Communications Principals (EE 320)

Lecture 2

Fourier Representation of Signals and Systems

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The Fourier Transform

Let $g(t)$ denote a nonperiodic deterministic signal, expressed as some function of time t. The Fourier transform of the signal is given by the integral

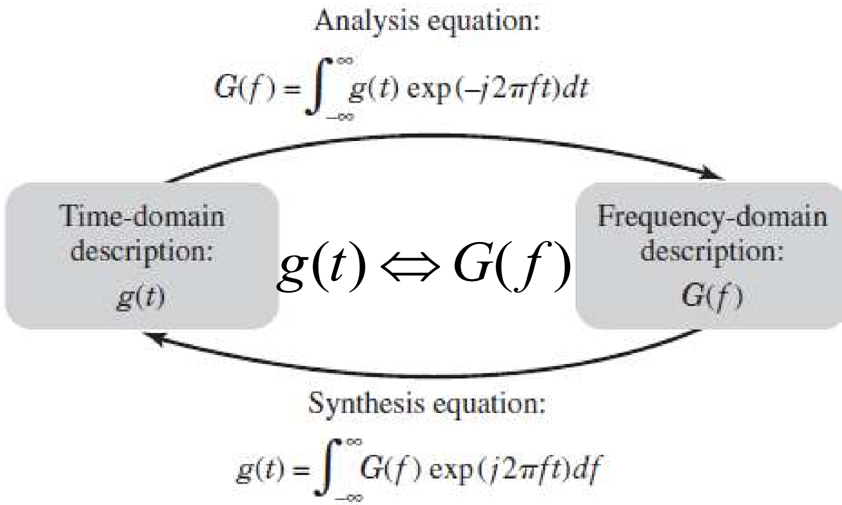
$$G(f) = \int_{-\infty}^{\infty} g(t)\exp(-j2\pi ft)dt$$

where $j = \sqrt{-1}$ and the variable f denotes *frequency.*

Given the Fourier transform $G(f)$, the original signal $g(t)$ is recovered exactly using the formula for the *inverse Fourier transform:*

$$g(t) = \int_{-\infty}^{\infty} G(f)\exp(j2\pi ft)df$$

$$g(t) = F^{-1}[G(f)]$$



$$G(f) = F[g(t)]$$

Continuous Spectrum

In general, the Fourier transform $G(f)$ is a complex function of frequency so that we may express it in the form

$$G(f) = |G(f)| \exp[j\theta(f)]$$

where $|G(f)|$ is called the continuous amplitude spectrum of $g(t)$, and $\theta(f)$ is called the continuous phase spectrum of $g(t)$.

For the special case of a real-valued function $g(t)$ we have

$$G(-f) = G^*(f)$$

$$|G(-f)| = |G(f)|$$

The magnitude spectrum is an even function of f.

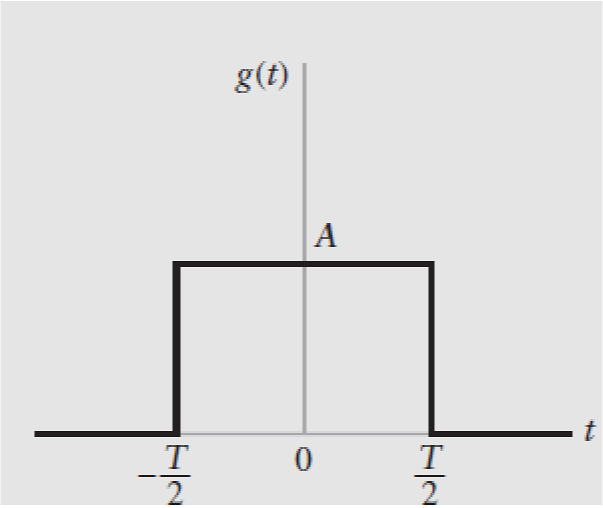
$$\theta(-f) = -\theta(f)$$

The phase spectrum is an odd function of f.

