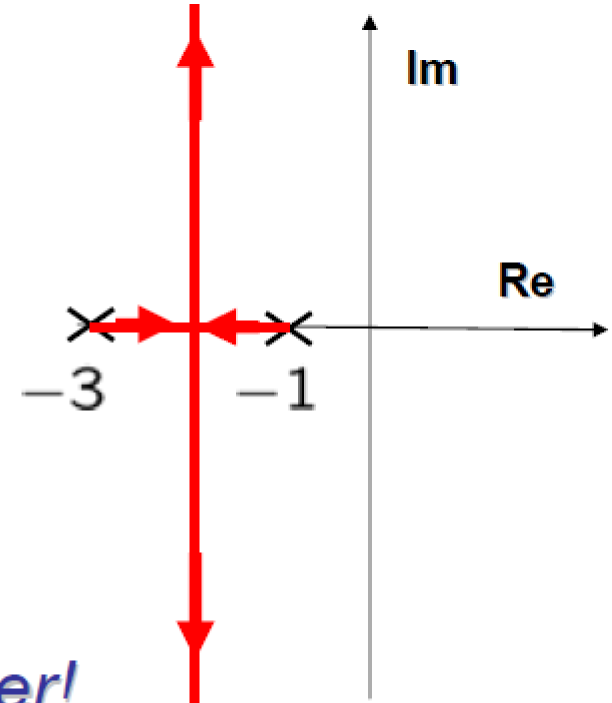
  (Being able to draw root locus is NOT good enough as a mechanical/control engineer!)

  - Example 1: Gain design to meet design specifications
  - Example 2: Pole location design
  - Example 3: Multiple parameter design
  - Example 4: Multiple parameter design

Example 1

- Design the gain K s.t.
  - Overshoot at most 5 %
  - 2% settling time at most 2 sec
  - Error constant Kp > 1





$$(\text{CL TF}) = \frac{K}{(s+1)(s+3)+K}$$

2nd order!

Example 1 (cont'd)

- Allowable region
  - Overshoot at most 5 %

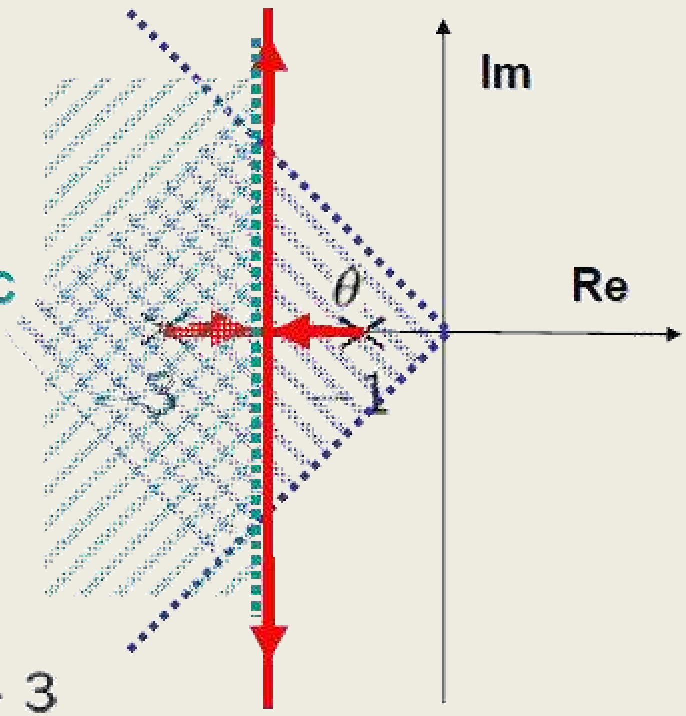
$$\theta \leq \frac{\pi}{4}$$

  - 2% settling time at most 2 sec

$$T_s = \frac{4}{\zeta\omega} \leq 2 \quad \Leftrightarrow \quad \zeta\omega \geq 2$$

- Error constant Kp > 1

$$K_p = KL(0) = \frac{K}{3} > 1 \quad \Leftrightarrow \quad K > 3$$



Example 1 (cont'd)

- Gain value computations

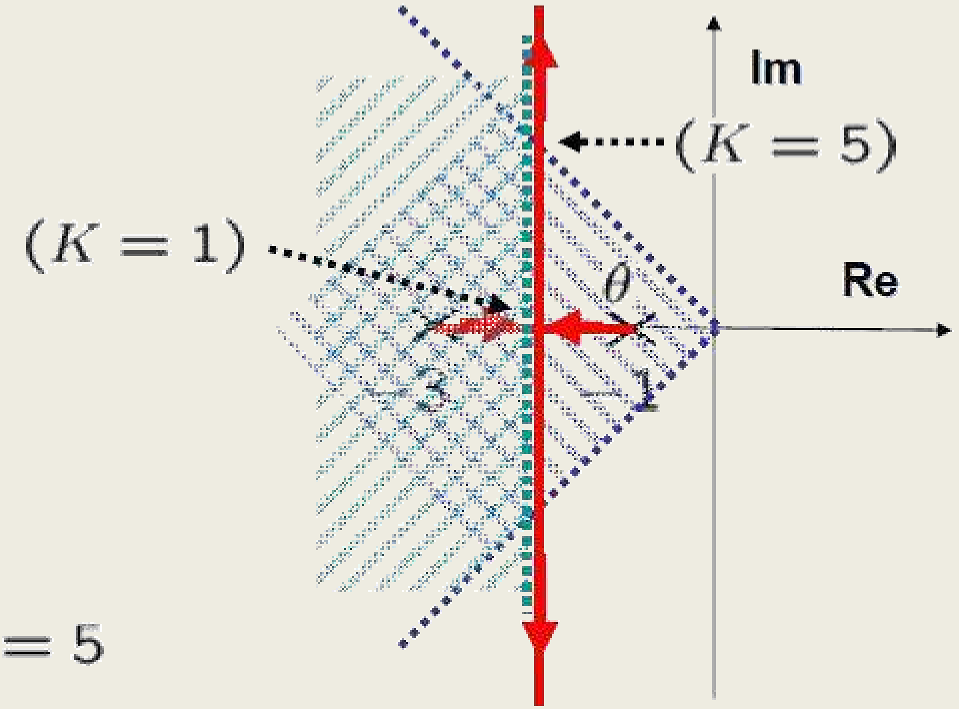
$$K = -\frac{1}{L(s)}$$

  - s=-2

$$K = -\frac{1}{L(-2)} = 1$$

  - s=-2+2j

$$K = -\frac{1}{L(-2+2j)} = 5$$



Example 1 (cont'd)

