



Taibah University
Mechanical Engineering

ME 202

Engineering Mechanics: Dynamics

Prepared by:

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Chapter 3: Particle Kinetics

- Introduction
- Force, Mass, and Accelerations **3 Lectures**
 - Newton's Second Law.
 - Equation of Motion and Kinetics Solution of Problems
 - Rectangular Motion and Curvilinear Motion
- Energy Method **2 Lectures**
 - Work and Kinetic Energy
 - Potential Energy
- Momentum Method **3 Lectures**
 - Linear Impulse and Linear Momentum
 - Angular Impulse and Angular Momentum

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3 Lectures

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Introduction

Particle Kinetics: is the study of the relationship between the applied forces and the resulting changes in motion.

This topic is based on Newton's Second Law:

$$\mathbf{F} = m\mathbf{a}$$

It combines the knowledge of force properties (**Statics**) and **Kinematics** of particles.

Solution Approaches

- Three techniques for solving the kinetics problems:
 1. Direct application of $\mathbf{F}=m \mathbf{a}$. (Force-Mass-Acceleration)
 2. Work and energy principles.
 3. Impulse and momentum methods.
- The three approaches are interrelated but each approach has unique characteristics and advantages:
 - Method 1 is simple and can be applied straight forward.
 - Method 2 involves *Spatial Integral* of: $\mathbf{F}=m \mathbf{a}$.
 - Method 3 involves *Time Integral* of: $\mathbf{F}=m \mathbf{a}$.

Force-Mass-Acceleration:

Newton's Second Law

- Subject a mass particle to the action of single force \mathbf{F}_1 and measure its acceleration \mathbf{a}_1 in the primary inertial system and record the ratio $\mathbf{F}_1/\mathbf{a}_1$ as C_1 .
- The experiment is repeated for $\mathbf{F}_2, \mathbf{a}_2; \mathbf{F}_3, \mathbf{a}_3; \text{etc...}$ and compare this ratio:

$$\frac{F_1}{a_1} = \frac{F_2}{a_2} = \dots = \frac{F}{a} = C$$

- The constant C is a measure of an invariable property that is the inertia (mass) of the particle.
- **Mass (Inertia)**: represents the particle's resistance to the rate of change in its velocity.

Force-Mass-Acceleration:

Newton's Second Law

Conclusions:

- For a given force, if the particle has high inertia (mass), the resulting acceleration will be low.
- For the same force, if the particle has small inertia (mass), the acceleration will be high.
- We can conclude that C is proportional to the particle's mass. Therefore: $C = k m$, where k is a constant.
- The Force-acceleration relationship can be written as:

$$F = k m a$$

Force-Mass-Acceleration:

Newton's Second Law

- The particle **acceleration** is in the same direction of the **applied force**.
- Newton's law can be written in vector form as:

$$\mathbf{F} = k m \mathbf{a}$$

- It is customary to write Newton's 2nd law in the form:

$$\mathbf{F} = m \mathbf{a}$$

- When the particle is acted upon by a system of forces, the resulting acceleration will be proportional to the vector-sum of the applied forces and in its direction.

$$\Sigma \mathbf{F} = m \mathbf{a}$$

Example: 3/1

- During a brake test, the rear-engine car is stopped from an initial velocity of 100km/h in a distance of 50m. If it is known that all four wheels contribute equally to the braking force, determine the force F at each wheel. Assume constant deceleration for the 1500-kg car.

$$v^2 = v_0^2 + 2a(s - s_1)$$

$$\mathbf{F} = m\mathbf{a}$$

Example: 3/1

- **Given:** $v_o=100\text{km/h}=27.7\text{m/s}$, $s=50\text{m}$,
 $m=1500\text{kg}$
- **Required:** force exerted by each wheel
- **Approach:** using Newton's 2nd law: $\mathbf{F=ma}$
- **Solution:**

$$v^2 = v_o^2 + 2a(s - s_1)$$

$$\mathbf{F = ma}$$

- $0=(27.7)^2+2a(0-50) \rightarrow a= 7.716\text{m/s}$
- Total braking force: $F=ma \rightarrow$
 $F_T=1500*7.716=11.574\text{KN}$
- Braking force per wheel: $F_w=F_T/4=2.893\text{KN}$

Equation of Motion

When a particle of mass m is acted upon by a system of concurrent forces, the resulting acceleration will be proportional to the resultant of the applied forces and in its direction.

$$\Sigma \mathbf{F} = m \mathbf{a}$$

Two Types of Kinetic Problems

1. **The acceleration is known** or can be determined from the kinematics. In this case, deriving the force function will be easy and straight forward.
2. **The force is specified** and the resulting motion need to be defined:
 - In this case, we need to perform integration in order to define the resulting motion.
 - Numerical integration and computer programs will be helpful for complex expressions.

Constrained and Unconstrained Motion

- **Unconstrained Motion:**

- The particle is free of any physical or mechanical guides.
- The particle motion path is determined by initial condition (velocity, displacement) and the external forces acting on it.
- Examples: airplane, rocket, piece of wood flying by tornado wind,...

- **Constrained Motion:**

- The motion path of the particle is totally or partially determined by restraining guides.
- Examples: train, pendulum, turbine shaft,...

Particle Degrees of Freedom (DOF)

- **Degrees of Freedom:** The number of coordinates required to fully define the position of the particle at any time.
- **Unconstrained Spatial Motion:** 3-DOF
 - The particle moves freely in the 3-Dimensional space .
- **Unconstrained Planar Motion:** 2-DOF
 - The particle can move freely in the 2 -Dimensional space or in-plane.
- **Constrained one DOF motion:**
 - The particle path is controlled by fully restraining guides, but can move freely in a rectilinear or curvilinear space - like a train on its track or collar on a shaft.
- How can we classify the projectile motion?

Solving Kinetics Problem

Step 1: Choose body to be isolated.

Step 2: Construct the **free body diagram** (FBD) for each body:

1. Cut free each environmental connection and replace it with a force. If the magnitude of the force is known, add label. If the magnitude is unknown, assign a variable label.
2. Label pertinent points, angles, etc. (Outline of the body is all that is needed). Contact forces touch the body outline; body forces will be shown where they act. (e.g. weights act at the center of gravity).
3. Decide upon and show a coordinate system.

Solving Kinetics Problem

Step 3: Apply $\mathbf{F}=m \mathbf{a}$ in the chosen coordinates system. For a 2D problem:

$$\Sigma F_x=ma_x \quad \text{and} \quad \Sigma F_y=ma_y$$

Step 4: List and count the unknowns. Compare the number of unknowns n to the number of equations m .
If $n \leq m$, : Solve for the unknowns.
If $n > m$: write additional relationships, such as kinematic constraint, Friction force $F=\mu N$, etc.. Then solve for the unknowns

Step 5: Apply common sense. Are the magnitude and direction of the results reasonable?

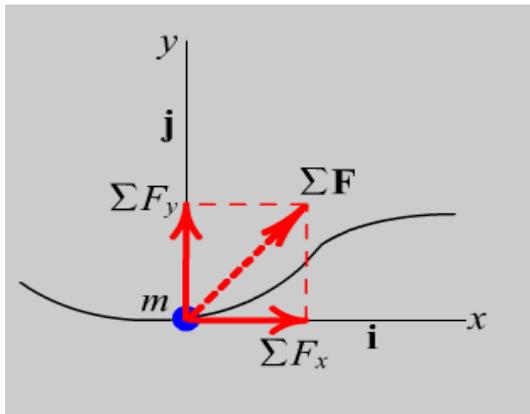
Rectangular Motion

- The particle **acceleration** is in the same direction of the resultant of the **applied forces**.
- *Newton's Second Law*: is a vector relation between the applied force and the resulting acceleration:

$$\Sigma \mathbf{F} = m \mathbf{a}$$

Equation of Motion

- In the rectangular coordinates, the force and acceleration vectors can be written as:



$$\mathbf{F} = F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k}$$

$$\mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}$$

- Newton's second law can be decomposed as follows:

$$\Sigma F_x \mathbf{i} + \Sigma F_y \mathbf{j} + \Sigma F_z \mathbf{k} = m(a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k})$$

Equation of Motion

- Equating the coefficients of like unit vectors yield:

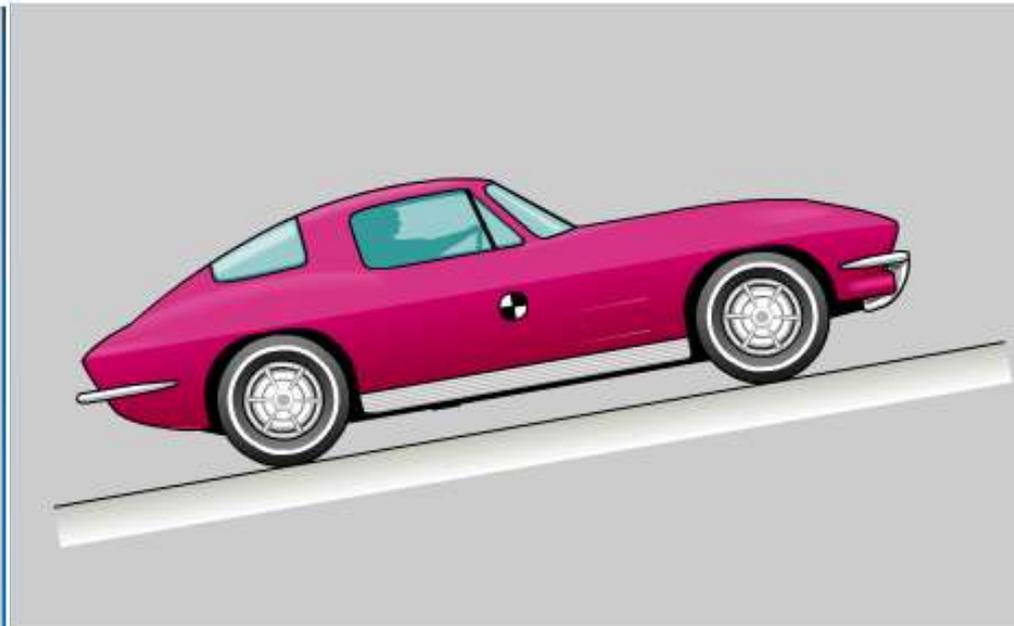
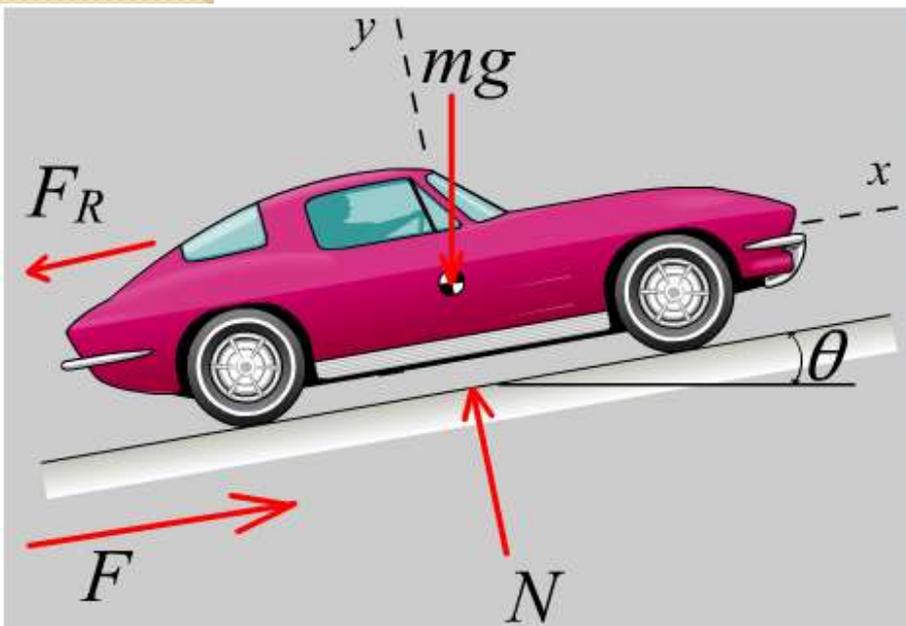
$$\Sigma F_x = ma_x \quad \Sigma F_y = ma_y \quad \Sigma F_z = ma_z$$

- Depending on the number of degrees of freedom (DOF) of the problem, we decide the number of differential equations that will be necessary to describe the motion.

Example

A large 1800-kg car has a total of 450 N of retarding force acting on it as it travels at a speed of 95 km/h. This is the total aerodynamic drag, tire rolling resistance and all internal friction forces.

- Compute the power required at steady speed of 95 km/h on a level ground.
- Repeat (a), but now the car is going uphill on a 6% incline.
- Repeat (b) but now the car is accelerating at 0.9 m/s^2 .



- Looking at the general case of (c), we can treat the other (a) and (b).

$$\Sigma F_x = ma_x:$$

$$F - F_R - mg \sin \theta = ma_x$$

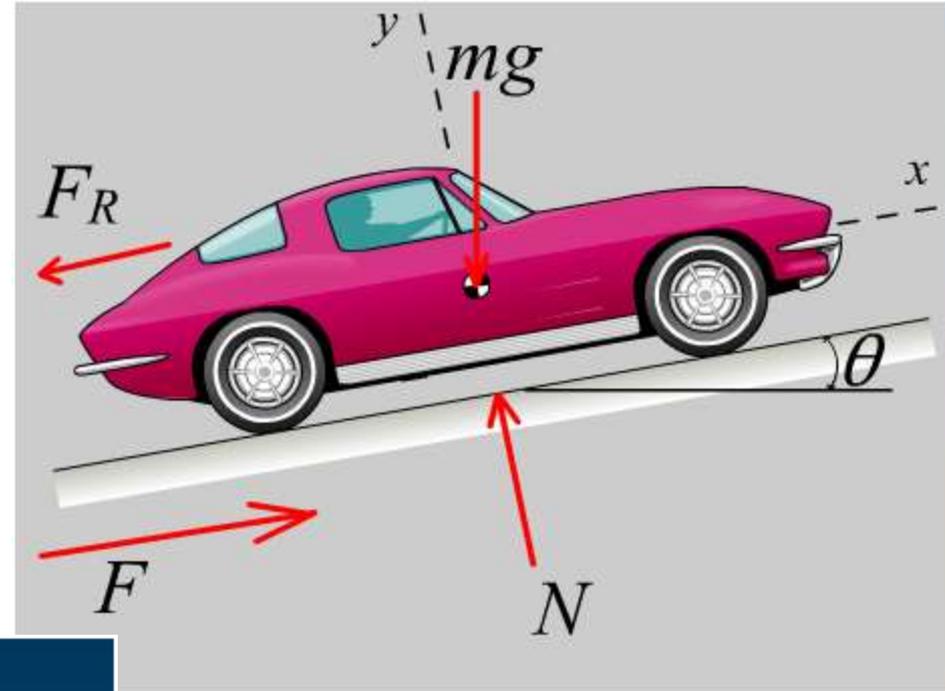
$$F = F_R + m(g \sin \theta + a_x)$$

$$(a) \theta = a_x = 0$$

$$F = F_R = 450 \text{ N}$$

$$P = \mathbf{F} \cdot \mathbf{v} = Fv$$

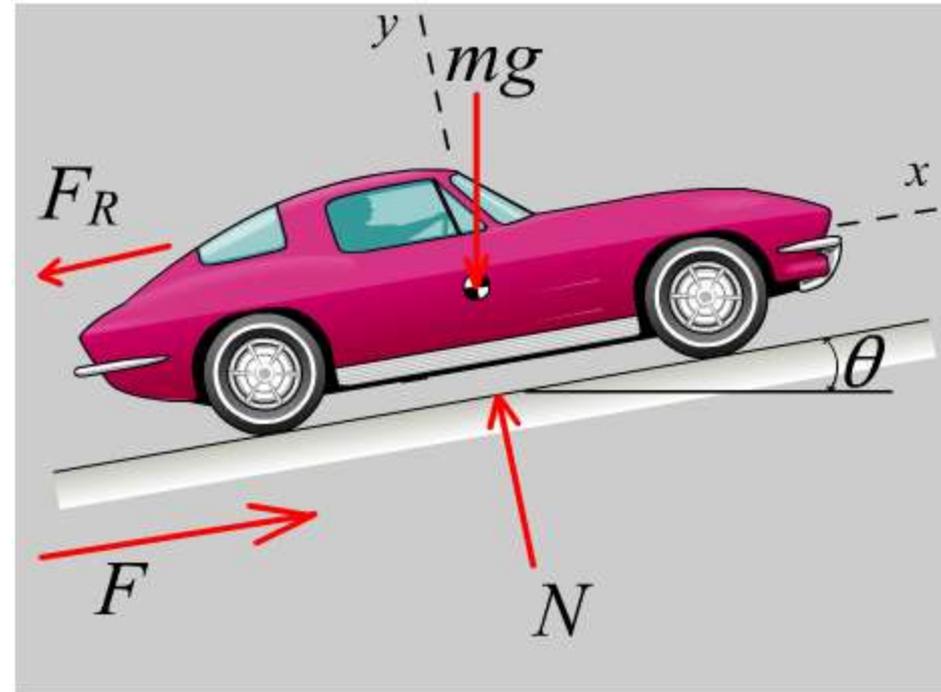
$$= 450(26.4) = 11\,880 \frac{\text{J}}{\text{s}}$$



$$\text{or } P = 11\,880 \text{ W}$$

$$= 11.88 \text{ kW}$$

$$F = F_R + m(g \sin \theta + a_x)$$

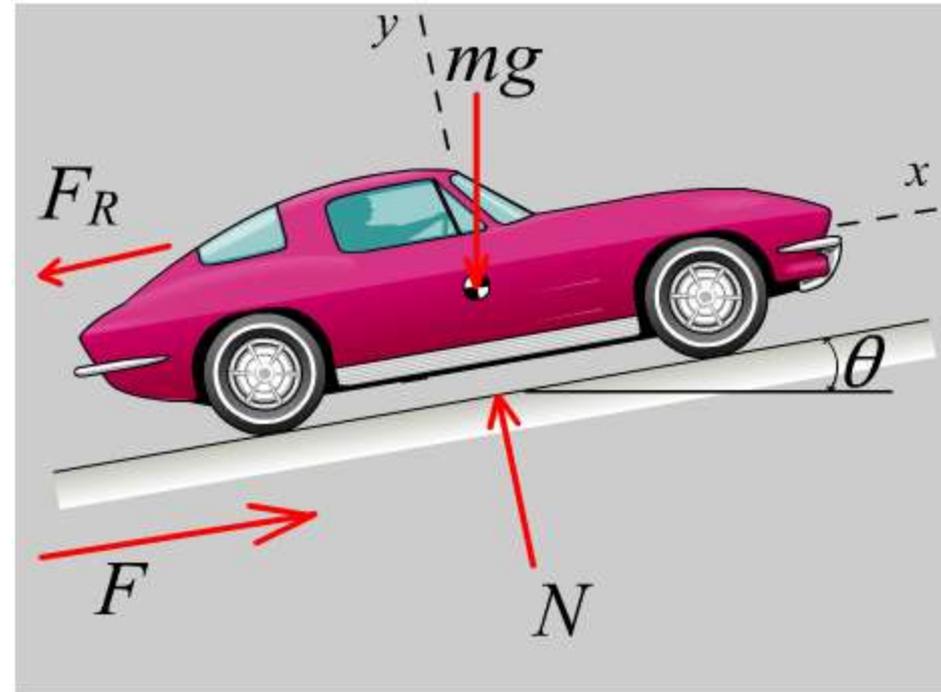


$$(b) a_x = 0, \theta = \tan^{-1} \frac{6}{100}$$

$$\begin{aligned} F &= F_R + mg \sin \theta \\ &= 450 + 1800(9.81) \sin \theta \\ &= 450 + 1058 \\ &= 1508 \text{ N} \end{aligned}$$