Fourier Transform of a Rectangular Pulse

A rectangular function of unit amplitude and unit duration centered at $t = 0$ is expressed as

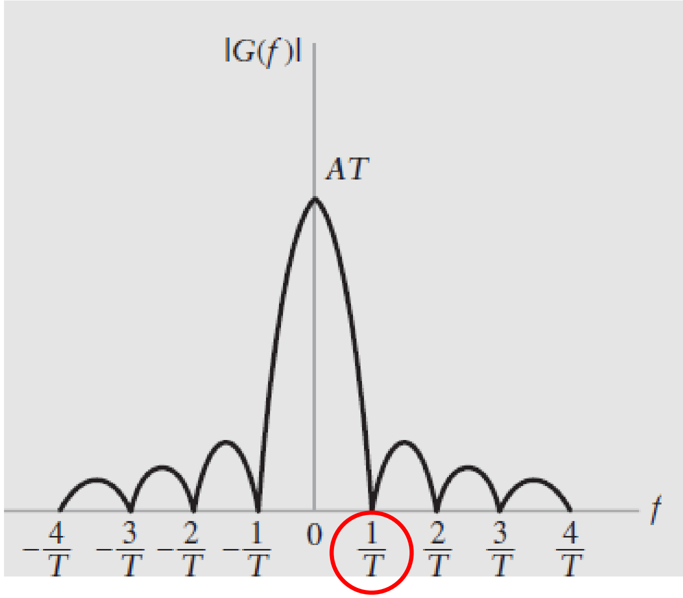
$$\text{rect}(t) = \begin{cases} 1 & -\frac{1}{2} \le t \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$



A rectangular pulse $g(t)$ of duration T and amplitude A centered at $t = 0$ is expressed as

$$g(t) = A\,\text{rect}(t/T)$$

$$G(f) = \int_{-T/2}^{T/2} A \exp(-j2\pi f t)dt = AT\left(\frac{\sin(\pi fT)}{\pi fT}\right)$$



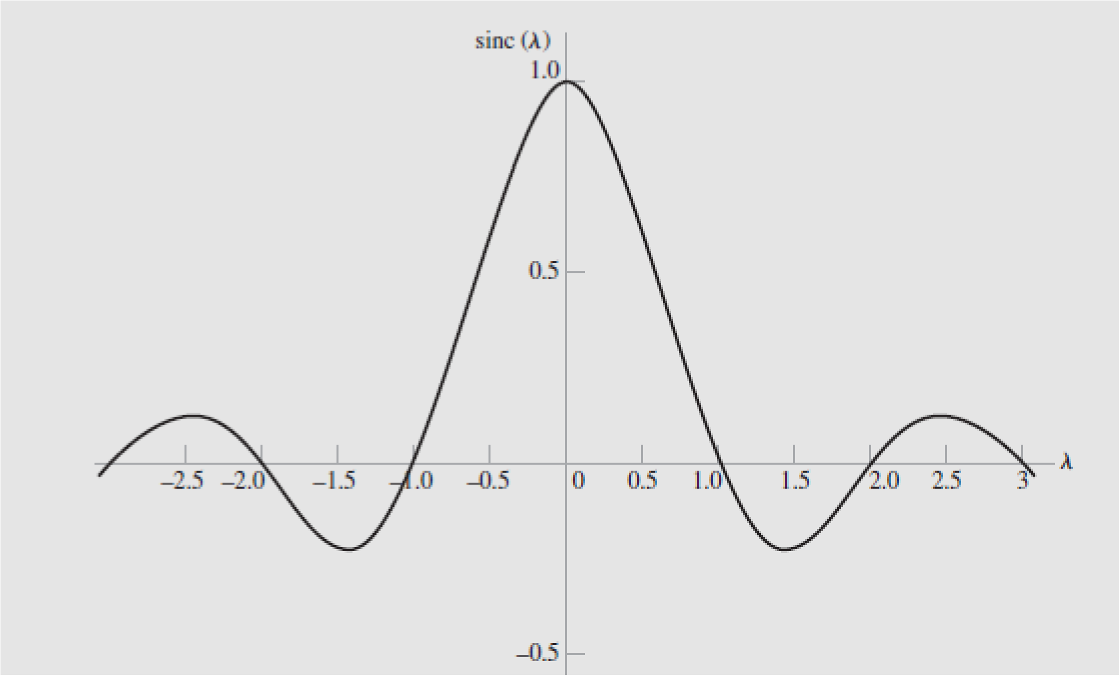
$G(f)$ can be expressed in terms of the sinc(.) function as follows

