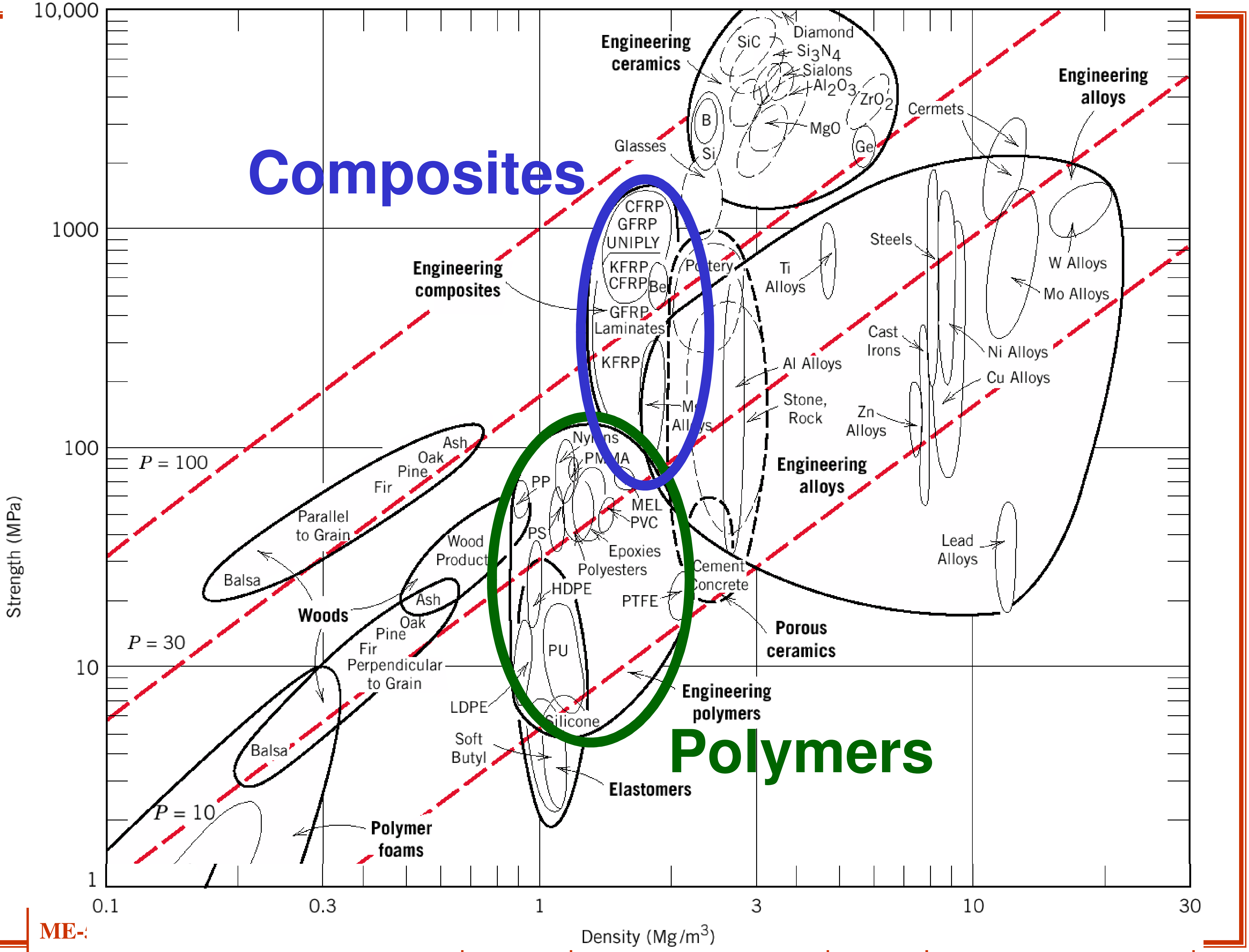
CHAPTER 2

The Deformation of an Elastic Solid

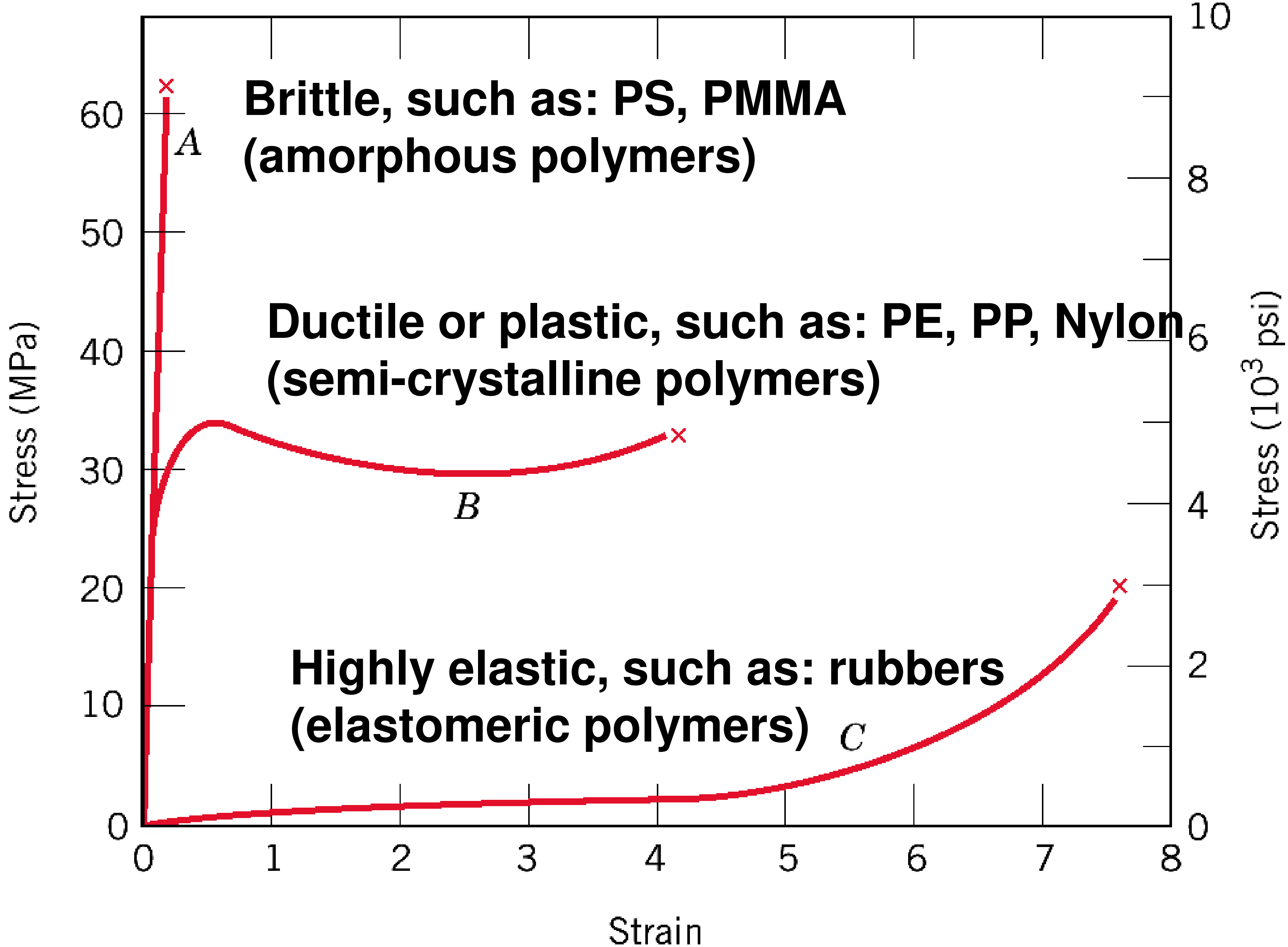
Outline

1. Overall Behavior of Polymers
2. The State of Stress
3. The State of Strain
4. Hooke's Law
5. Finite Strain Elasticity

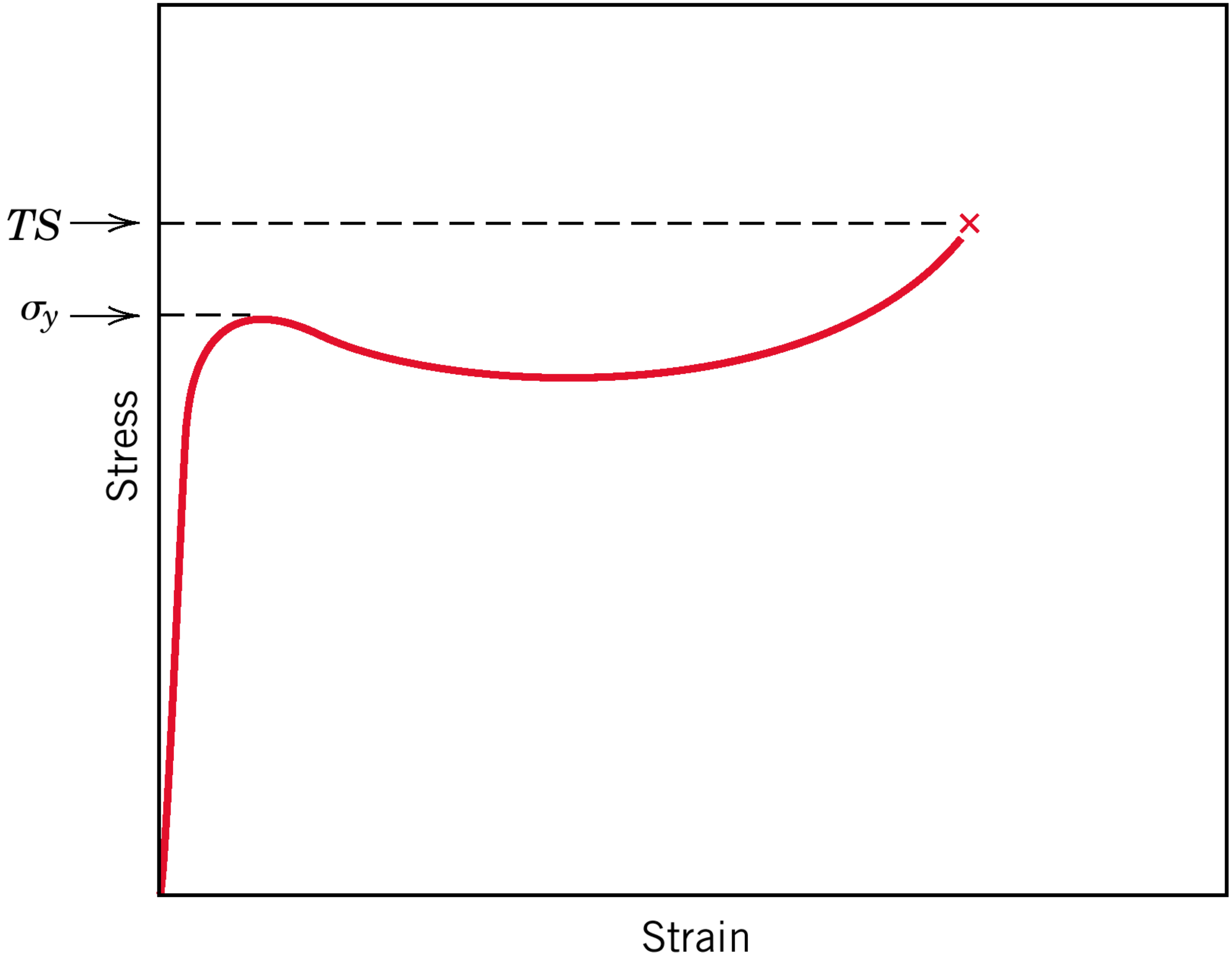


Composites
Polymers
Engineering ceramics
Diamond
SiC
Si_3N_4
Sialons
Al_2O_3
ZrO_2
B
Si
MgO
Ge
Glasses
Cermets
Engineering alloys
CFRP
GFRP
UNIPLY
KFRP
CFRP
Be
GFRP
Laminates
KFRP
Engineering composites
Pottery
Ti Alloys
Steels
W Alloys
Mo Alloys
Cast Irons
Ni Alloys
Cu Alloys
Al Alloys
Stone, Rock
Zn Alloys
Engineering alloys
Lead Alloys
Ash
Oak
Pine
Fir
Parallel to Grain
Balsa
Woods
Wood Products
Ash
Oak
Pine
Fir
Perpendicular to Grain
Balsa
$P = 100$
$P = 30$
$P = 10$
Nylons
PMMA
PP
PS
MEL
PVC
Epoxies
Polyesters
HDPE
PU
PTFE
LDPE
Silicone
Soft Butyl
Cement Concrete
Porous ceramics
Engineering polymers
Elastomers
Polymer foams
Strength (MPa)
Density (Mg/m³)
10,000
1000
100
10
1
0.1
0.3
1
3
10
30