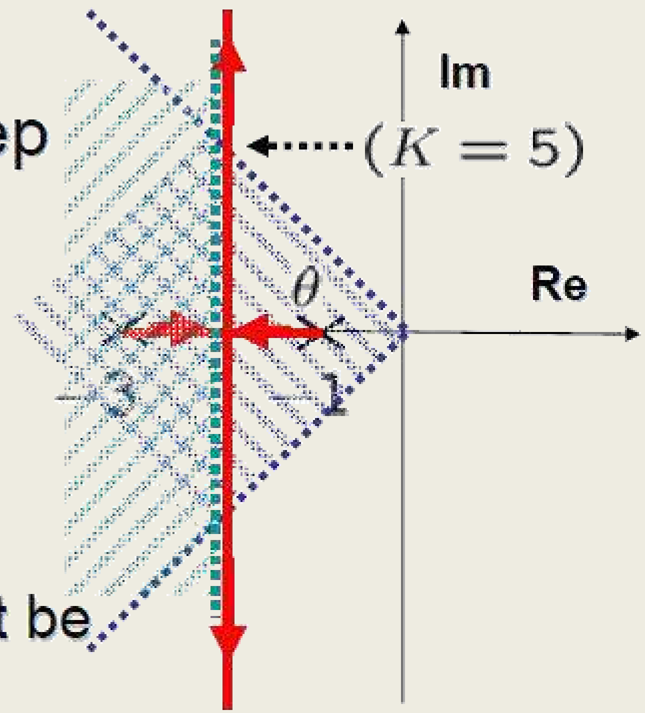
- Acceptable gain

  $$3 < K \leq 5$$

- Minimum SS error for unit step

  $$e_{ss} = \frac{1}{1+5/3} = \frac{3}{8}$$

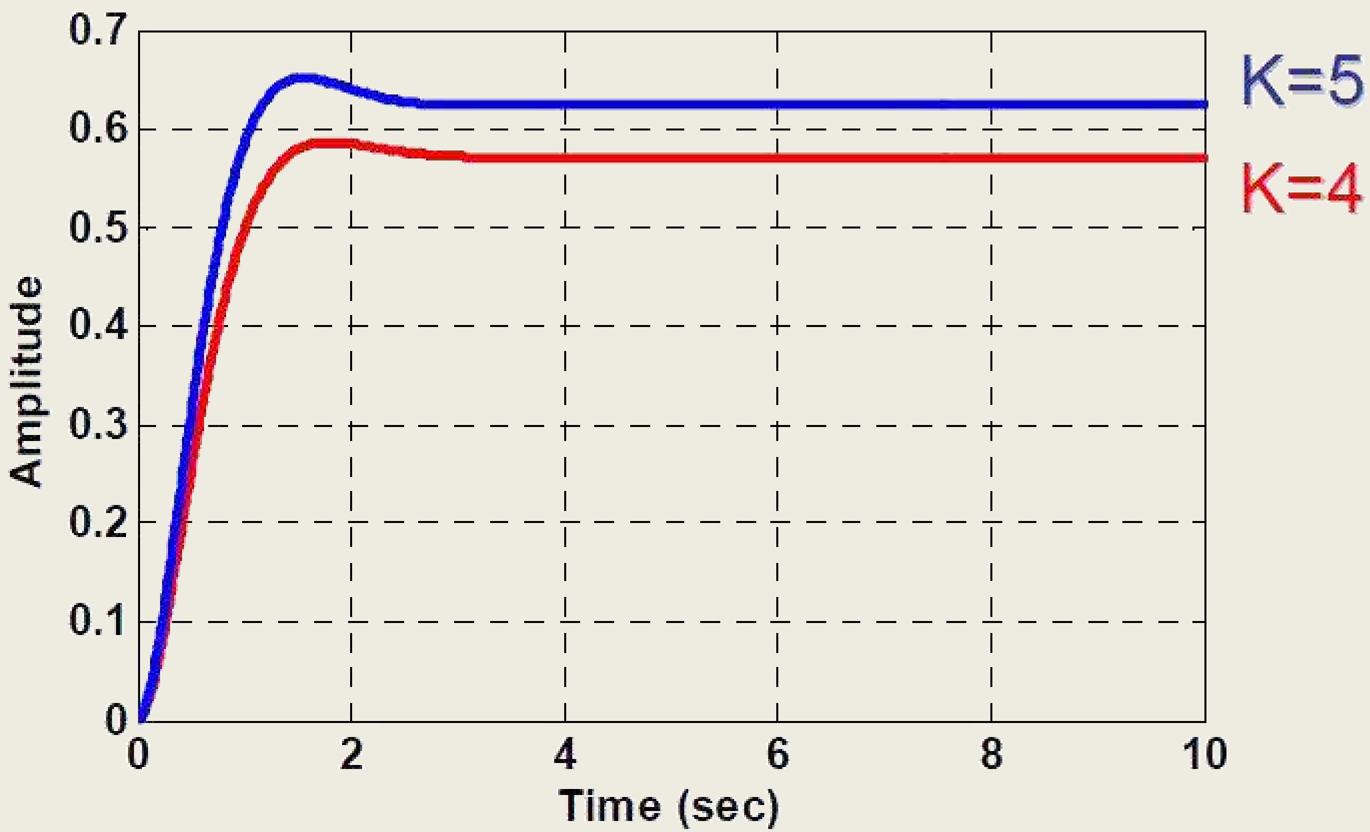
- Limitations of gain controller
  - Ts cannot be less than 2 sec
  - Overshoot and SS error cannot be improved simultaneously.



➡ *Lead-lag compensator design! (Next lecture)*

Example 1 (cont'd)

- Step responses



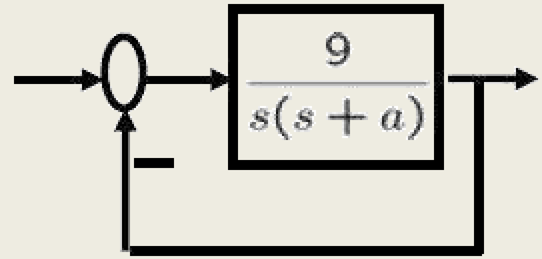
- How to use the root locus

(Being able to draw root locus is NOT good enough as a control engineer!)

  - Example 1: Gain design to meet design specifications
  - Example 2: Pole location design
  - Example 3: Multiple parameter design
  - Example 4: Multiple parameter design

Example 2

Draw root locus for a>0.