$$G(f) = AT\text{sinc}(fT)$$

$$A\,\text{rect}(t/T) \Leftrightarrow AT\text{sinc}(fT)$$

Properties of the sinc(.) function

$$\text{sinc}(\lambda) = \frac{\sin(\pi\lambda)}{\pi\lambda}$$

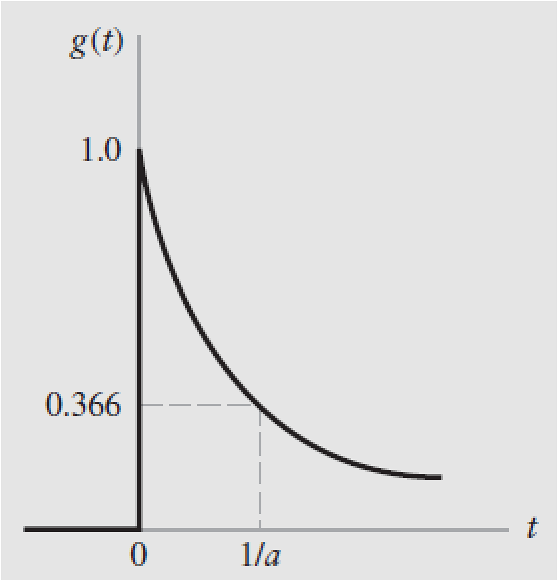


- it has its maximum value of unity at $\lambda=0$
- it approaches zero as λ approaches infinity,
- It is oscillating through positive and negative values.
- It goes through zero at ± 1, $\pm$ 2, $\pm$ 3, and so on.

Fourier Transform of a Decaying Exponential Pulse

A decaying exponential pulse starting at $t = 0$ can be expressed as

$$g(t) = \exp(-at)u(t)$$



where $u(t)$ is the unit step function defined as

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

$$G(f) = \int_0^{\infty} \exp(-at)\ \exp(-j2\pi f t)dt$$

$$= \int_0^{\infty} \exp\big(-(a + j2\pi f)t\big)dt$$

$$= \frac{1}{-(a + j2\pi f)} \exp\big(-(a + j2\pi f)t\big)\Big|_0^{\infty}$$

$$= \frac{1}{(a + j2\pi f)}$$

$$|G(f)| = \frac{1}{\sqrt{a^2 + (2\pi f)^2}}$$

$$\theta(f) = -\tan^{-1}\big(2\pi f / a\big)$$

Properties of the Fourier Transform

1- Linearity (Superposition)

Let $g_1(t) \rightleftharpoons G_1(f)$ *and* $g_2(t) \rightleftharpoons G_2(f)$. *Then for all constants* c_1 *and* c_2, *we have*

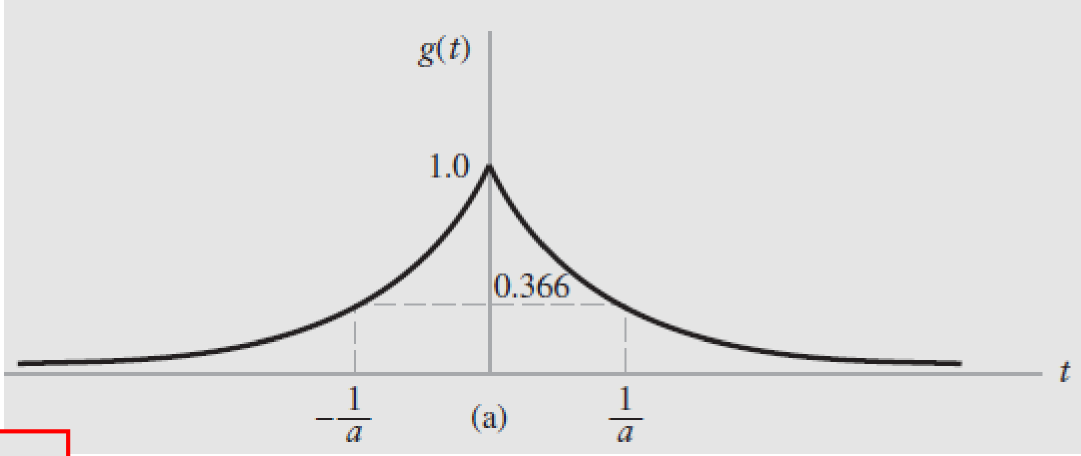
$$c_1 g_1(t) + c_2 g_2(t) \rightleftharpoons c_1 G_1(f) + c_2 G_2(f)$$

Example: Given that

$$\exp(-at)u(t) \rightleftharpoons \frac{1}{a + j2\pi f}$$ and $$\exp(at)u(-t) \rightleftharpoons \frac{1}{a - j2\pi f}$$

use the linearity property to find the FT of the signal

$$g(t) = \exp(-a|t|) = \begin{cases} \exp(-at), & t > 0 \\ 1, & t = 0 \\ \exp(at), & t < 0 \end{cases}$$



Since $g(t)$ equals the summation of the above two functions, we have

$$G(f) = \frac{1}{a + j2\pi f} + \frac{1}{a - j2\pi f} = \frac{2a}{a^2 + (2\pi f)^2}$$