$$\begin{aligned} P &= \mathbf{F} \cdot \mathbf{v} = Fv \\ &= 1508(26.4) \\ &= 39\,800 \text{ W} \\ &= 39.8 \text{ kW} \end{aligned}$$

$$F = F_R + m(g \sin \theta + a_x)$$



$$(c) a_x = 0.9 \text{ m/s}^2, \theta = \tan^{-1} \frac{6}{100}$$

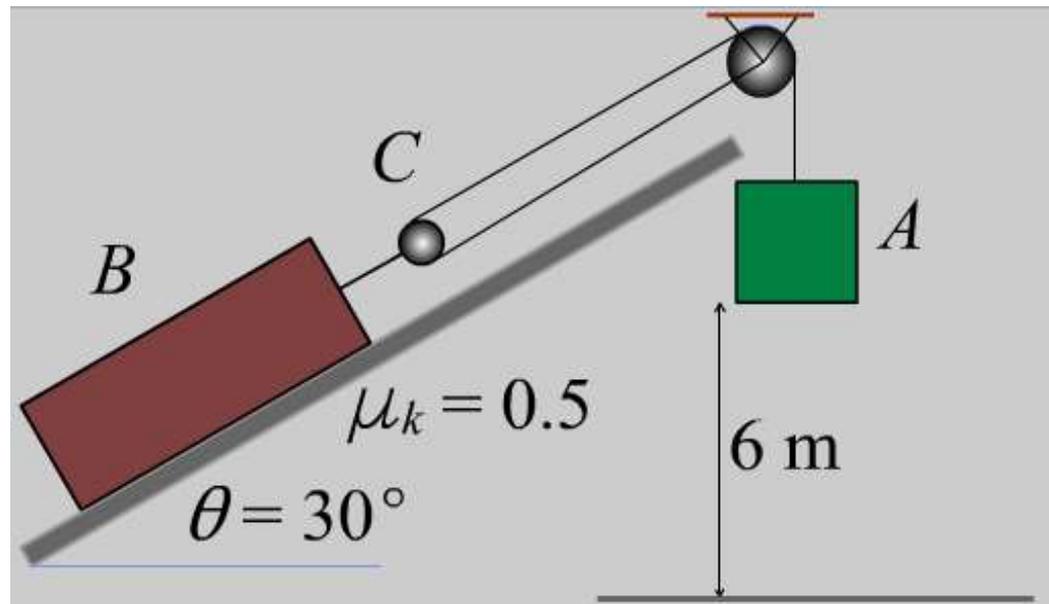
$$\begin{aligned} F &= F_R + m(g \sin \theta + a_x) \\ &= 450 + 1800(9.81 \sin \theta + 0.9) \\ &= 450 + 1800(0.588 + 0.9) \\ &= 3130 \text{ N} \end{aligned}$$

$$\begin{aligned} P &= \mathbf{F} \cdot \mathbf{v} = Fv \\ &= 3127(26.4) \\ &= 82\,500 \text{ W} \\ &= 82.5 \text{ kW} \end{aligned}$$

Example:

A 125-kg block A is released from rest in the position shown and pulls the 200-kg block B up the 30° ramp. If coefficient of kinetic friction between the block B and the ramp is 0.5, find the velocity of block A as it hits the ground.

Follow our standard solution procedure.

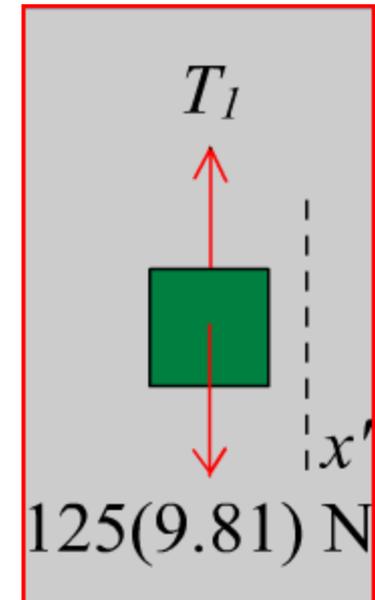
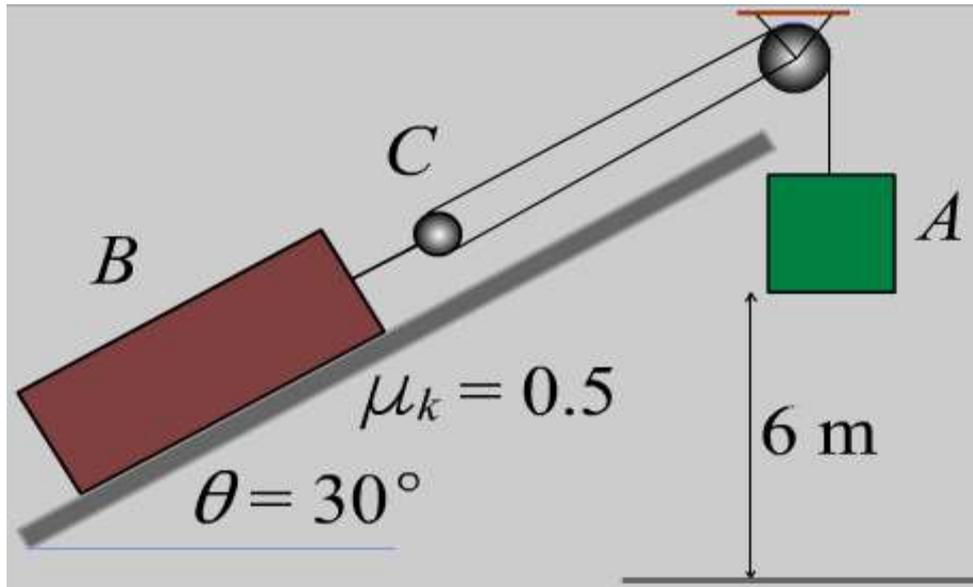


Solution: Block A

$$\Sigma F_{x'} = ma_{x'}$$

$$W_A - T_1 = m_A a_A$$

- $125(9.81) - T_1 = 125 a_A \quad (1)$



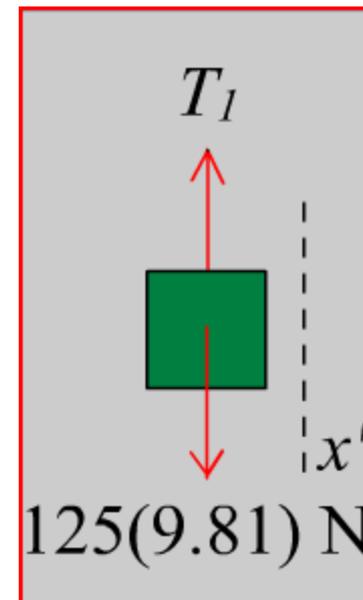
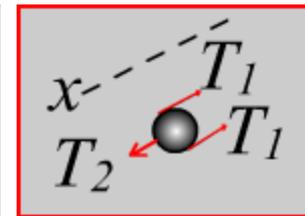
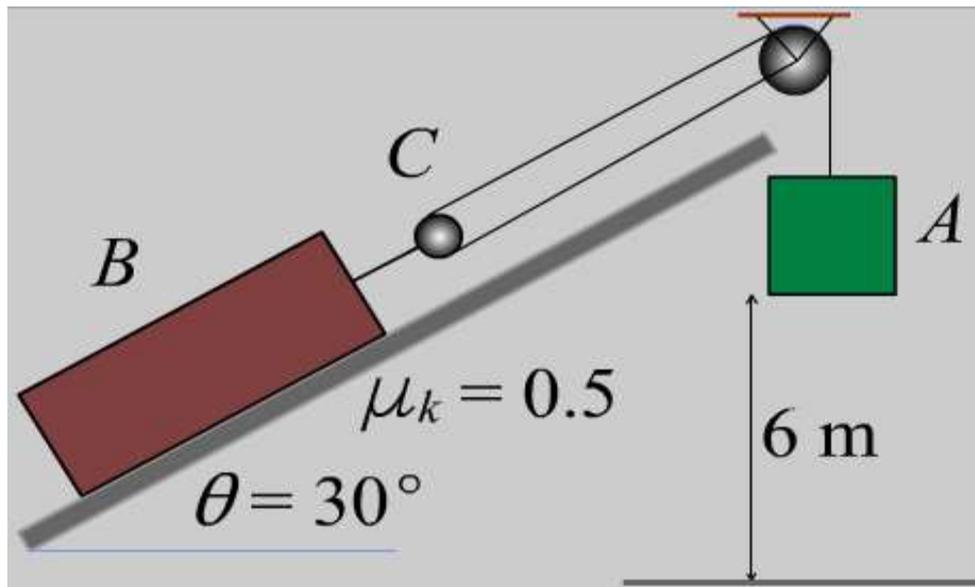
Solution: Pulley C

$$\Sigma F_x = 0$$

$$T_2 - 2T_1 = 0$$

- $125(9.81) - T_1 = 125 a_A$ (1)

- $T_2 - 2T_1 = 0$ (2)



Solution: Block B

- $125(9.81) - T_1 = 125 a_A$ (1)

- $T_2 - 2T_1 = 0$ (2)