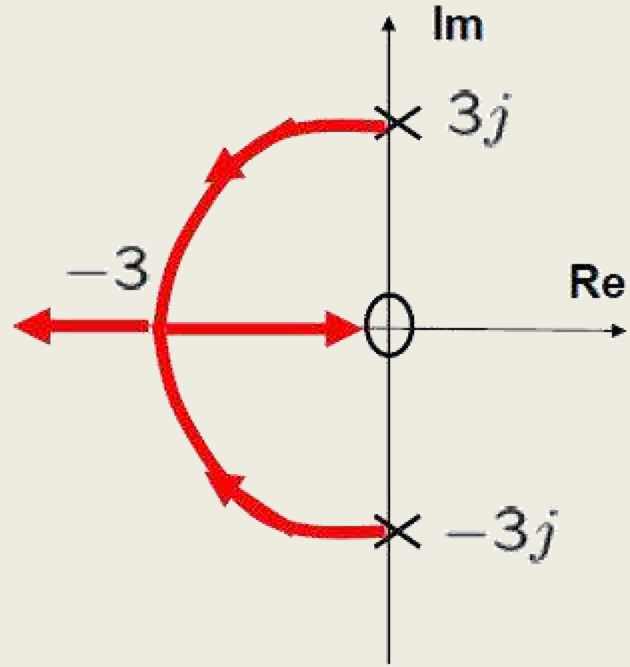


Characteristic eq.

$$1+\frac{9}{s(s+a)}=0$$

$$\Leftrightarrow \quad \underbrace{s^2+9}_{\text{Term without a}}+\underbrace{sa}_{\text{Term with a}}=0$$

$$\Leftrightarrow \quad 1+a\underbrace{\frac{s}{s^2+9}}_{L(s)}=0$$



Example 2 (cont'd)

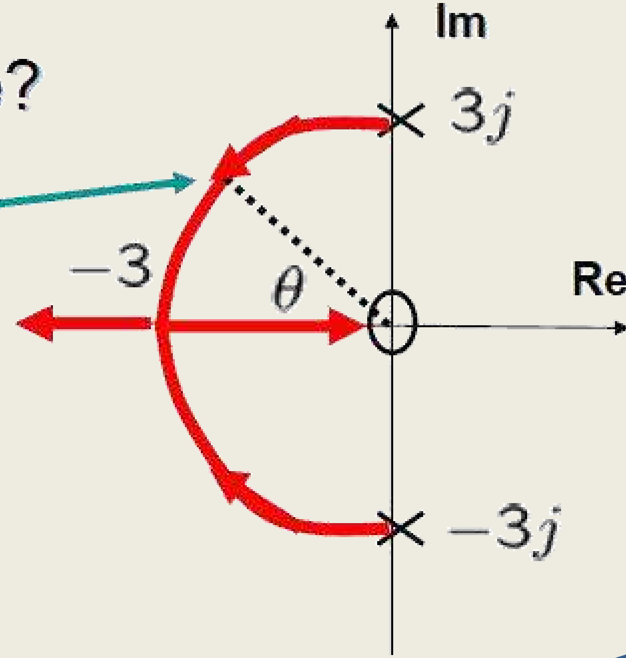
If you want damping ratio

$$1 + a\underbrace{\frac{s}{s^2+9}}_{L(s)} = 0$$

$$\zeta = \cos\theta, \quad \theta = \frac{\pi}{4}$$

What should the "a"-value be?

$$s_0 = -3cos\theta + 3jsin\theta$$



$$a = -\left.\frac{1}{L(s)}\right|_{s=s_0} = \cdots = 3\sqrt{2}$$

- How to use the root locus

  (Being able to draw root locus is NOT good enough as a control engineer!)

  - Example 1: Gain design to meet design specifications
  - Example 2: Pole location design
  - Example 3: Multiple parameter design
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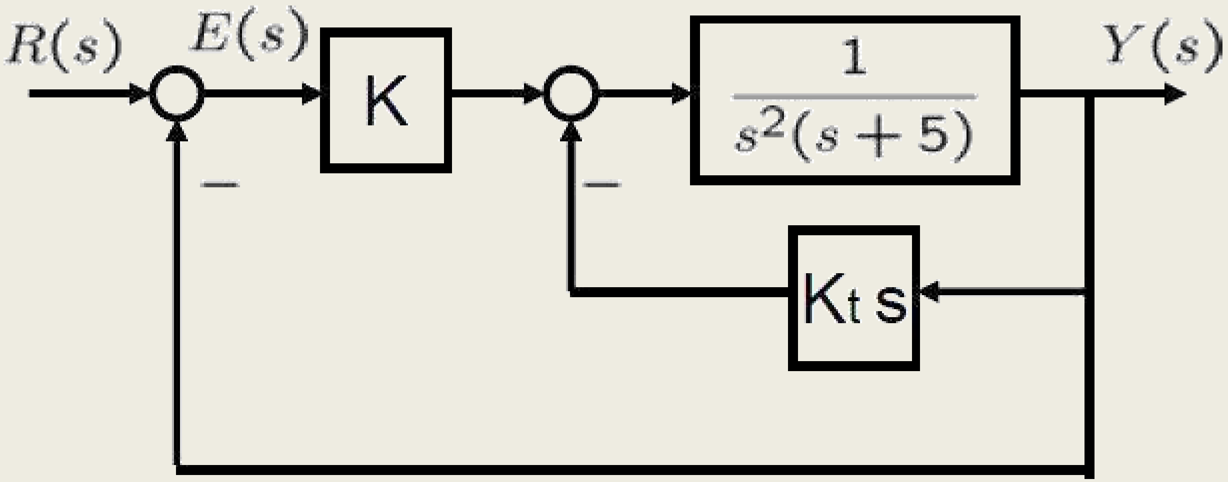
Example 3