Properties of the Fourier Transform

2- Dilation (Similarity) Property

Let $g(t) \rightleftharpoons G(f)$. *Then, the dilation property or similarity property states that*

$$g(at) \rightleftharpoons \frac{1}{|a|} G\left(\frac{f}{a}\right)$$

For the special case when $a = -1$, the dilation rule reduces to the reflection property, which states that if $g(t) \Leftrightarrow G(f)$, then $g(-t) \rightleftharpoons G(-f)$

Example: The Fourier transform of the signal $g_1(t) = \exp(at)\, u(-t)$ is given by $G_1(f) = \frac{1}{a+j2\pi f}$. Find the Fourier transform of the signal $g_2(t) = \exp(-at)\, u(t)$

Solution: Since $g_2(t) = g_1(-t)$, then we have $G_2(f) = G_1(-f)$ that is

$$G_2(f) = \frac{1}{(a - j2\pi f)}$$

Properties of the Fourier Transform

3- Duality Property

If $g(t) \rightleftharpoons G(f)$, *then*

$$G(t) \rightleftharpoons g(-f)$$

Example: The Fourier transform of the signal $g_1(t) = A\ \mathrm{rect}(t/T)$ is given by $A\,\mathrm{rect}(t/T) \Leftrightarrow AT\,\mathrm{sinc}(fT)$. Use the duality property to calculate the Fourier transform of the signal $g_2(t) = A\ \mathrm{sinc}(2Wt)$.

Solution:

$$A\,\mathrm{rect}(t/T) \Leftrightarrow AT\,\mathrm{sinc}(fT)$$

Duality property →

$$AT\,\mathrm{sinc}(tT) \Leftrightarrow A\,\mathrm{rect}(f/T)$$

Dividing by T →

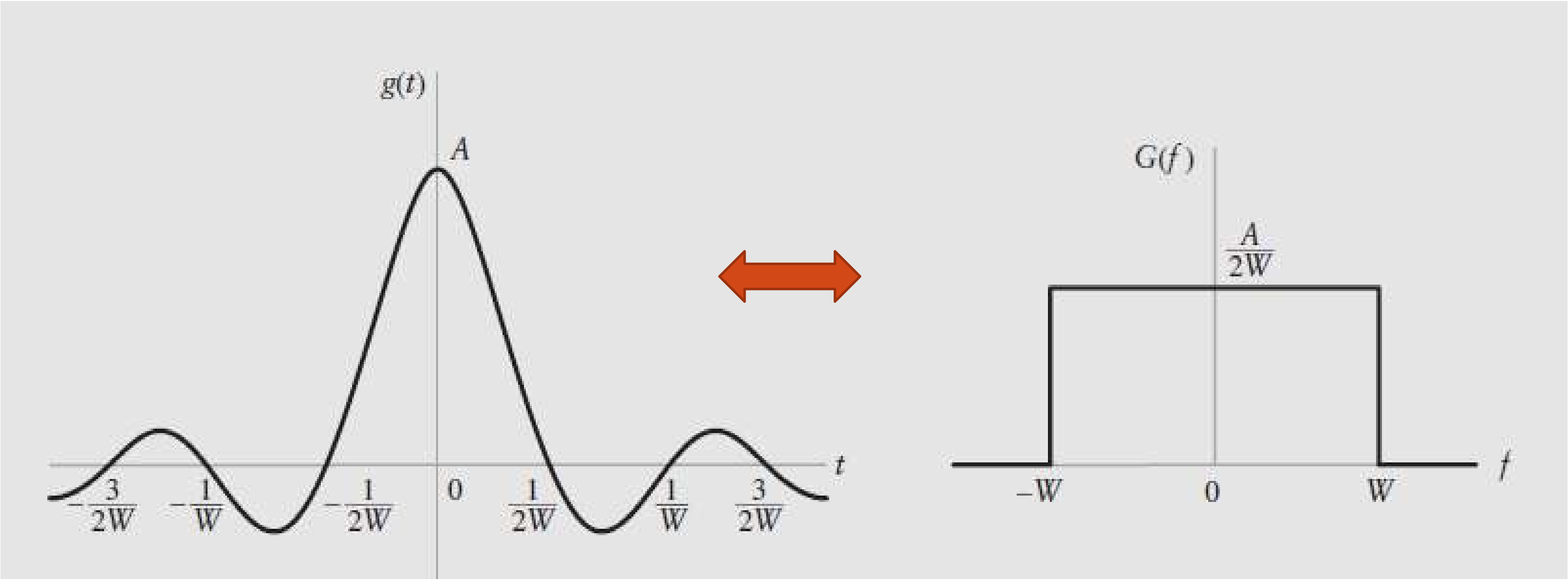
$$A\,\mathrm{sinc}(tT) \Leftrightarrow \frac{A}{T}\mathrm{rect}(f/T)$$

Replacing T → 2W →

$$A\,\mathrm{sinc}(2tW) \Leftrightarrow \frac{A}{2W}\mathrm{rect}(f/2W)$$

Properties of the Fourier Transform

3- Duality Property (Cont)



Properties of the Fourier Transform

4- Time-Shift Property

If $g(t) \rightleftharpoons G(f)$, *then*

$$g(t - t_0) \rightleftharpoons G(f) \exp(-j2\pi f t_0)$$

where t_0 is a real constant time shift.