- $N - 1699 = 0$ (3)

- $-T_2 + 0.5N + 200(9.81) \sin 30 = 200a_B$ (4)

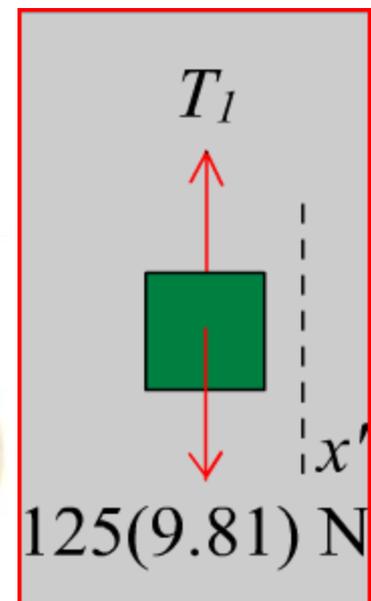
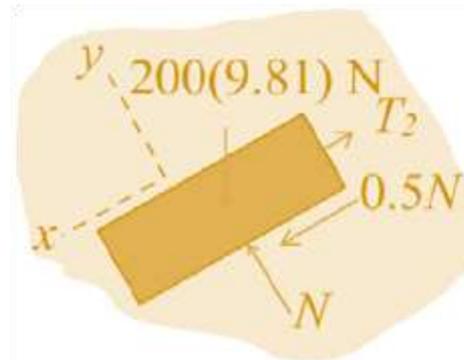
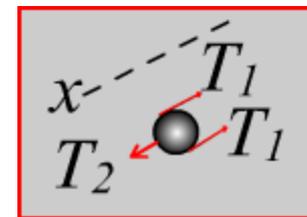
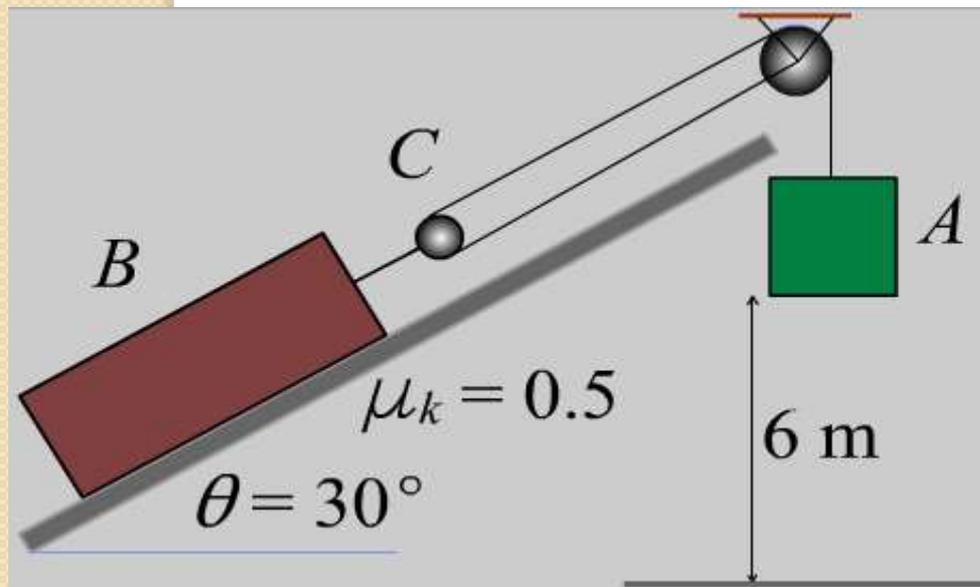
$$\Sigma F_y = 0 \rightarrow N - W_b \cos \theta = 0$$

$$N - 1699 = 0 \quad (3)$$

$$\Sigma F_x = m_B a_B$$

$$-T_2 + \mu_k N + W_b \sin \theta = m_B a_B$$

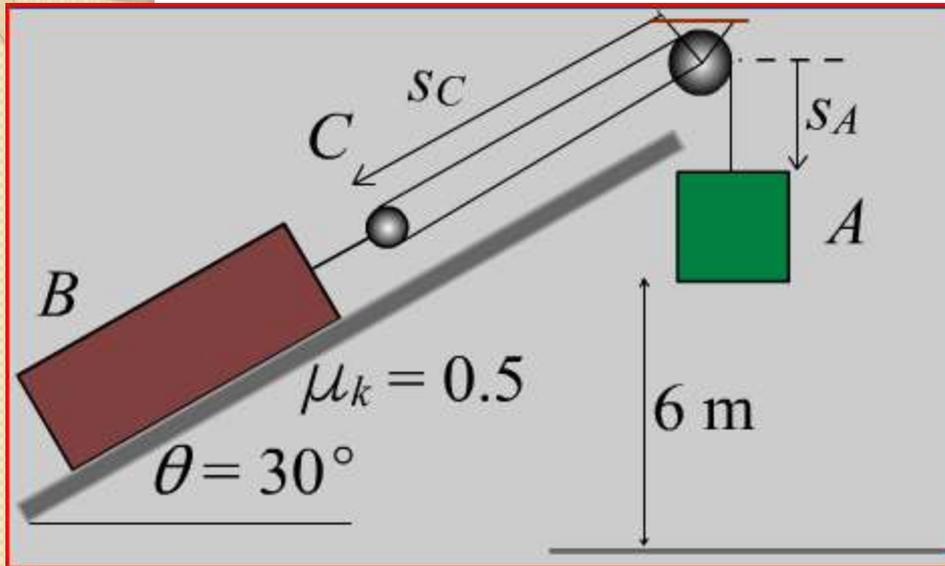
$$-T_2 + 0.5N + 200(9.81) \sin 30 = 200a_B$$



Solution:

- $125(9.81) - T_1 = 125 a_A$ (1)
- $T_2 - 2T_1 = 0$ (2)
- $N - 1699 = 0$ (3)
- $-T_2 + 0.5N + 200(9.81)\sin 30 = 200a_B$ (4)
- **Number of unknowns:** T_1, a_A, T_2, N, a_B
- **5 Unknowns with 4 equations. Look for geometric constraints.**

Kinematic constraint



- String Length:
 $L = 2S_C + S_A + \text{constant}$
- Differentiating twice:
 $2a_C + a_A = 0$
 $a_B = a_C$
 $2a_B + a_A = 0 \quad (5)$

5 Unknowns and 5 Equations.

Solution

- $T_1 = 1004 \text{ N}$
- $T_2 = 2008 \text{ N}$
- $N = 1699 \text{ N}$
- $a_A = 1.777 \text{ m/s}^2$ (down)
- $a_B = -0.888 \text{ m/s}^2$ (up the ramp)
- For constant acceleration:

$$v_A^2 = 2v_0^2 + 2a_A(s - s_0) \quad \longrightarrow \quad v_A = 4.62 \text{ m/s down}$$
$$v_A^2 = 0^2 + 2(1.777)(6)$$

Summary:

- When a particle of mass m is subjected to the action of concurrent unbalanced forces $\mathbf{F}_1, \mathbf{F}_2, \dots$, its equation of motion is written as:

$$\Sigma \mathbf{F} = m\mathbf{a}$$

- Equating the coefficients of like unit vectors yield:

$$\Sigma F_x = ma_x \qquad \Sigma F_y = ma_y \qquad \Sigma F_z = ma_z$$

- Depending on the number of degrees of freedom (DOF) of the problem, we decide the number of differential equations that will be necessary to describe the motion.

$$\Sigma F_x = ma_x$$

$$\Sigma F_y = 0$$

$$\Sigma F_z = 0$$

Normal & Tangential Coordinates

- Here, we merely componentiate $\Sigma \mathbf{F}$ and \mathbf{a} in $(\Sigma \mathbf{F} = m\mathbf{a})$ along the normal and tangential axes n and t .

$$\Sigma \mathbf{F} = m\mathbf{a}$$

$$\Sigma F_n \mathbf{e}_n + \Sigma F_t \mathbf{e}_t = m(a_n \mathbf{e}_n + a_t \mathbf{e}_t)$$

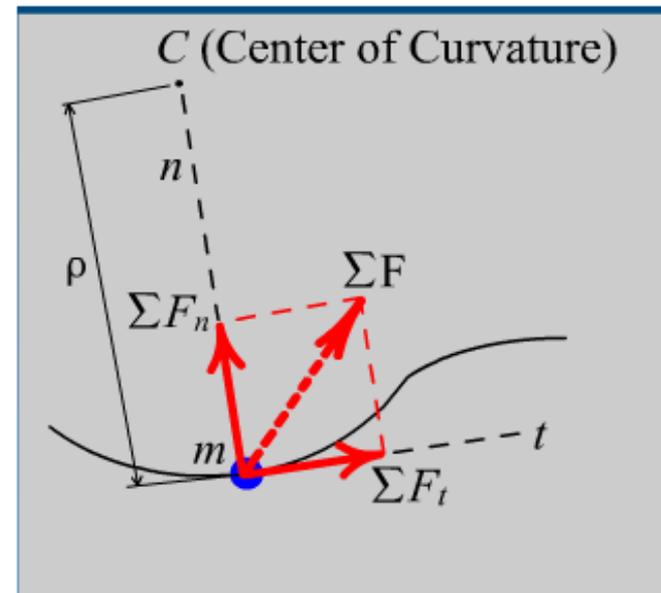
- Equating the coefficients of like unit vectors yields