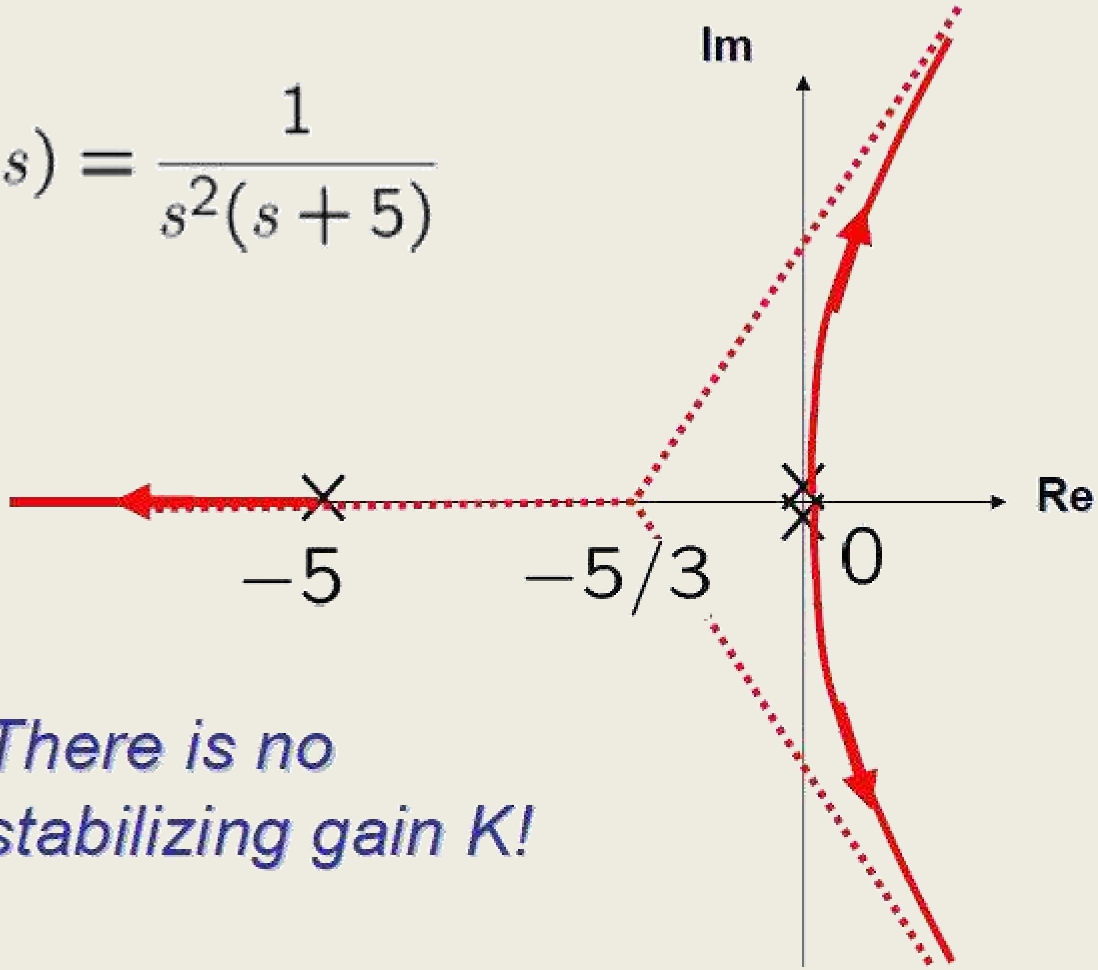
a) Set K_t=0. Draw root locus for K>0.

b) Set K=10. Draw root locus for K_t>0.

c) Set K=5. Draw root locus for K_t>0.

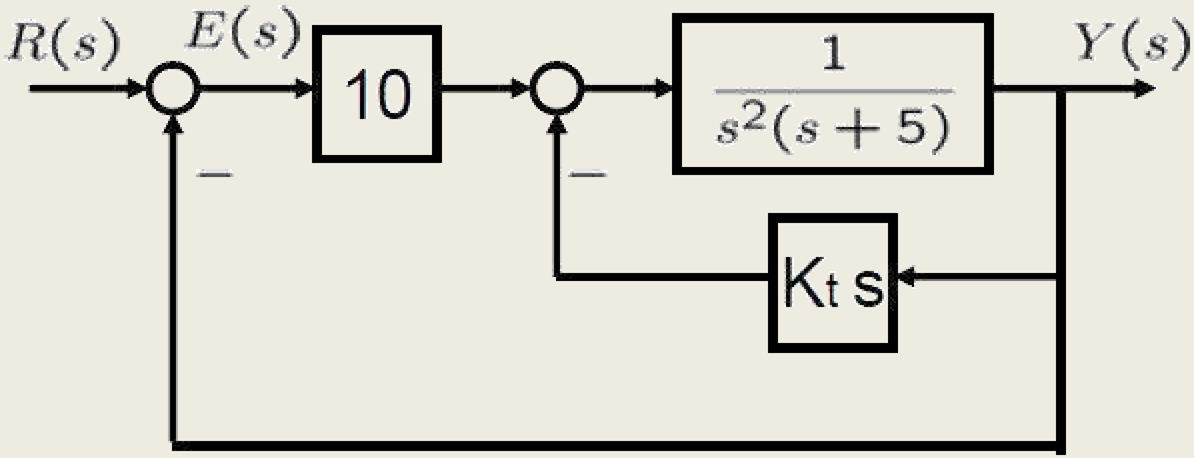
Example 3 (a): K_t=0

$$L(s) = \frac{1}{s^2(s+5)}$$



There is no stabilizing gain K!