The time-shifting property states that:

- if a function $g(t)$ is shifted along the time axis by an amount t_0, the effect is equivalent to multiplying its Fourier transform $G(f)$ by the factor $\exp(-j2\pi f t_0)$.
- This means that the amplitude of $G(f)$ is unaffected by the time shift, but its phase is changed by the linear factor $-j2\pi f t_0$ which varies linearly with frequency f.

Properties of the Fourier Transform

5- Frequency-Shift Property

If $g(t) \rightleftharpoons G(f)$, *then*

$$\exp(j2\pi f_c t)g(t) \rightleftharpoons G(f - f_c) \qquad (1)$$

where f_c is a real constant frequency.

The time-shifting property states that multiplication of a function $g(t)$ by the factor $\exp(j2\pi f_c t)$ is equivalent to shifting its Fourier transform $G(f)$ along the frequency axis by the amount f_c. Similarly, we have

$$\exp(-j2\pi f_c t)g(t) \rightleftharpoons G(f + f_c) \qquad (2)$$

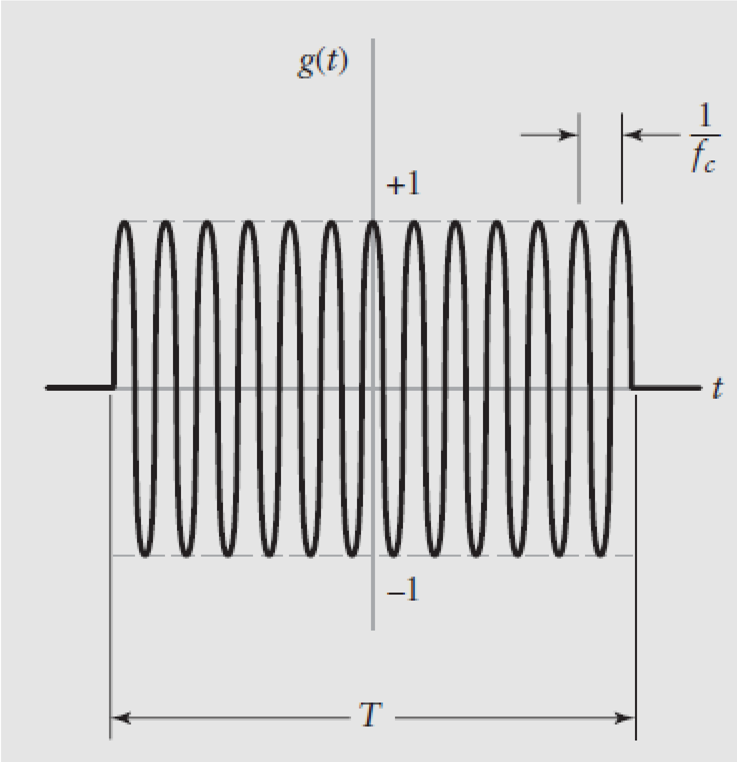
Since $\cos(2\pi f_c t) = 0.5\{\exp(j2\pi f_c t) + \exp(-j2\pi f_c t)\}$ Eq. (1) and (2) can be combined to produce the following important relation