STRESS–STRAIN BEHAVIOR



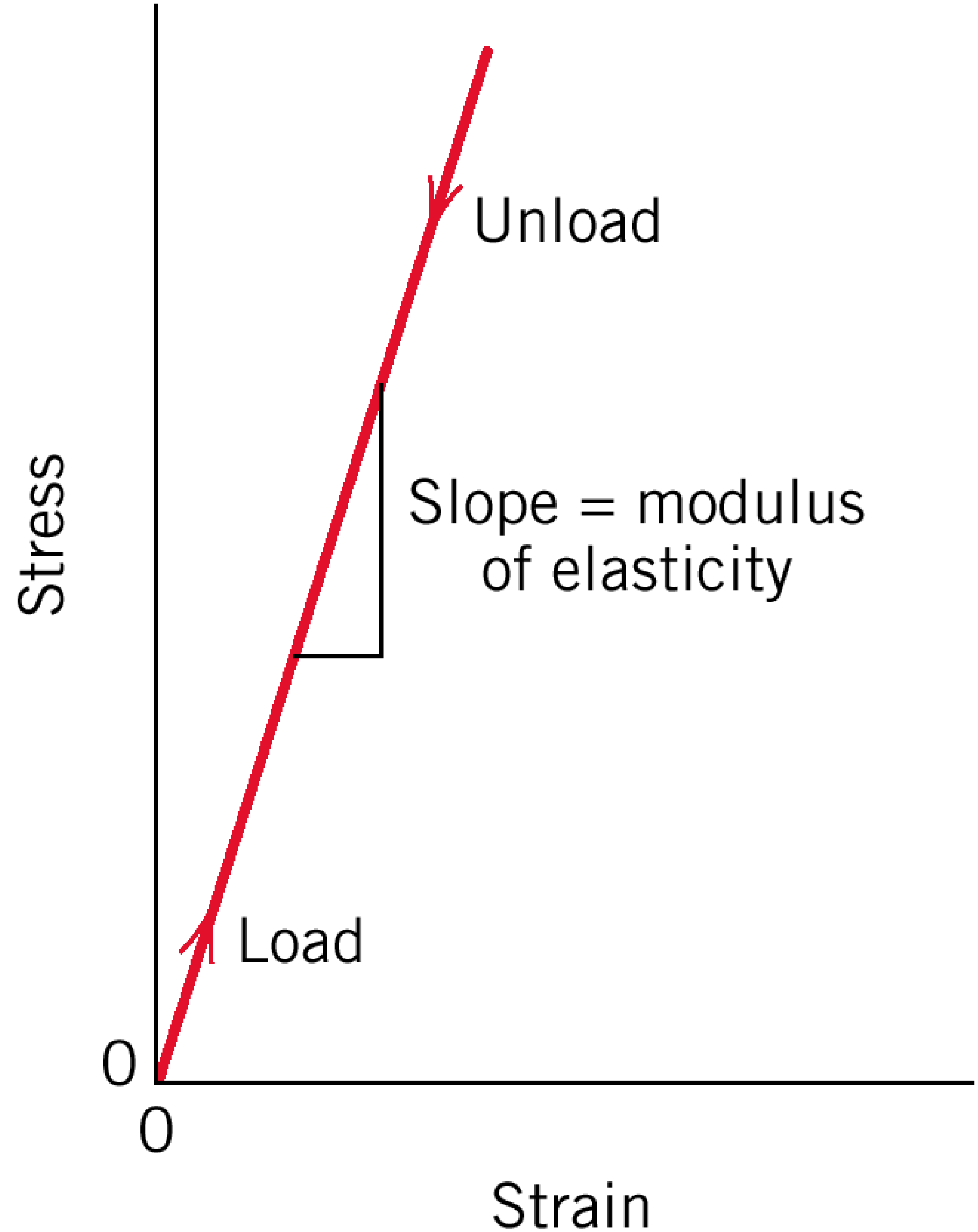
Yield and Tensile Stresses



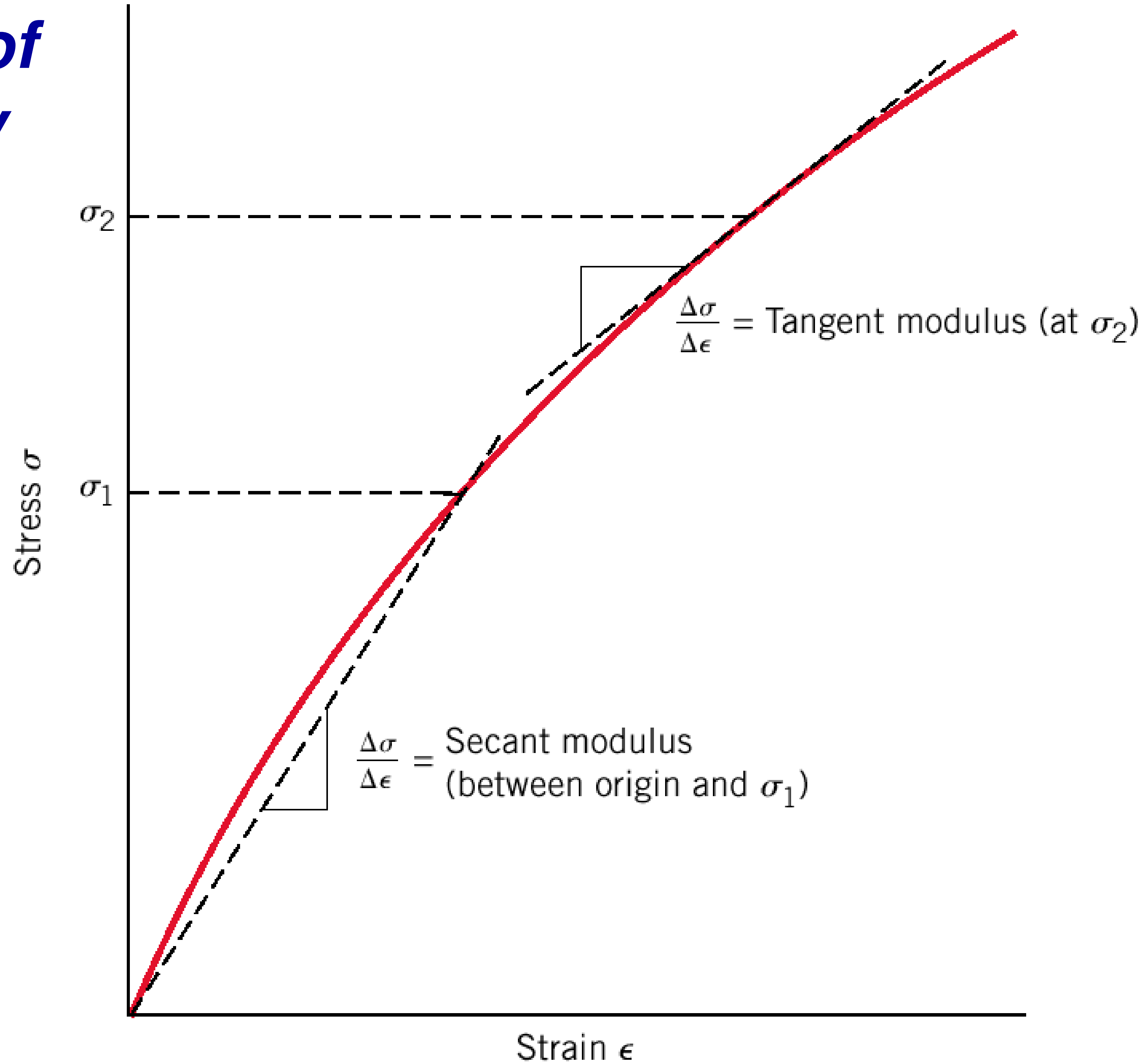
Modulus of Elasticity

$$E = \frac{\Delta\sigma}{\Delta\varepsilon}$$

Young's modulus



Modulus of Elasticity



Material	*Specific Gravity*	*Tensile Modulus [GPa (ksi)]*	*Tensile Strength [MPa (ksi)]*	*Yield Strength [MPa (ksi)]*	*Elongation at Break (%)*
Polyethylene (low density)	0.917–0.932	0.17–0.28 (25–41)	8.3–31.4 (1.2–4.55)	9.0–14.5 (1.3–2.1)	100–650
Polyethylene (high density)	0.952–0.965	1.06–1.09 (155–158)	22.1–31.0 (3.2–4.5)	26.2–33.1 (3.8–4.8)	10–1200
Polyvinyl chloride	1.30–1.58	2.4–4.1 (350–600)	40.7–51.7 (5.9–7.5)	40.7–44.8 (5.9–6.5)	40–80
Polytetrafluoroethylene	2.14–2.20	0.40–0.55 (58–80)	20.7–34.5 (3.0–5.0)	—	200–400
Polypropylene	0.90–0.91	1.14–1.55 (165–225)	31–41.4 (4.5–6.0)	31.0–37.2 (4.5–5.4)	100–600
Polystyrene	1.04–1.05	2.28–3.28 (330–475)	35.9–51.7 (5.2–7.5)	—	1.2–2.5
Polymethyl methacrylate	1.17–1.20	2.24–3.24 (325–470)	48.3–72.4 (7.0–10.5)	53.8–73.1 (7.8–10.6)	2.0–5.5
Phenol-formaldehyde	1.24–1.32	2.76–4.83 (400–700)	34.5–62.1 (5.0–9.0)	—	1.5–2.0
Nylon 6,6	1.13–1.15	1.58–3.80 (230–550)	75.9–94.5 (11.0–13.7)	44.8–82.8 (6.5–12)	15–300
Polyester (PET)	1.29–1.40	2.8–4.1 (400–600)	48.3–72.4 (7.0–10.5)	59.3 (8.6)	30–300
Polycarbonate	1.20	2.38 (345)	62.8–72.4 (9.1–10.5)	62.1 (9.0)	110–150