Example 3 (b): K=10

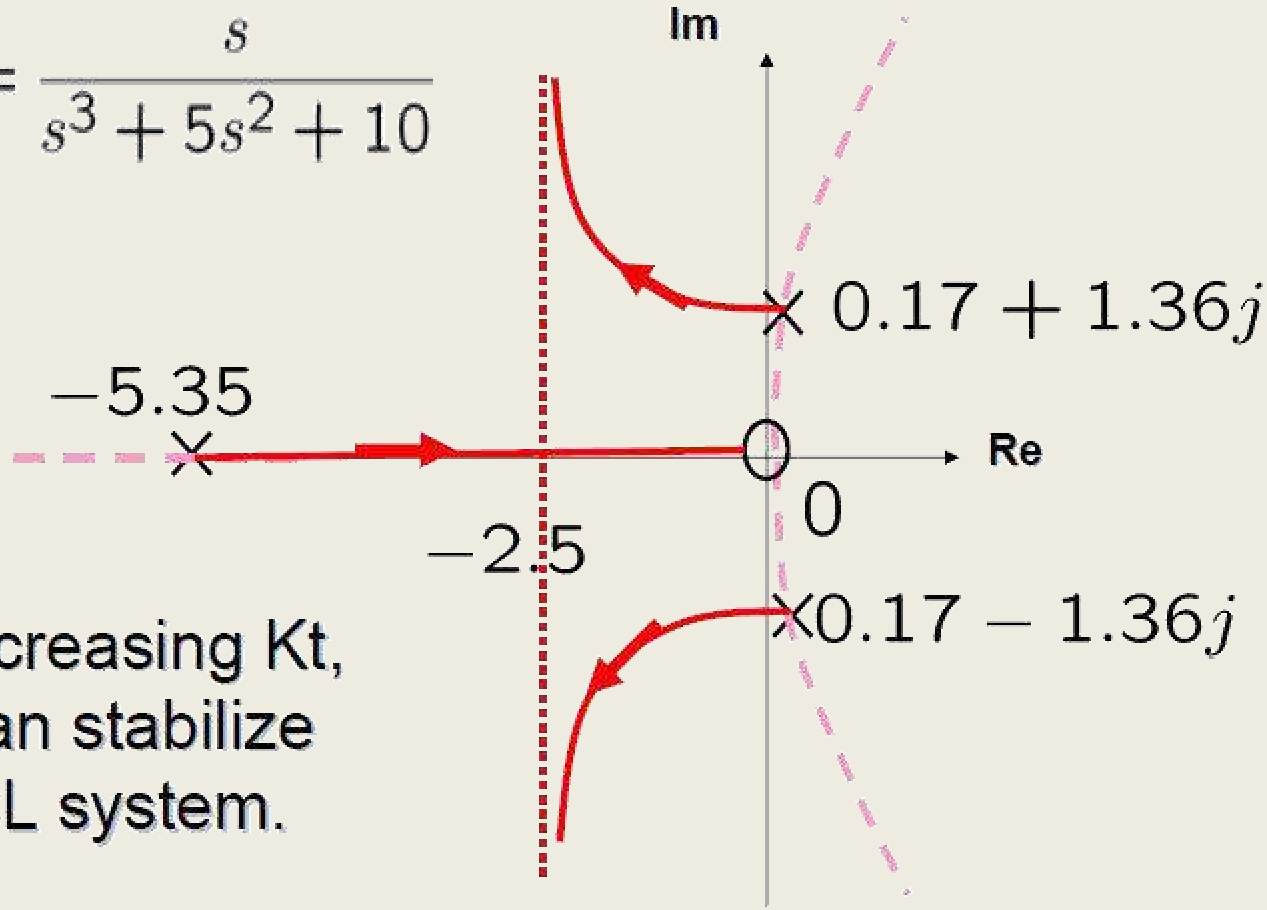


Characteristic eq. $$1+10\left(\frac{\frac{1}{s^2(s+5)}}{1+\frac{K_t s}{s^2(s+5)}}\right)=0$$

$$\Leftrightarrow \; s^2(s+5)+K_t s+10=0 \;\Leftrightarrow\; 1+K_t\underbrace{\frac{s}{s^3+5s^2+10}}_{L(s)}=0$$

Example 3 (b)

$$L(s) = \frac{s}{s^3 + 5s^2 + 10}$$



By increasing Kt, we can stabilize the CL system.

Finding Kt for marginal stability

- Characteristic equation

$$1+\frac{K_t s}{s^3+5s^2+10}=0 \Leftrightarrow s^3+5s^2+K_t s+10=0$$

- Routh array

$$\begin{array}{c|cc} s^3 & 1 & K_t \\ s^2 & 5 & 10 \\ s^1 & \frac{5K_t-10}{5} & \\ s^0 & 10 & \end{array}$$

Stability condition

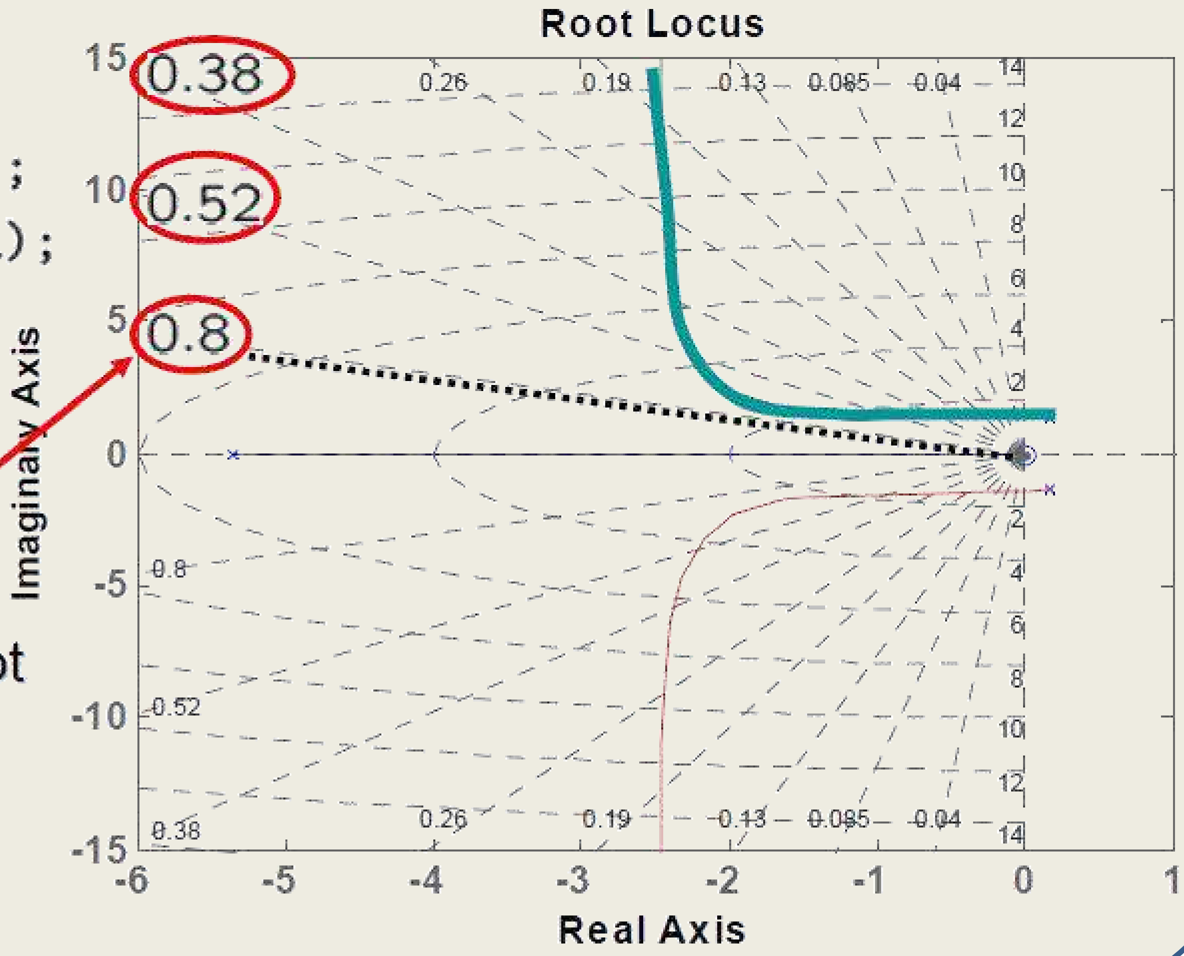
$K_t > 2$

- When Kt=2

$$5s^2+10=0 \Rightarrow s=\pm\sqrt{2}j$$

Matlab command “rlocus.m”