$$g(t)\cos(2\pi f_c t) \Leftrightarrow \frac{1}{2}\{G(f - f_c) + G(f + f_c)\}$$

Properties of the Fourier Transform

5- Frequency-Shift Property

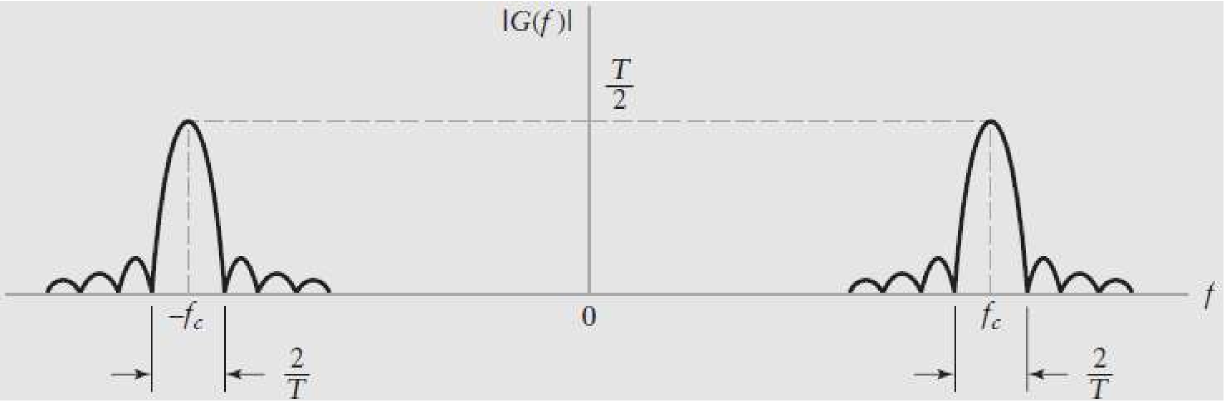
Example: Calculate The Fourier transform of the signal $g(t)$ shown in the following figure. Note that $g(t)$ can be expressed as $g(t) = g_1(t)\cos(2\pi f_c t)$, where $g_1(t) = \mathrm{rect}(t/T)$.



$$G_1(f) = T\,\mathrm{sinc}(fT)$$

$$g(t) = g_1(t)\cos(2\pi f_c t) \Leftrightarrow \frac{1}{2}\{G_1(f - f_c) + G_1(f + f_c)\}$$

$$G(f) = \frac{T}{2}\{\mathrm{sinc}((f - f_c)T) + \mathrm{sinc}((f + f_c)T)\}$$



Properties of the Fourier Transform

6- Area Under *g*(*t*) *If* $g(t) \rightleftharpoons G(f)$, *then*

$$\int_{-\infty}^{\infty} g(t)\, dt = G(0)$$

That is, the area under a function $g(t)$ is equal to the value of its Fourier transform $G(f)$ at $f = 0$.

7- Area Under *G*(*f*) *If* $g(t) \rightleftharpoons G(f)$, *then*

$$g(0) = \int_{-\infty}^{\infty} G(f)\, df$$

That is, the value of the function $g(t)$ at $t = 0$ is equal to the area under its Fourier transform $G(f)$.

Properties of the Fourier Transform

8- Modulation Property *Let* $g_1(t) \rightleftharpoons G_1(f)$ *and* $g_2(t) \rightleftharpoons G_2(f)$. *Then*

$$g_1(t)g_2(t) \rightleftharpoons \int_{-\infty}^{\infty} G_1(\lambda)G_2(f-\lambda)\, d\lambda$$

That is, the multiplication of two signals in the time domain is transformed into the convolution of their individual Fourier transforms in the frequency domain.

$$g_1(t)g_2(t) \rightleftharpoons G_1(f) \star G_2(f)$$

9- Convolution Property *Let* $g_1(t) \rightleftharpoons G_1(f)$ *and* $g_2(t) \rightleftharpoons G_2(f)$. *Then*

$$g_1(t) \star g_2(t) \rightleftharpoons G_1(f)G_2(f)$$

$$\int_{-\infty}^{\infty} g_1(\tau)g_2(t-\tau)\, d\tau \rightleftharpoons G_1(f)G_2(f)$$

That is, the convolution of two signals in the time domain is transformed into the multiplication of their individual Fourier transforms in the frequency domain.

Properties of the Fourier Transform

10- Rayleigh's Energy Theorem

Let $g(t) \rightleftharpoons G(f)$. *Then*

$$\int_{-\infty}^{\infty} |g(t)|^2\, dt = \int_{-\infty}^{\infty} |G(f)|^2\, df$$

The Rayleigh's energy theorem states that the total energy of a Fourier-transformable signal equals the total area under the curve of squared amplitude spectrum of this signal.

Example: Calculate the energy of the function $g(t) = A\ \mathrm{sinc}(2Wt)$.

$$E = \int_{-\infty}^{\infty} g^2(t)dt = A^2 \int_{-\infty}^{\infty} \mathrm{sinc}^2(2Wt)dt$$

Very difficult to evaluate

Since we have $G(f) = (A/2W)\ \mathrm{rect}(f/2W)$, we can calculate E as follows

$$E = \int_{-\infty}^{\infty} |G(f)|^2 df = \int_{-W}^{W} \left(\frac{A}{2W}\right)^2 df = \frac{A^2}{2W}$$

Dirac Delta Function δ(t)

Properties of δ(t)