$$\Sigma F_n = ma_n$$

$$a_n = \rho \dot{\beta}^2 = v^2 / \rho = v \dot{\beta}$$

$$\Sigma F_t = ma_t$$

$$a_t = \dot{v}$$

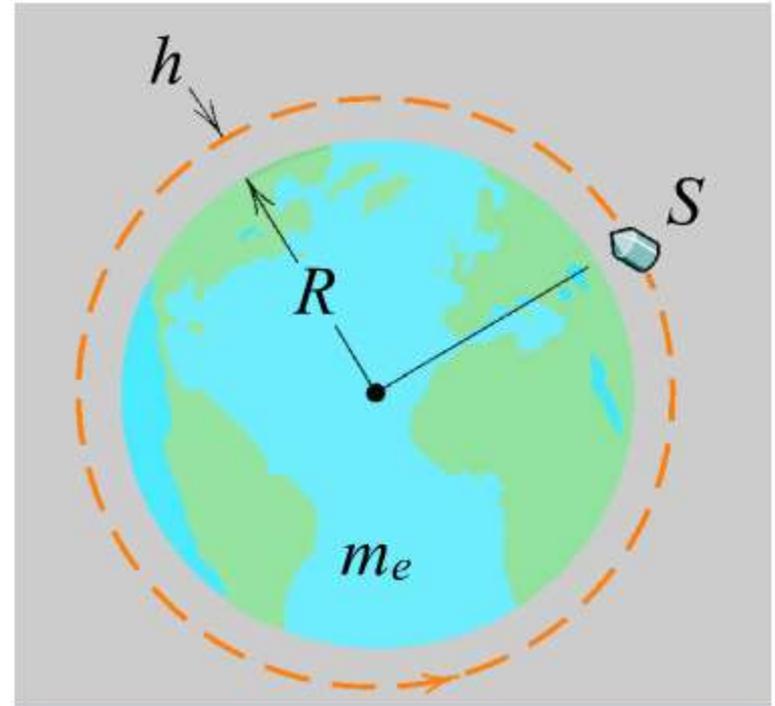
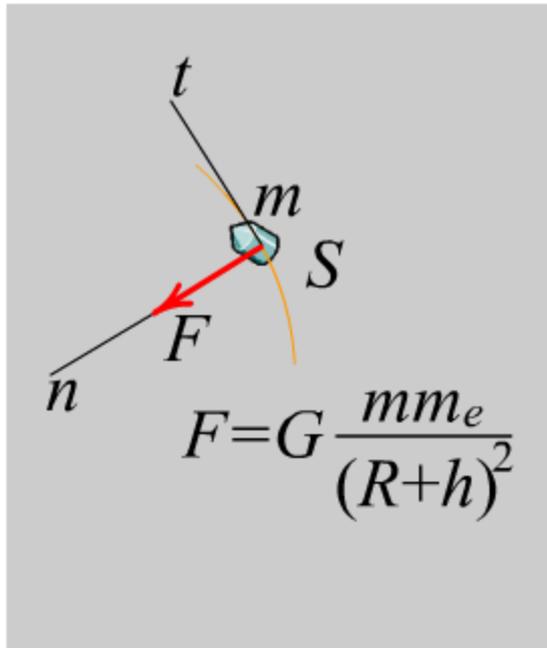


Curvilinear Motion

Coordinates	Newton's Law	Acceleration
Rectilinear	$\Sigma F_x = ma_x$	$a_x = \ddot{x}$
	$\Sigma F_y = ma_y$	$a_y = \ddot{y}$
Normal and Tangential	$\Sigma F_n = ma_n$	$a_n = \rho \dot{\beta}^2 = v^2 / \rho = v \dot{\beta}$
	$\Sigma F_t = ma_t$	$a_t = \dot{v}$
Polar coordinates	$\Sigma F_r = ma_r$	$a_r = \ddot{r} - r\dot{\theta}^2$
	$\Sigma F_\theta = ma_\theta$	$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$

Review Examples

Compute the magnitude v of the velocity for the spacecraft S to maintain a circular orbit of altitude 320 km above the surface of the earth.



Review Examples

$$\Sigma F_n = ma_n: G \frac{mm_e}{(R+h)^2} = m \frac{v^2}{(R+h)}$$

$$v = \sqrt{\frac{Gm_e}{(R+h)}} = R \sqrt{\frac{g}{(R+h)}} \quad (gR^2 = Gm_e)$$

For:

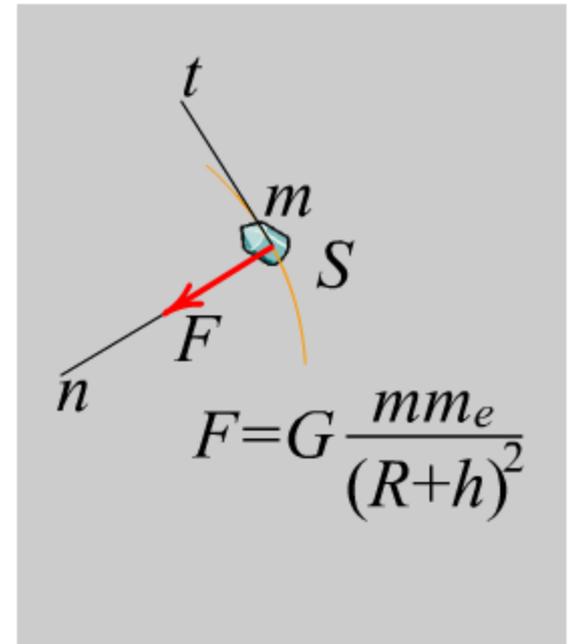
$$R = 6371 \text{ km} = 6\,371\,000 \text{ m}$$

$$h = 320 \text{ km} = 320\,000 \text{ m}$$

$$g = 9.825 \text{ m/s}^2$$

$$v = R \sqrt{\frac{g}{(R+h)}}$$

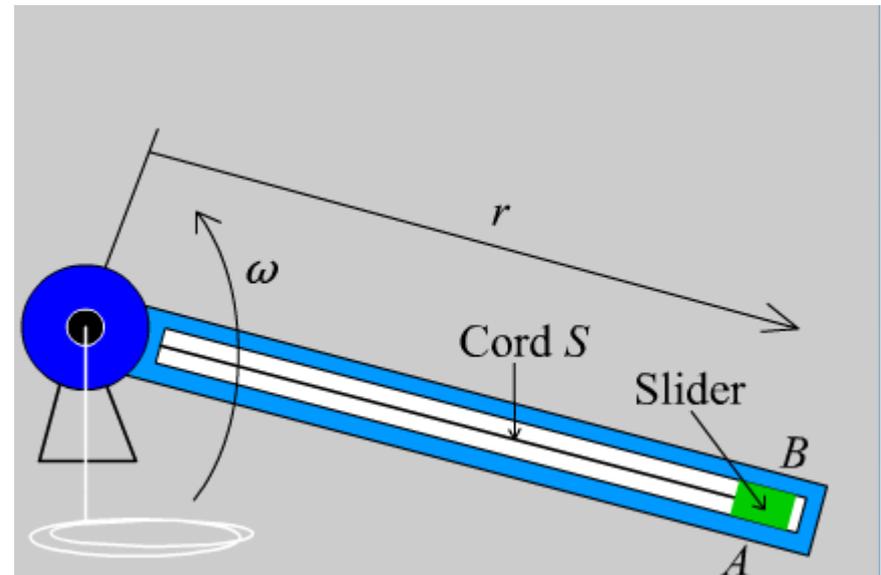
$$v = 7720 \text{ m/s}$$



Example 2

As the slotted arm rotates about a vertical axis, the 2-kg slider is moved along the slot by pulling the cord S . At the end of the illustrated motion the conditions are $r = 225$ mm, $\dot{r} = -50$ mm/s (constant) $\omega = 6$ rad/s and $\dot{\omega} = -2$ rad/s².

In the final instant depicted, determine the tension T in the cord and the normal force N exerted by the side of the slot, on the slider. Which side of the slot A or B , contacts the slider?



Example 2

$$\Sigma F_r = ma_r = m(\ddot{r} - r\dot{\theta}^2)$$

$$\Sigma F_\theta = ma_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

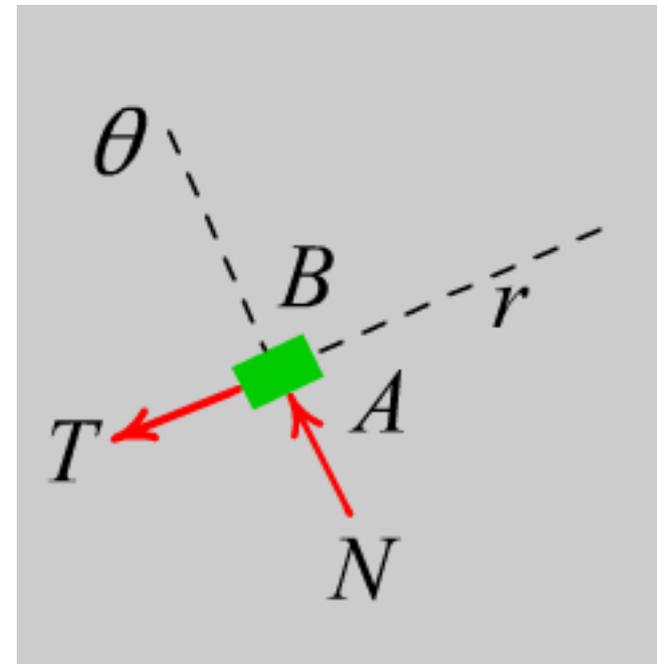
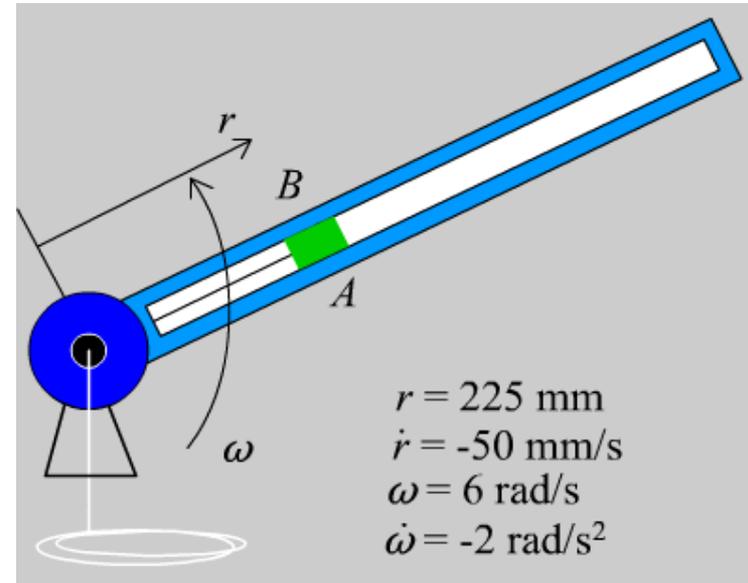
$$-T = 2[0 - 0.225(6)^2]$$

$$N = 2 [0.225(-2) + 2(-0.05)(6)]$$

$$T = 16.2 \text{ N}$$

$$N = -2.1 \text{ N}$$

Contact is on side *B*, *N*: ↘



Thank you very much.