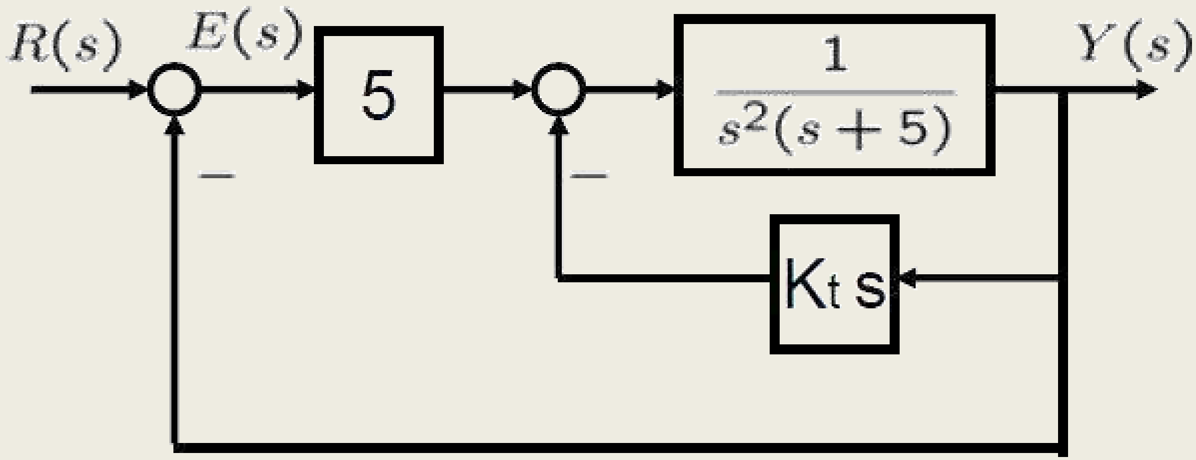
```
num=[1 0];
den=[1 5 0 10];
sys=tf(num,den);
rlocus(sys)
grid on
```



If K=10, we cannot achieve $\zeta = 0.8$

for any K_t>0.

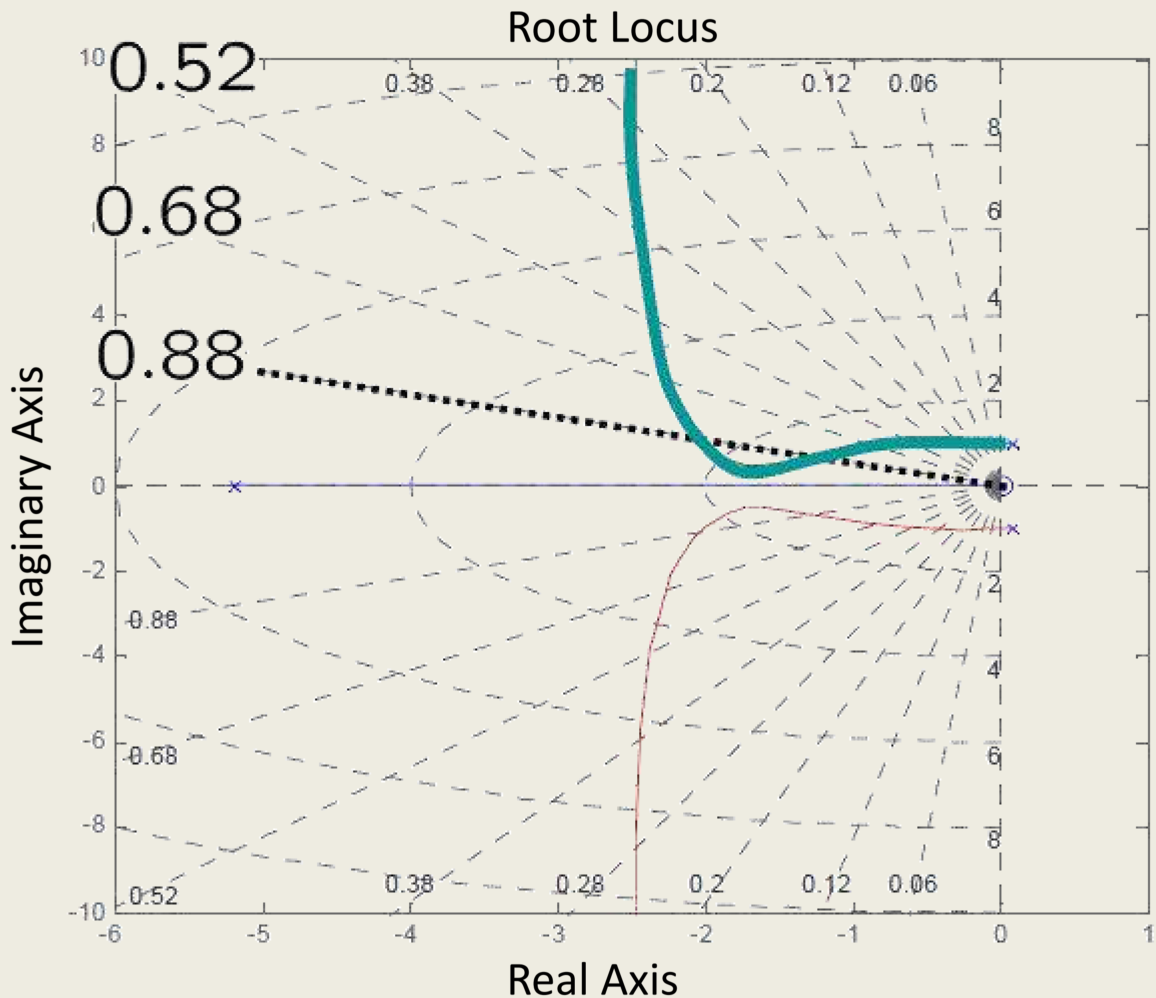
Example 3 (c): K=5



Characteristic eq. $$1+5\left(\frac{\frac{1}{s^2(s+5)}}{1+\frac{K_t s}{s^2(s+5)}}\right)=0$$

$$\Leftrightarrow \quad s^2(s+5)+K_t s+5=0 \quad \Leftrightarrow \quad 1+K_t\underbrace{\frac{s}{s^3+5s^2+5}}_{L(s)}=0$$

Example 3 (c): Root locus plot

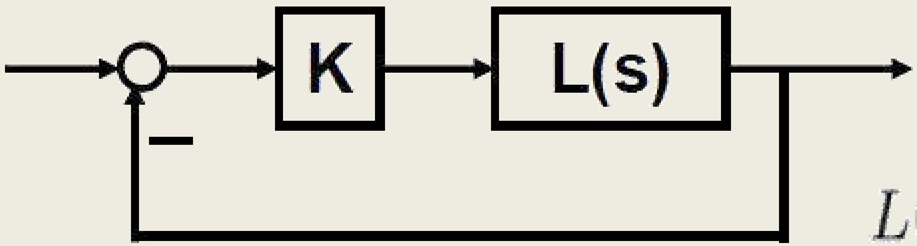


- How to use the root locus

  (Being able to draw root locus is NOT good enough as a control engineer!)