1- $\delta(t) = 0 \quad t \neq 0$

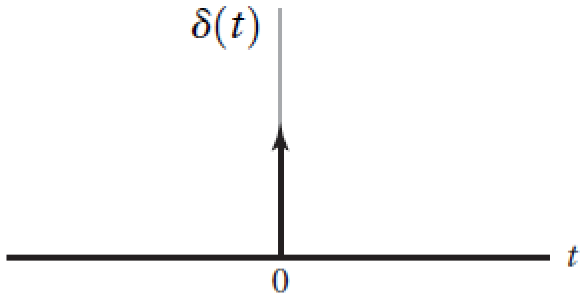
2- $\int_{-\infty}^{\infty} \delta(t)dt = 1$

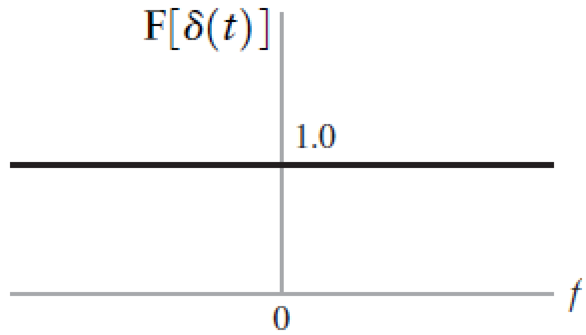
3- $\int_{-\infty}^{\infty} g(t)\delta(t - t_0)dt = g(t_0)$

4- $g(t) * \delta(t) = \int_{-\infty}^{\infty} g(\tau)\delta(t-\tau)d\tau = g(t)$

5- $g(t) * \delta(t - t_0) = g(t - t_0)$

6- $F[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)\exp(-j2\pi ft)dt = \exp(0) = 1$



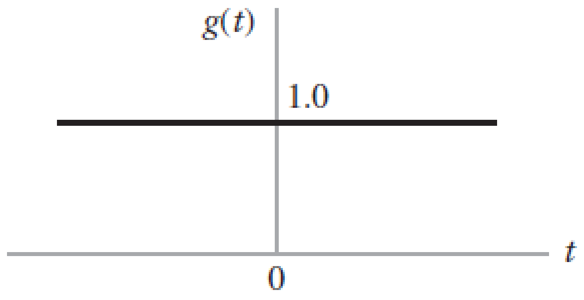


$\delta(t) \rightleftharpoons 1$

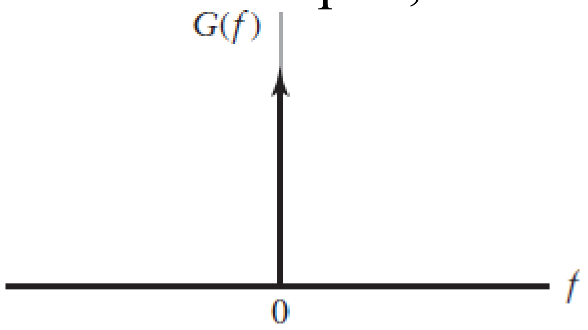
Applications of the Dirac Delta Function

1- Spectrum of a DC signal

By applying the duality property to the Fourier-transform pair, we obtain



$$1 \rightleftharpoons \delta(f)$$



2- Spectrum of a Complex Exponential Function

Applying the frequency-shifting property to the above equation, we obtain the Fourier transform pair