**THAT IS ALL FOR
TODAY.**

Topics

- Work:
 - Definition: Energy, Force, Work, Power
 - Calculation of work
 - Work associated with a constant external force
 - Work associated with a spring force
 - Work associated with weight
- Work and Curvilinear Motion.
- Principle of Work and Kinetic Energy.
- Power.
- Efficiency.

Definition of Work

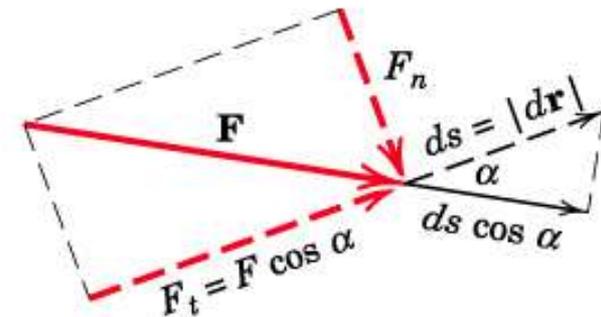
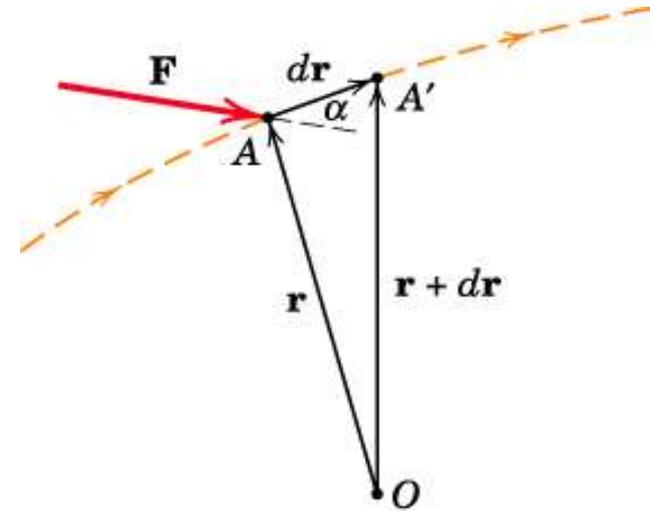
- When a force \mathbf{F} acts on a particle A (at position \mathbf{r}) and cause it to move to position A' and have a net differential displacement $d\mathbf{r}$, **it exerts work**.
- The work done by \mathbf{F} on the particle during the displacement $d\mathbf{r}$ is defined as:

$$dU = \mathbf{F} \cdot d\mathbf{r}$$

- Work is a scalar quantity – (the dot product) and can be written as:

$$dU = |\mathbf{F}| |d\mathbf{r}| \cos \alpha = F ds \cos \alpha$$

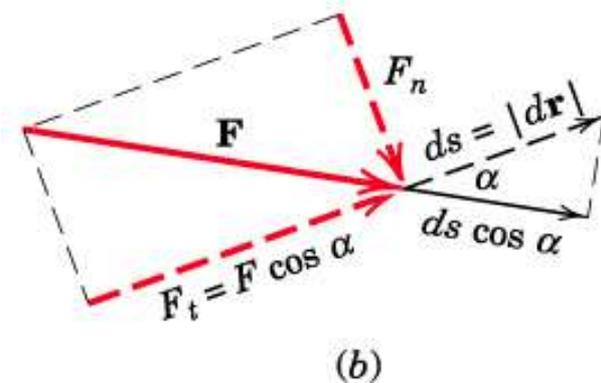
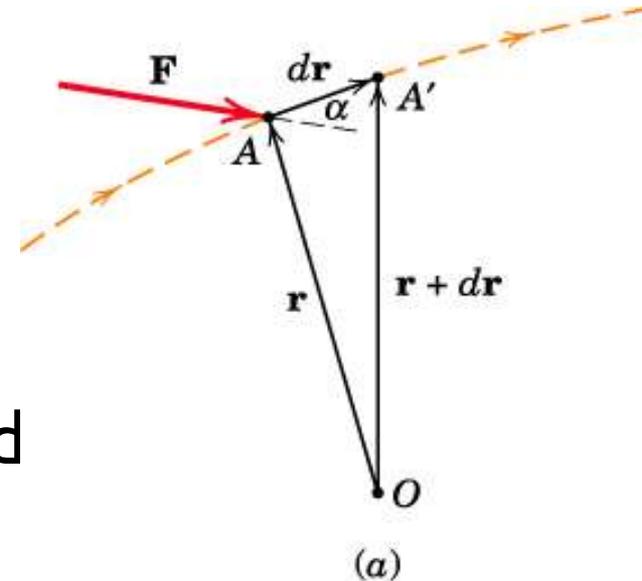
- ds is magnitude of $d\mathbf{r}$ and along the path.



Work Definition

$$dU = F ds \cos \alpha = F_t ds$$

- Work is positive if F_t is in the direction of the displacement and is negative if it is in the opposite direction.
- Tangent forces (F_t) are **Active forces** and produce work.
- Normal forces (F_n) are **Passive forces** and produce no work.

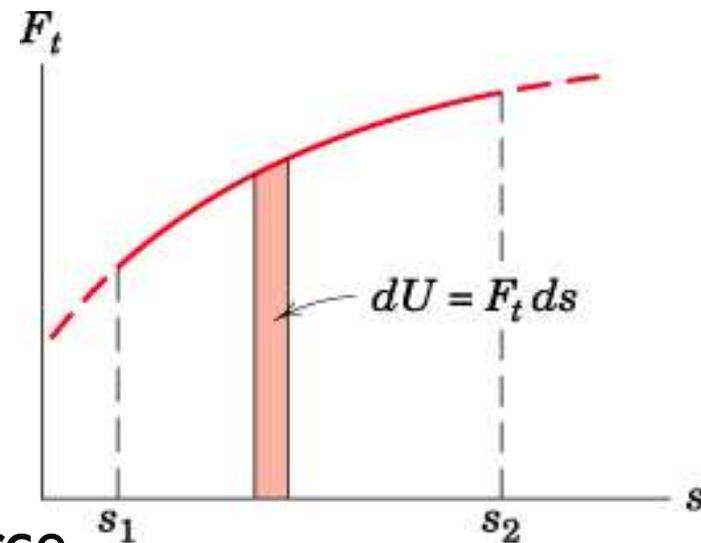


Calculation of Work:

- During a finite displacement of the point of application of the force $d\mathbf{r}$, the force does a finite work $dU = \mathbf{F} \cdot d\mathbf{r}$.

$$U = \int_1^2 \mathbf{F} \cdot d\mathbf{r} = \int_1^2 (F_x dx + F_y dy + F_z dz)$$

$$\text{or } U = \int_{s_1}^{s_2} F_t ds$$



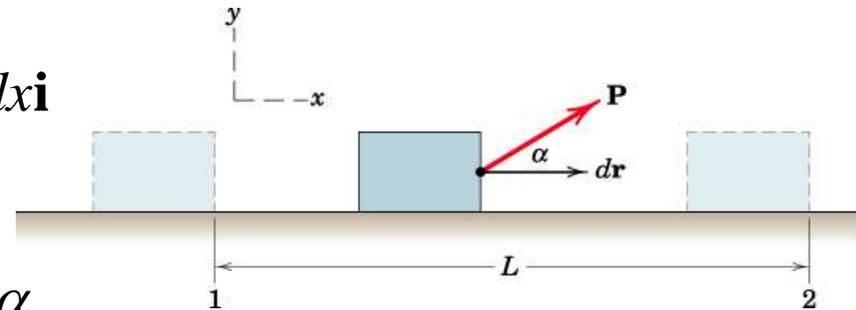
- The units of work are that of force times displacement.
- In the SI system (N.m) (called Joule).
- In the US system lb-ft or ft-lb.