Range of Mechanical Properties for Various Engineering Plastics

TABLE 7.1

Material	UTS (MPa)	E (GPa)	Elongation (%)	Poisson's ratio (ν)
ABS	28–55	1.4–2.8	75–5	—
ABS, reinforced	100	7.5	—	0.35
Acetal	55–70	1.4–3.5	75–25	—
Acetal, reinforced	135	10	—	0.35–0.40
Acrylic	40–75	1.4–3.5	50–5	—
Cellulosic	10–48	0.4–1.4	100–5	—
Epoxy	35–140	3.5–17	10–1	—
Epoxy, reinforced	70–1400	21–52	4–2	—
Fluorocarbon	7–48	0.7–2	300–100	0.46–0.48
Nylon	55–83	1.4–2.8	200–60	0.32–0.40
Nylon, reinforced	70–210	2–10	10–1	—
Phenolic	28–70	2.8–21	2–0	—
Polycarbonate	55–70	2.5–3	125–10	0.38
Polycarbonate, reinforced	110	6	6–4	—
Polyester	55	2	300–5	0.38
Polyester, reinforced	110–160	8.3–12	3–1	—
Polyethylene	7–40	0.1–1.4	1000–15	0.46
Polypropylene	20–35	0.7–1.2	500–10	—
Polypropylene, reinforced	40–100	3.5–6	4–2	—
Polystyrene	14–83	1.4–4	60–1	0.35
Polyvinyl chloride	7–55	0.014–4	450–40	—

Engineering Stress and Strain

Engineering Stress, σ:

$$\sigma = \frac{F}{A_0}$$

F **= applied load perpendicular to A_0.**

A_0 **= the original cross-sectional area before any load is applied.**

Engineering Strain, ε:

$$\varepsilon = \frac{\ell - \ell_0}{\ell_0} = \frac{\Delta\ell}{\ell_0}$$

ℓ **= instantaneous length.**

ℓ_0 **= the original length before any load is applied.**

$\Delta\ell$ **= elongation.**

Ductility

Ductility: is the measure of the degree of plastic deformation that has been sustained at fracture. Can be expressed as:

Percent elongation

$$\%EL = \left(\frac{\ell_f - \ell_o}{\ell_o}\right) \times 100$$

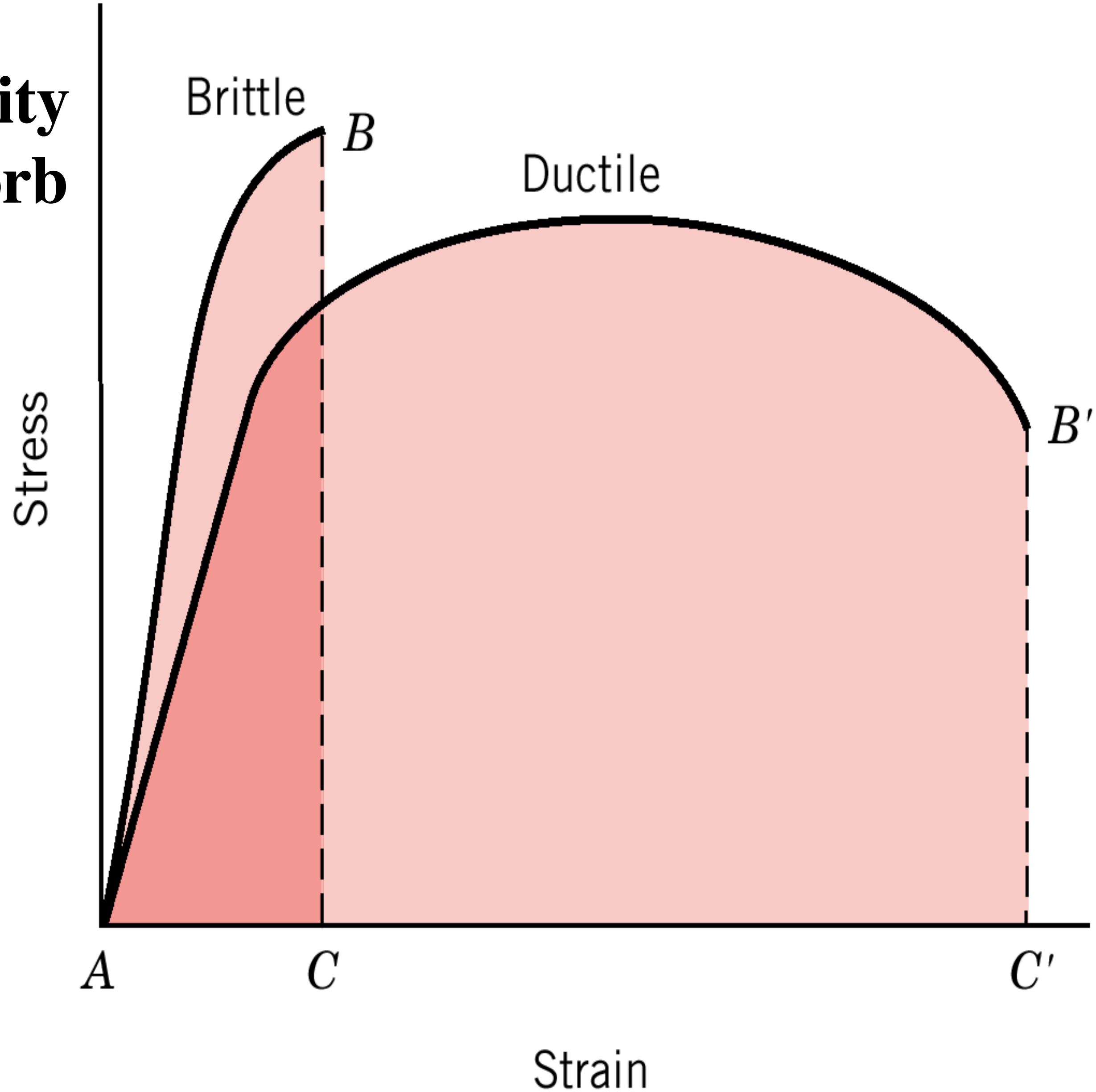
Percent area reduction

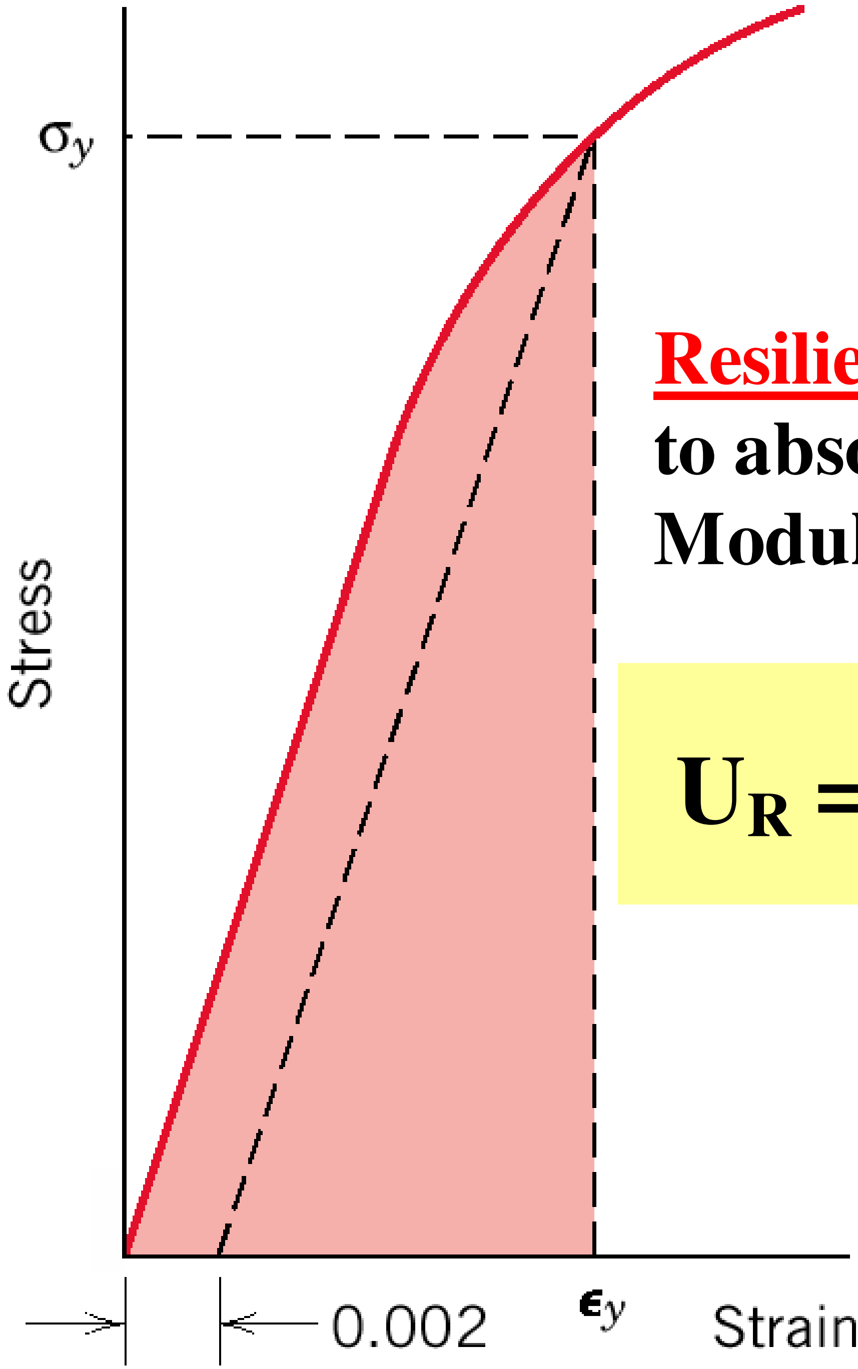
$$\%AR = \left(\frac{A_o - A_f}{A_o}\right) \times 100$$

Toughness

Toughness: is the ability of a material to absorb energy up to fracture.

$$U_T = \int_0^{\varepsilon_f} \sigma \, d\varepsilon$$





Resilience

Resilience: is the capacity of a material to absorb energy up to the yield point.

Modulus of resilience:

$$U_R = \int_0^{\varepsilon_y} \sigma \, d\varepsilon = \frac{1}{2}\sigma_y \varepsilon_y = \frac{\sigma_y^2}{2E}$$

True Stress and True Strain

Engineering stress:

$$\sigma = \frac{F_i}{A_o}$$

Engineering strain:

$$\varepsilon = \int_{\ell_o}^{\ell_i} \frac{d\ell}{\ell_o} = \frac{\ell_i - \ell_o}{\ell_o}$$

In the elastic region:

$$\sigma = E\,\varepsilon$$

If no volume change $(A_i \ell_i = A_o \ell_o)$

$$\sigma_T = \sigma\,(1+\varepsilon)$$

$$\varepsilon_T = \ln(1+\varepsilon)$$

True stress:

$$\sigma_T = \frac{F_i}{A_i}$$

True strain:

$$\varepsilon_T = \int_{\ell_o}^{\ell_i} \frac{d\ell}{\ell} = \ln(\ell_i) - \ln(\ell_o)$$

$$= \ln\left(\frac{\ell_i}{\ell_o}\right)$$

In the plastic region:

$$\sigma_T = K\,\varepsilon_T^n$$

The State of Stress

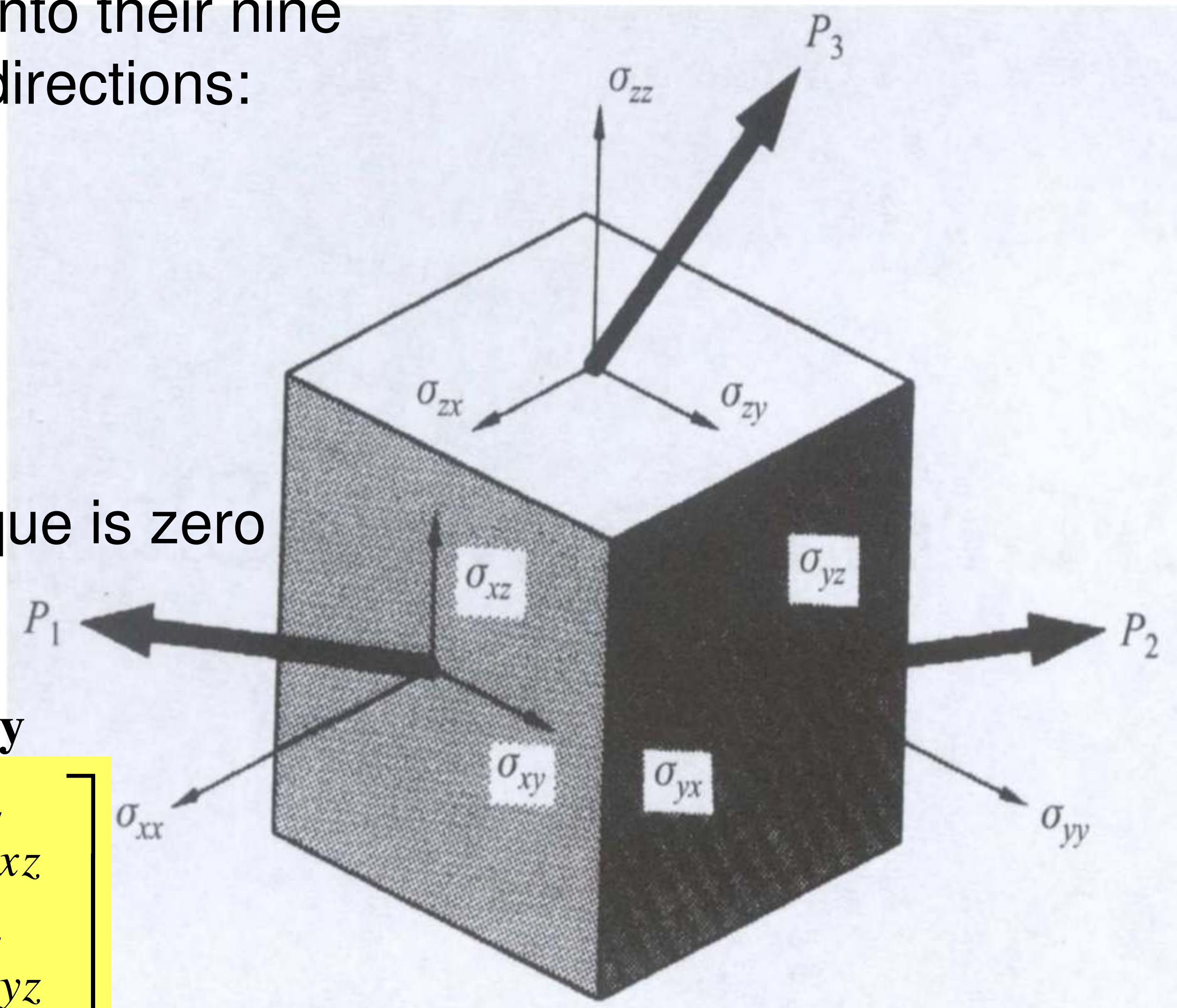
The forces are resolved into their nine componets in the x, y, z directions:

P_1: σ_{xx}, σ_{xy}, σ_{xz}
P_2: σ_{yy}, σ_{yx}, σ_{yz}
P_3: σ_{zz}, σ_{zx}, σ_{zy}

At equilibrium => net torque is zero

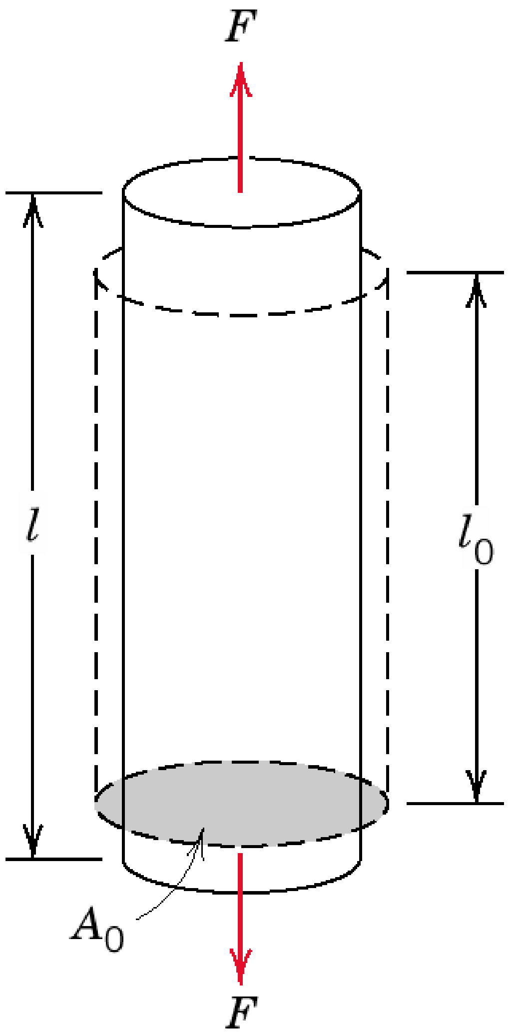
$\sigma_{xy} = \sigma_{yx}$
$\sigma_{xz} = \sigma_{zx}$ $\sigma_{yz} = \sigma_{zy}$

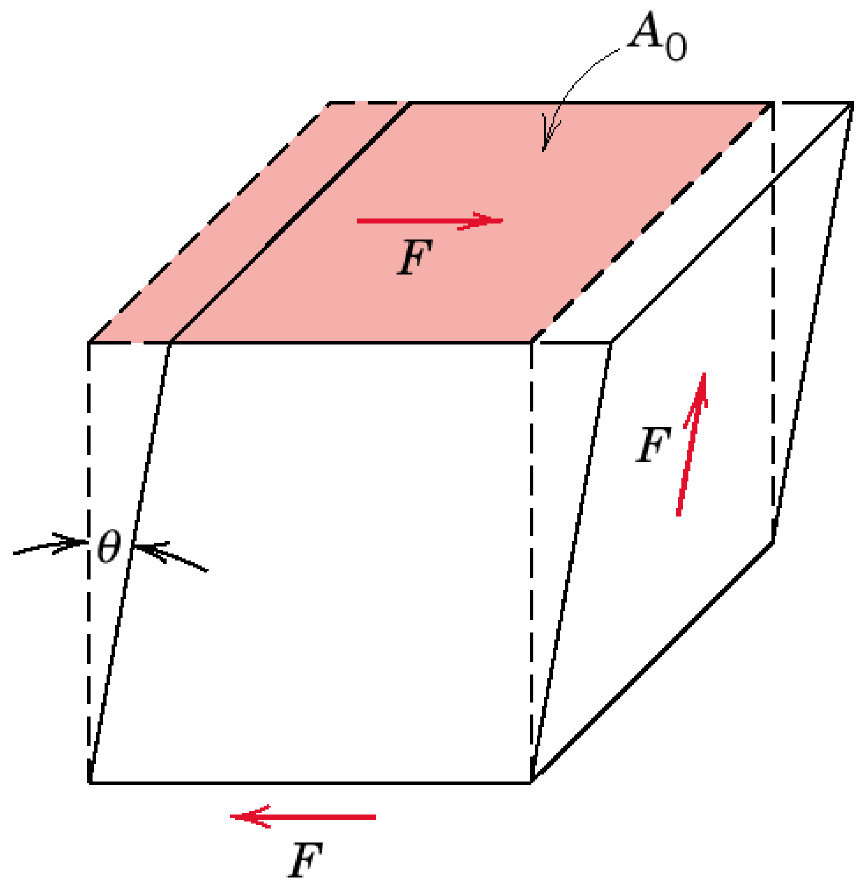
$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix}$$



The State of Strain

In the elastic behavior of isotropic materials, it is usual to consider two types of train only.