  - Example 1: Gain design to meet design specifications
  - Example 2: Pole location design
  - Example 3: Multiple parameter design
  - Example 4: Multiple parameter design

Example 4



$$L(s) = \frac{1 + Ts}{s(s+1)(s+2)}$$

a) Set T=0. Draw root locus for K>0.

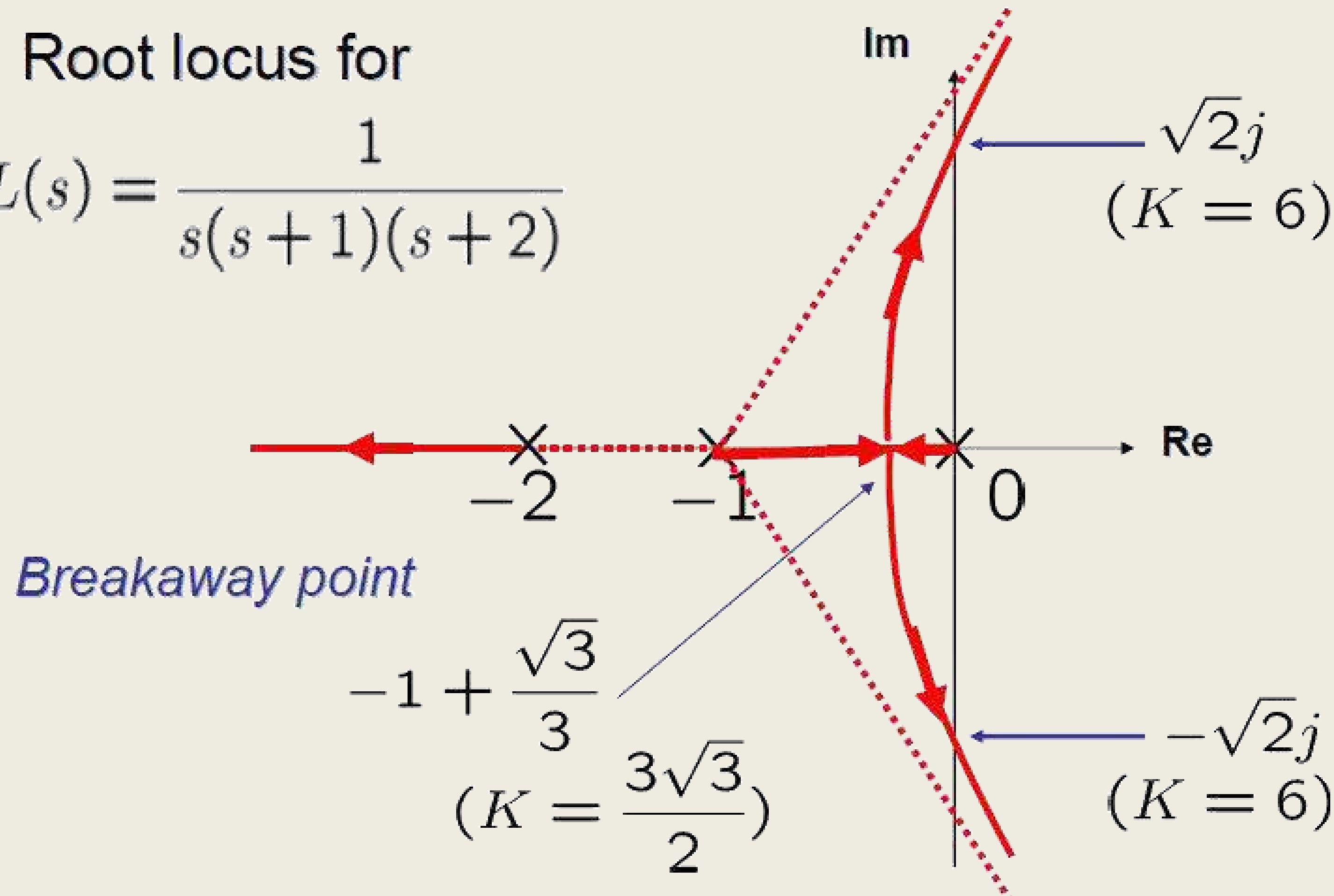
$$L(s) = \frac{1}{s(s+1)(s+2)}$$

b) Vary T to see the effect of a zero on root locus.

Example 4 (a)

- Root locus for

$$L(s) = \frac{1}{s(s+1)(s+2)}$$



Example 4 (b)

- When K is fixed and T is a positive parameter, the characteristic equation can be written as

$$1+K\frac{1+Ts}{s(s+1)(s+2)}=0$$

$$\Leftrightarrow \quad \underbrace{s(s+1)(s+2)+K}_{\text{Term without T}}+\underbrace{TKs}_{\text{Term with T}}=0$$

$$\Leftrightarrow \quad 1+T\frac{Ks}{s(s+1)(s+2)+K}=0$$

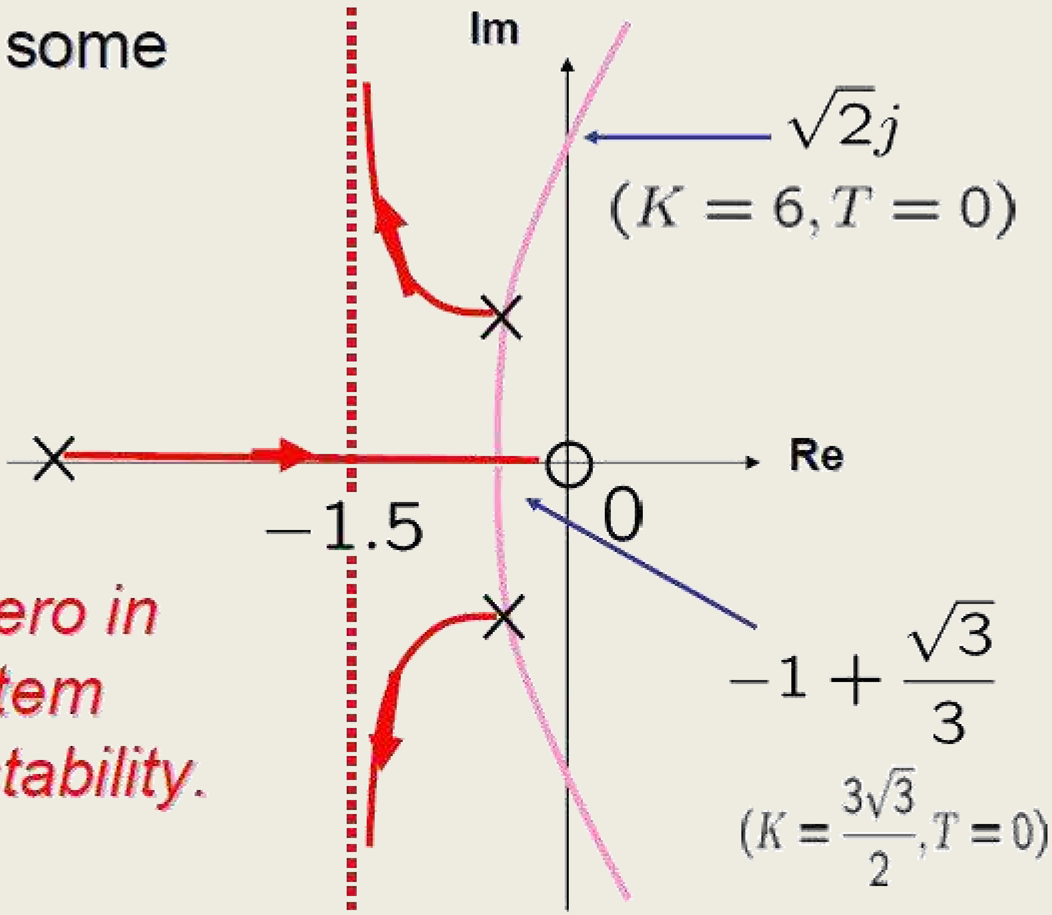
Example 4 (b) (cont'd)