Work Done: Constant External Force

- Constant force \mathbf{P} is applied to body and it moves from position 1 to position 2.
- The work done by the force can be determined as:

$$U = \int_1^2 \mathbf{F} \cdot d\mathbf{r} = \int_1^2 [(P \cos \alpha)\mathbf{i} + (P \sin \alpha)\mathbf{j}] \cdot dx\mathbf{i}$$

$$U = \int_{x_1}^{x_2} P \cos \alpha dx = P \cos \alpha (x_2 - x_1) = PL \cos \alpha$$



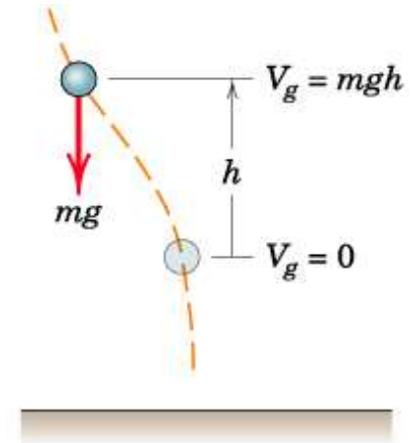
- The work is the component $P_x = P \cos \alpha$ of P in the direction of movement times the distance L traveled.

$$U_{1-2} = (P \cos \alpha)L$$

Work Associated with Weight

- Constant body weight (force) associated with constant gravity.

$$\begin{aligned} U &= \int_1^2 \mathbf{F} \cdot d\mathbf{r} = \int_1^2 -mg\mathbf{j} \cdot (dx\mathbf{i} + dy\mathbf{j}) \\ &= -mg(y_2 - y_1) = mg(y_1 - y_2) \end{aligned}$$



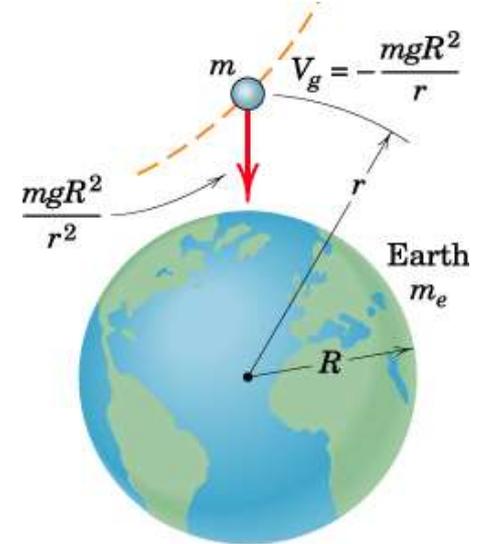
- The work done on the particle by its constant weight is $U_{1-2} = mg(y_1 - y_2)$.
- It is clear that U_{1-2} is positive if the altitude of the particle decreases and U_{1-2} is negative if the altitude increases.

Work Associated with Weight

- Particle moves in a variable gravity field:
 - changes in the altitude leads to changes in the gravitational acceleration.

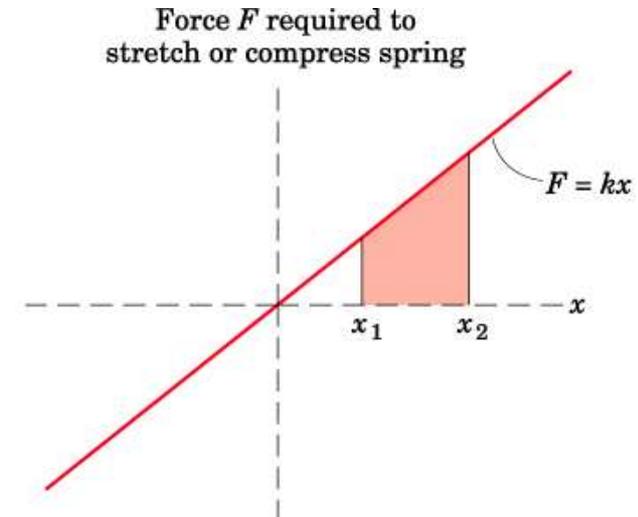
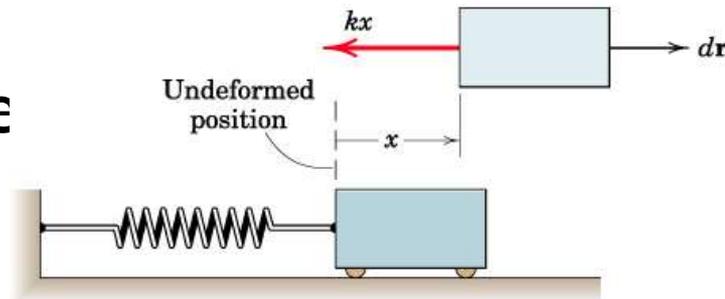
$$U_{1-2} = \int_1^2 \mathbf{F} \cdot d\mathbf{r} = \int_1^2 -G \frac{mm_e}{r^2} \mathbf{e}_r \cdot (dr \mathbf{e}_r)$$
$$= -Gmm_e \int_{r_1}^{r_2} \frac{dr}{r^2} = Gmm_e \left[\frac{1}{r_2} - \frac{1}{r_1} \right]$$

- The work done on the particle by its variable weight F is U_{1-2} .
- Again, the work is negative if the altitude increases and positive if the altitude decreases.



Work Associated with Spring Force

- Springs are usually attached to bodies to either generate forces or store energy in the system.
- The spring has a stiffness k .
- A spring stretched or compressed a distance x will have a force proportional to its deflection (deformation) as:
$$F = kx$$

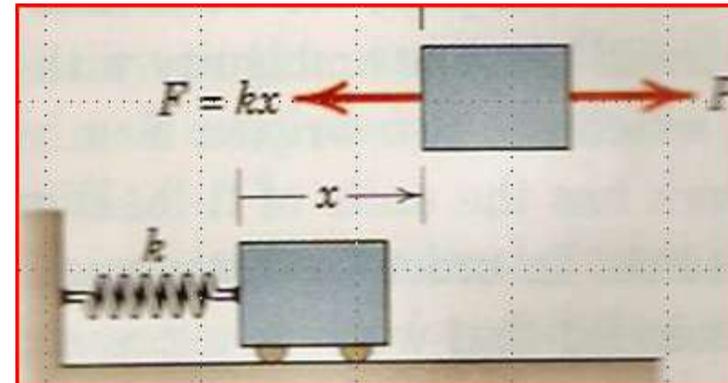
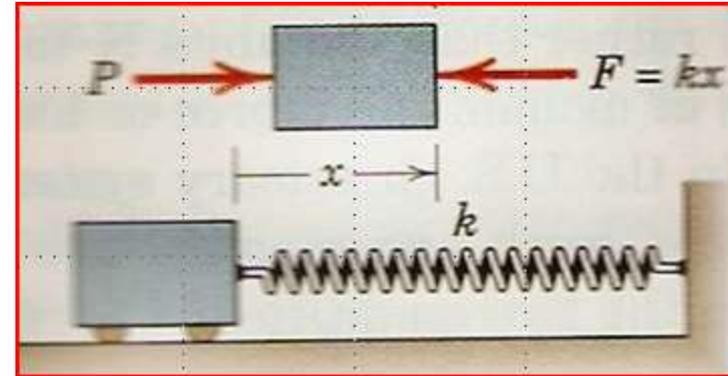


Work Associated with Spring Force

- Figure shows the results of compressing and stretching a spring and the spring reaction forces.
- ***The force exerted by a spring on the body is always in a direction opposite to the spring displacement.***
- The spring then does negative work on the body. Thus the work done by the spring on the body is written as:

$$U_{1-2} = \int_1^2 \mathbf{F} \cdot d\mathbf{r} = \int_1^2 (-kx) \mathbf{i} \cdot dx \mathbf{i}$$

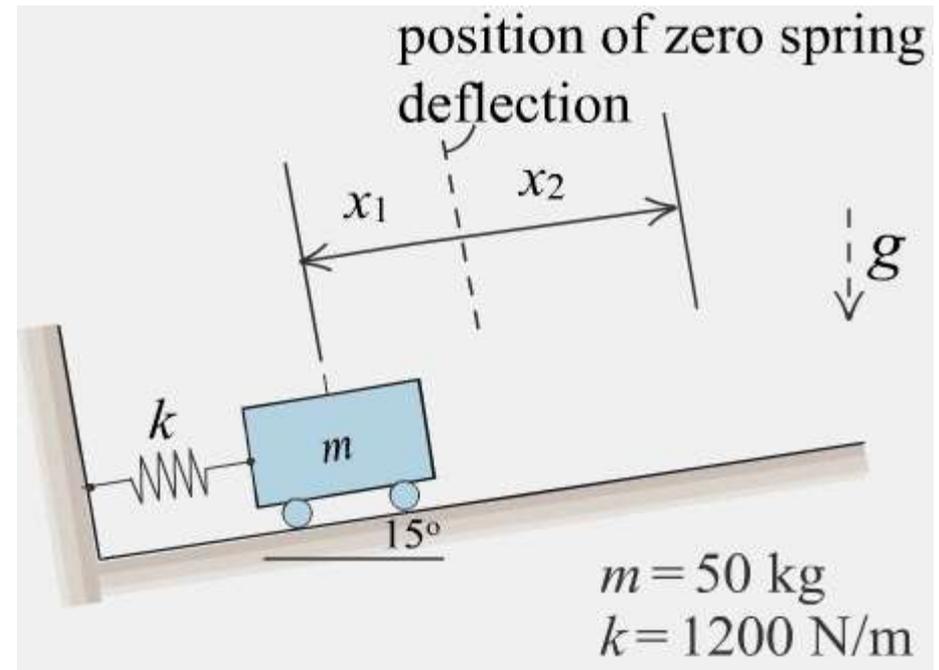
$$U_{1-2} = -\int_{x_1}^{x_2} kx dx = -\frac{1}{2}k(x_2^2 - x_1^2)$$



Example

For reasons that are not important here, the box moves from $x_1 = -100$ mm at position 1 to $x_2 = 150$ mm at position 2. Determine the following:

- (1) The work done on the box by its weight.
- (2) The work done on the box by the linear spring.
- (3) The change (final minus initial) in gravitational potential energy.
- (4) The change in elastic (spring) potential energy.