The State of Strain

These nine components can then define the state of strain as:

$$\varepsilon_{ij} = \begin{bmatrix} e_{xx} & \frac{1}{2}e_{xy} & \frac{1}{2}e_{xz} \\ \frac{1}{2}e_{xy} & e_{yy} & \frac{1}{2}e_{yz} \\ \frac{1}{2}e_{xz} & \frac{1}{2}e_{yz} & e_{zz} \end{bmatrix}$$

Poisson's Ratio

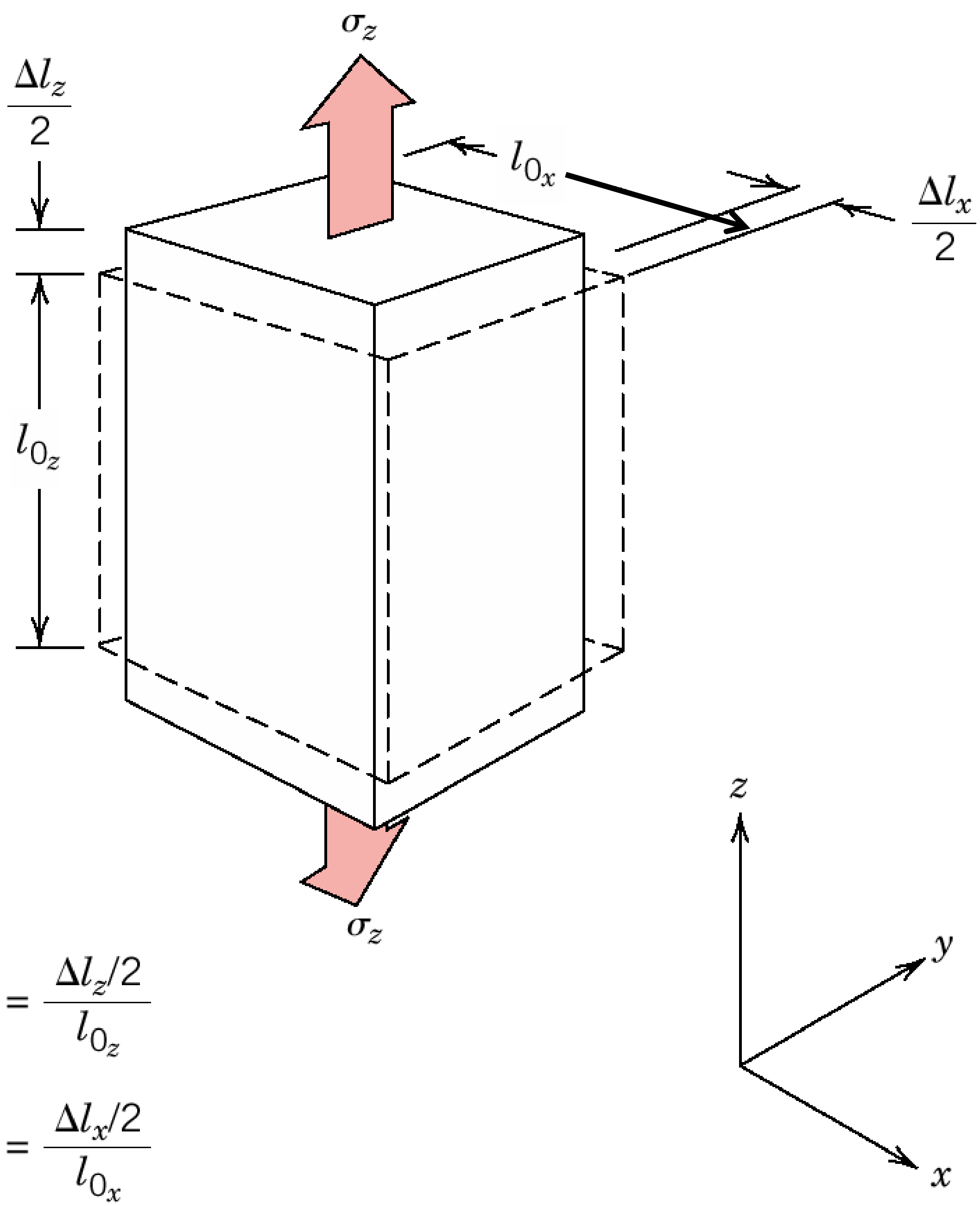
Poisson's Ratio, ν (nu):
It is the ratio of the lateral and axial strains.

$$\nu = -\frac{\varepsilon_x}{\varepsilon_z} = -\frac{\varepsilon_y}{\varepsilon_z}$$

$$0.25 \le \nu \le 0.5$$

Typical Values of ν:

- **No volume change ν = 0.5**
- **Polymer ν = 0.4**
- **Metal ν = 0.3**
- **Ceramic ν = 0.2**



$$\frac{\epsilon_z}{2} = \frac{\Delta l_z/2}{l_{0_z}}$$

$$-\frac{\epsilon_x}{2} = \frac{\Delta l_x/2}{l_{0_x}}$$

Elastic Deformation

Elastic Deformation:

Elastic deformation is nonpermanent.

$$\sigma = E\,\varepsilon$$

E = modulus of elasticity, or Young's modulus.

shear stress and strain are proportional:

$$\tau = G\,\gamma$$

G = shear modulus.

Hooke's Law

In the elastic range, the strains are represented by the general following equations:

$$e_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right]$$

$$e_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right]$$

$$e_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right]$$

$$e_{xz} = \frac{1}{G}\sigma_{xz} \qquad e_{yz} = \frac{1}{G}\sigma_{yz} \qquad e_{xy} = \frac{1}{G}\sigma_{xy}$$

where: $G = \dfrac{E}{2(1+\nu)}$

Also a bulk modulus K, related to the fractional change in volume is defined as:

$$K = \frac{E}{3(1-2\nu)}$$

Volume Strain (Dilatation)

By summing the three expression of Hooke's law:

$$\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{1-2\nu}{E}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right) = \Delta$$

Bulk Modulus

$$K = \frac{\sigma_m}{\Delta} = \frac{E}{3\left(1-2\nu\right)}$$

Mean Stress

$$\sigma_m = \frac{1}{3}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)$$

Rubber-like State

In the rubber-like state a polymer may be subject to large deformation and still show complete recovery. This is elastic behavior at large strains affects the definition of stress and strain.