- Root locus for some K & various T
- Zero of L(s):

$$s = -\frac{1}{T}$$

- Generally,

addition of a zero in open-loop system improves CL stability.



Example 4 (b)

- How to obtain intersections of asymptotes for

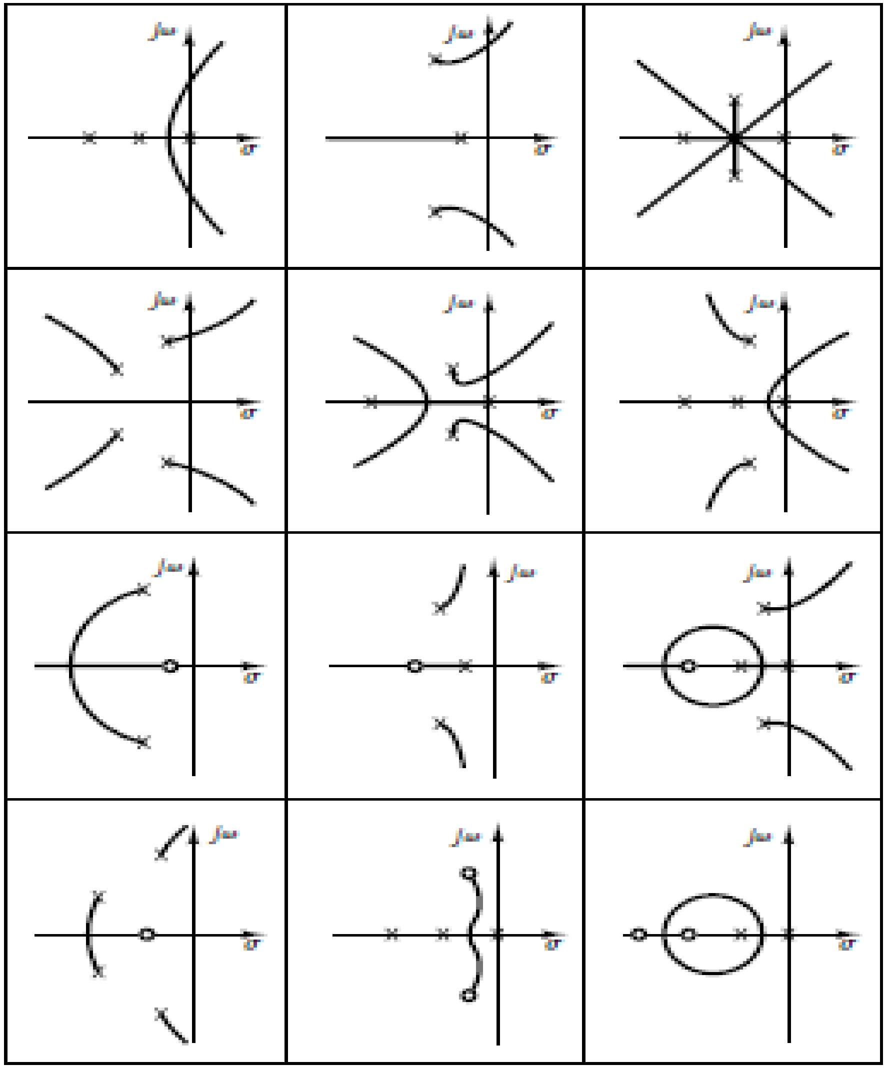
$$L(s) = \frac{s}{s(s+1)(s+2)+K}$$

$$\frac{\sum \text{pole} - \cancel{\sum \text{zero}}}{r} = \frac{\sum \text{pole}}{2}$$

$$(s-p_1)(s-p_2)(s-p_3) = s^3 - \underbrace{(p_1+p_2+p_3)}_{\sum \text{pole}} s^2 + \cdots$$

$$\boxed{\text{The coefficient of } s^2 \text{ is } -\sum \text{pole}}$$

Table 6–1 Open-Loop Pole–Zero Configurations and the Corresponding Root Loci



Summary

- Root Locus
  - What is root locus
  - How to roughly sketch root locus
  - Sketching root locus relies heavily on experience (PRACTICE)
  - To accurately draw root locus, use Matlab
- How to use the root locus
  - Example 1: Gain design to meet design specifications
  - Example 2: Pole location design
  - Example 3, 4: Multiple parameter design

  ==

- Next
  - esign Based on Root Locus
  - Lecture5: Assignment (Soon)

Assignment

- Sketch the root locus of the following open-loop transfer functions $(L(s))$.

(a) $L(s) = \frac{1}{s}$

(b) $L(s) = \frac{1}{s^2}$

(c) $L(s) = \frac{1}{s^3}$

(d) $L(s) = \frac{1}{s(s+4)}$

(e) $L(s) = \frac{s+1}{s(s+2)}$

(f) $L(s) = \frac{1}{s(s+1)(s+5)}$

(g) $L(s) = \frac{s}{s^2+s+1}$

(h) $L(s) = \frac{s+1}{s^2}$

(i) $L(s) = \frac{s}{(s+1)(s^2+2)}$

(j) $L(s) = \frac{s+3}{(s+1)(s^2+4s+5)}$

End of Lecture 6