$$\exp(j2\pi f_c t) \rightleftharpoons \delta(f - f_c)$$

Similarly, we have

$$\exp(-j2\pi f_c t) \rightleftharpoons \delta(f + f_c)$$

Applications of the Dirac Delta Function

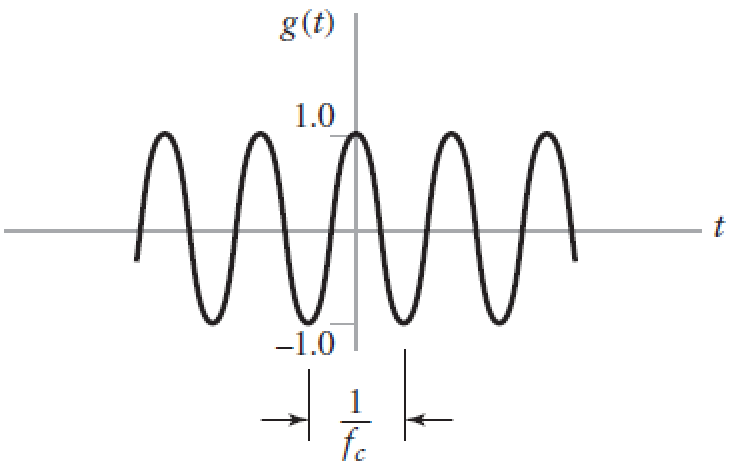
3- Spectrum of a Sinusoidal Signal

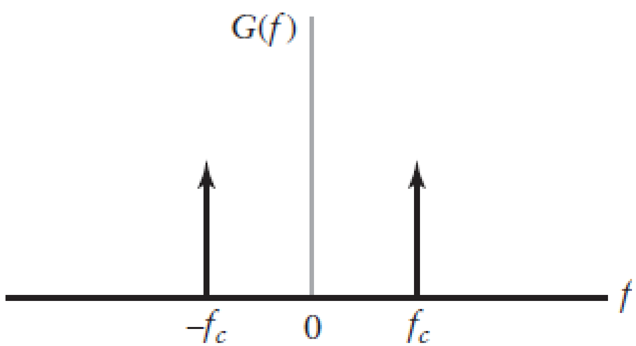
Recall that $\cos(2\pi f_c t)$ can be expressed as

$$\cos(2\pi f_c t) = \frac{1}{2}[\exp(j2\pi f_c t) + \exp(-j2\pi f_c t)]$$

Therefore, utilizing the expression of the Fourier transform of a complex exponential function we get

$$\cos(2\pi f_c t) \rightleftharpoons \frac{1}{2}[\delta(f - f_c) + \delta(f + f_c)]$$





Applications of the Dirac Delta Function

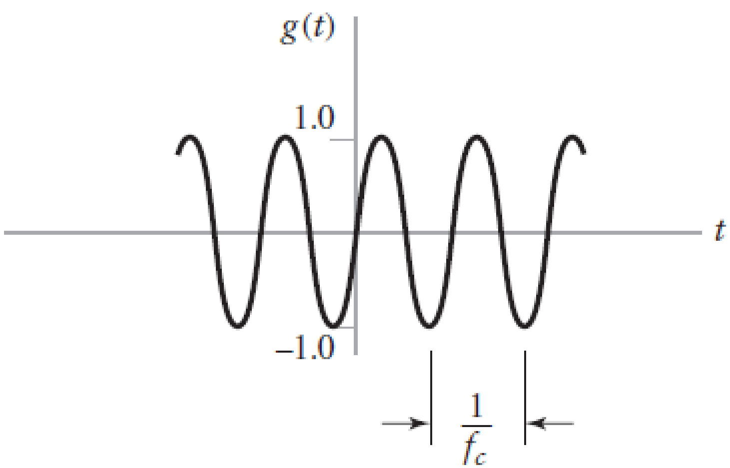
3- Spectrum of a Sinusoidal Signal (Cont.)

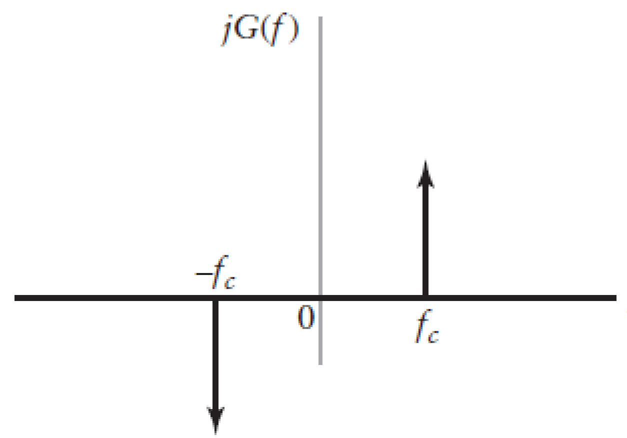
Recall that $\sin(2\pi f_c t)$ can be expressed as

$$\sin(2\pi f_c t) = \frac{1}{2j}[\exp(j2\pi f_c t) - \exp(-j2\pi f_c t)]$$

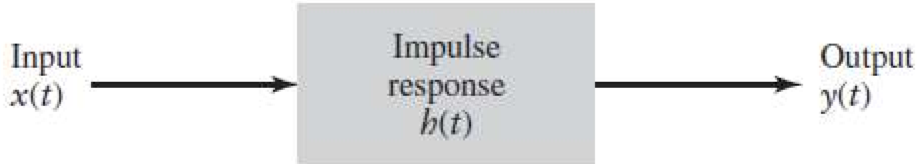
Therefore, utilizing the expression of the Fourier transform of a complex exponential function we get

$$\sin(2\pi f_c t) \rightleftharpoons \frac{1}{2j}[\delta(f - f_c) - \delta(f + f_c)]$$





Transmission of Signals Through Linear Systems



Time Domain

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

h(t) is called the *impulse response* of the system.

Frequency Domain

$$Y(f) = X(f)H(f)$$

H(f) is the Fourier transform of *h(t)*, and is called the *transfer function* or *the frequency response* of the system.