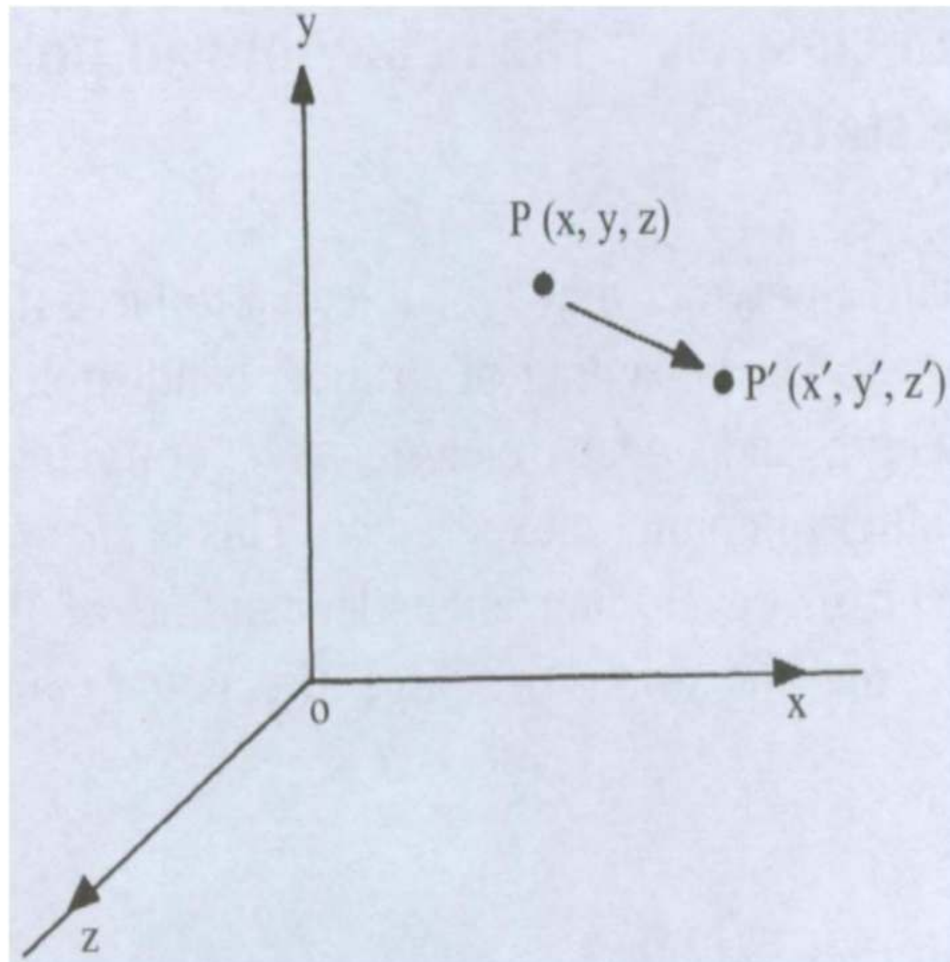
Consider a point P (x,y,z) to be deformed and moved to a point P'(x',y',z'). Where:

$$x' = \lambda_1 x$$

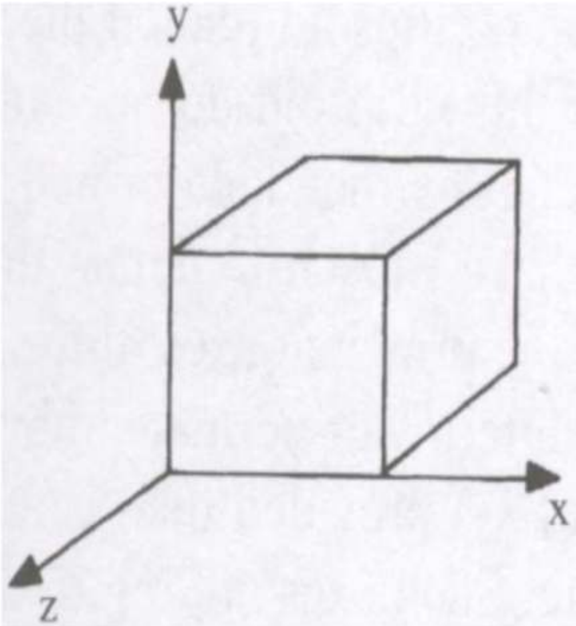
$$y' = \lambda_2 y$$

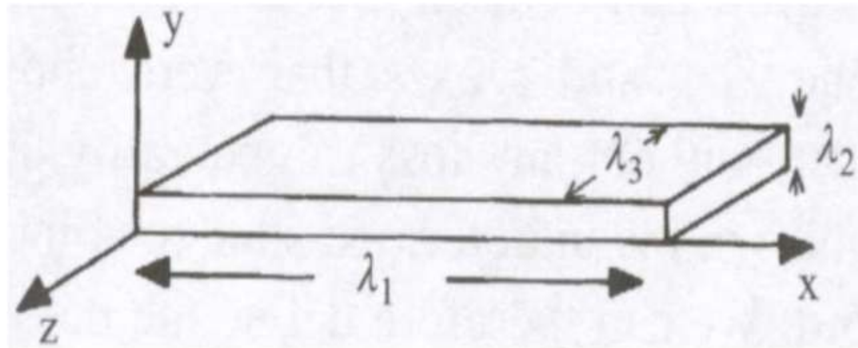
$$z' = \lambda_3 z$$

The quantities λ_1, λ_2, λ_3, are called the deformation ratios



Finite Strain





Finite strain are defined by these deformation ratios, λ_1, λ_2, λ_3, as follows:

$$\varepsilon_{xx} = \tfrac{1}{2}\left(\lambda_1^2 - 1\right) \qquad \varepsilon_{yy} = \tfrac{1}{2}\left(\lambda_2^2 - 1\right) \qquad \varepsilon_{zz} = \tfrac{1}{2}\left(\lambda_3^2 - 1\right)$$

For small strains λ_1 = (1+e_{xx}) and

$$\lambda_1^2 = \left(1 + e_{xx}\right)^2 \rightarrow 1 + 2e_{xx} \qquad therefore \qquad \varepsilon_{xx} = e_{xx}$$

The Strain Energy Function

If f_1, f_2, f_3, are stresses acting on the undeformed unit faces then the stresses acting on the deformed faces are:

$$\sigma_{xx} = \frac{f_1}{\lambda_2 \times \lambda_3} = \frac{f_1}{\lambda_2 \times \lambda_3} \times \frac{\lambda_1 \times \lambda_2 \times \lambda_3}{1} = \lambda_1 \times f_1$$

Also: $\sigma_{yy} = \frac{f_2}{\lambda_1 \times \lambda_3} = \lambda_2 \times f_2$ **and** $\sigma_{zz} = \frac{f_3}{\lambda_1 \times \lambda_2} = \lambda_3 \times f_3$

The work (W) of a small deformation is:

$$dW = f_1 d\lambda_1 + f_2 d\lambda_2 + f_3 d\lambda_3$$

$$dW = \frac{\sigma_{xx}}{\lambda_1} d\lambda_1 + \frac{\sigma_{yy}}{\lambda_2} d\lambda_2 + \frac{\sigma_{zz}}{\lambda_3} d\lambda_3$$

The Strain Energy Function

A solution of the previous equation is known as the strain energy function (U) which has to satisfy the following conditions:

1. **For zero strain U = 0;**
2. **For small strain, Hooke's law should be obtained**

An equation which satisfies these requirements is:

$$U = C_1\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right)$$

Under simple tension force where $\lambda_1=\lambda$ and $\lambda_2=\lambda_3=\lambda^{-1/2}$ and With the incompressible assumption ($\lambda_1\lambda_2\lambda_3=1$) U becomes:

$$U = C_1\left(\lambda^2 + \frac{2}{\lambda} - 3\right)$$

and

$$f = \frac{\partial U}{\partial \lambda} = 2C_1\left(\lambda - \frac{1}{\lambda^2}\right)$$

The Strain Energy Function

Materials which obey this relationship are sometimes called neo-Hookean:

$$f = 2C_1\left(\lambda - \frac{1}{\lambda^2}\right)$$

For small strain, where λ=1+e the equation becomes:

$$f = 2C_1\left(1+e-\frac{1}{(1+e)^2}\right) = 2C_1\left(1+e-\frac{1}{1+2e+\cancel{e^2}}\right)$$

$$= 2C_1\left(1+e-\frac{1-2e}{(1+2e)(1-2e)}\right) = 2C_1\left(1+e-\frac{1-2e}{1-\cancel{4e^2}}\right)$$

$$f = 2C_1\{1+e-(1-2e)\} = 6\,C_1\,e \qquad = E\;e$$

That is Hooke's law