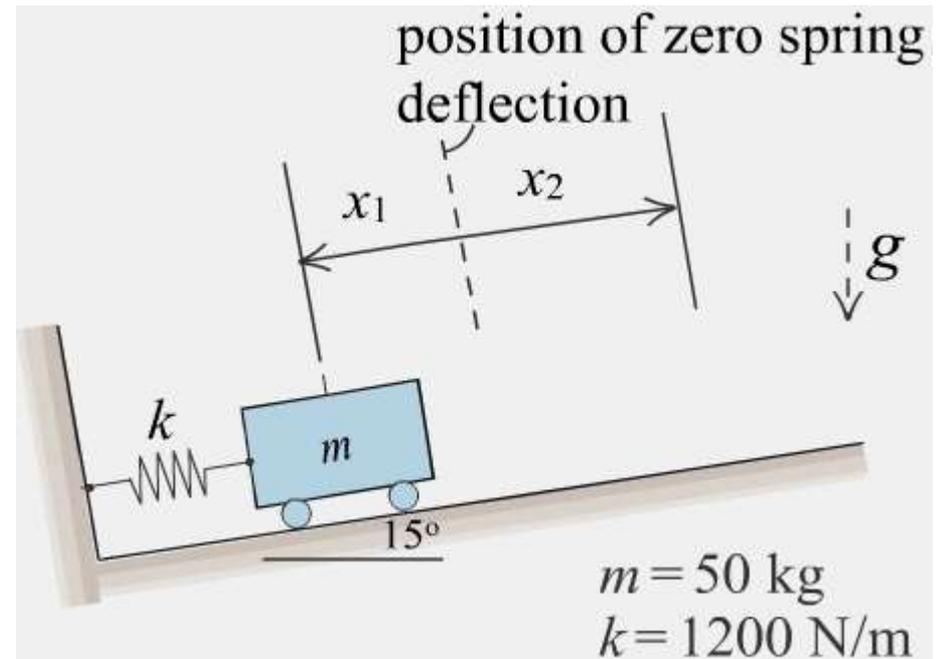


The work done on the box by its weight is:

$$\begin{aligned}(U_{1-2})_W &= -Wh \\ &= -50(9.81)[0.25\sin 15^\circ] \\ &= -31.7 \text{ J}\end{aligned}$$

The work done on the box by the linear spring is:

$$\begin{aligned}(U_{1-2})_s &= \frac{1}{2} k (x_1^2 - x_2^2) \\ &= \frac{1}{2} (1200) [0.1^2 - 0.15^2] \\ &= -7.5 \text{ J}\end{aligned}$$



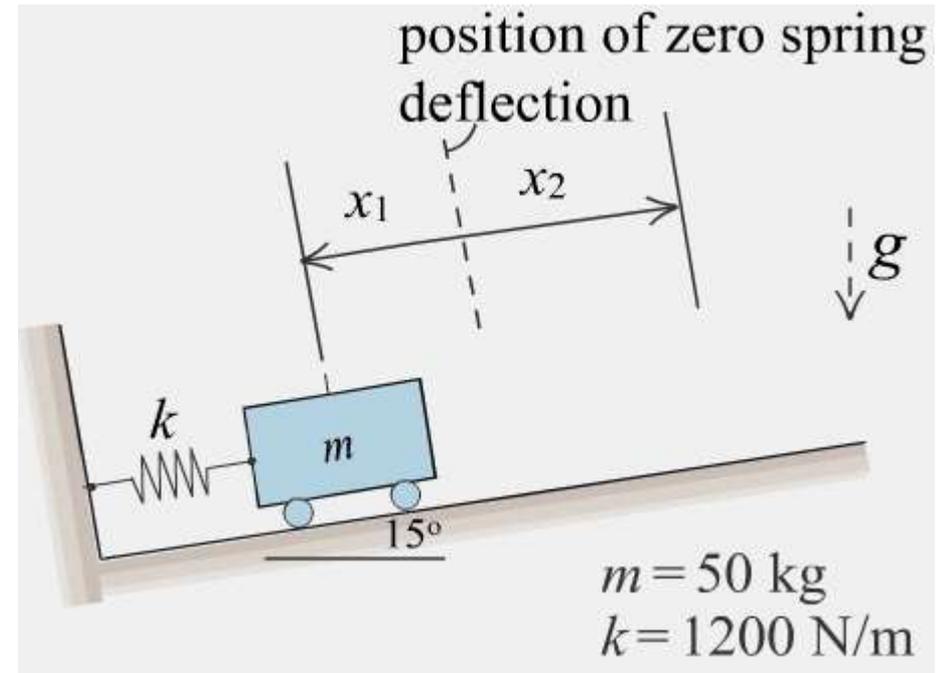
The change (final minus initial) in gravitational potential energy is:

$$\begin{aligned}\Delta V_g &= -(U_{1-2})_W \\ &= -(-31.7) \\ &= 31.7 \text{ J}\end{aligned}$$

Alternatively, we define a **datum** at position 1, so that

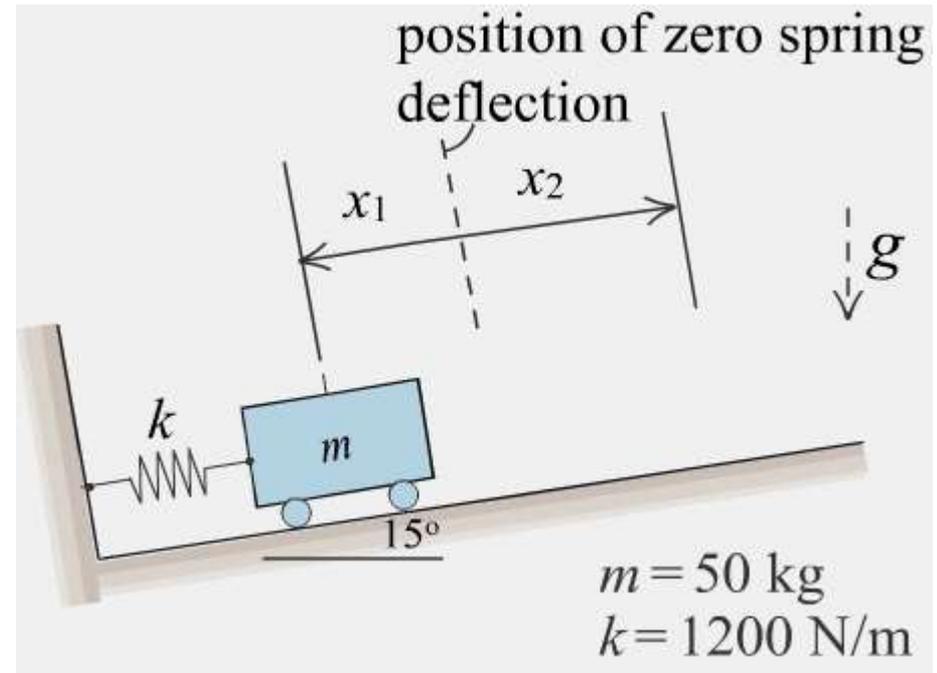
$$\begin{aligned}\Delta V_g &= mgy_2 - mgy_1 \\ &= 50(9.81)[0.25\sin 15^\circ] - 0 \\ &= 31.7 \text{ J}\end{aligned}$$

Recall that a datum is defined to be the altitude at which the gravitational potential energy is zero.



The change in elastic
(spring) potential energy is:
 $\Delta V_e = -(U_{1-2})_s = -(-7.5)$
 $= 7.5 \text{ J}$

Alternatively,
 $(U_{1-2})_s = \frac{1}{2} k (x_2^2 - x_1^2)$
 $= \frac{1}{2} (1200) [0.15^2 - 0.1^2]$
 $= 7.5 \text{ J}$



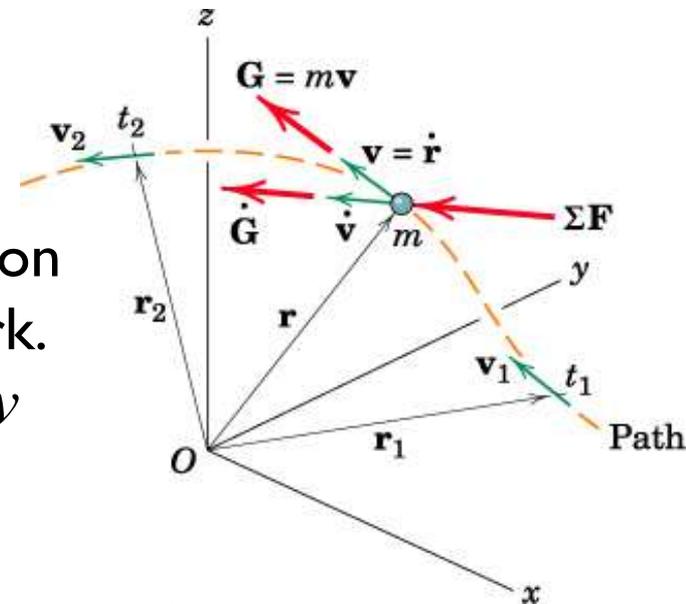
Work and Curvilinear Motion

- When a work is exerted on particle of mass m to move it along a curvilinear path from a position defined by vector \mathbf{r} under the action of force \mathbf{F} ($\Sigma\mathbf{F}$), it attains a displacement $d\mathbf{r}$ in time dt .
- The work done on the particle to move it from position 1 to 2 is given by:

$$U_{1-2} = \int_1^2 \mathbf{F} \cdot d\mathbf{r} = \int_{s_1}^{s_2} F_t ds$$

- Now, consider substituting Newton's relation (i.e $\mathbf{F} = m \mathbf{a}$) in the above equation for work.
- Recall that: $\mathbf{a} \cdot d\mathbf{r} = a_t ds$ and $a_t ds = v dv$

$$F_t ds = ma_t ds = m \left(v \frac{dv}{ds} \right) ds = mv dv$$



Work and Curvilinear Motion

- Thus, the expression of work due to \mathbf{F} is:

$$\int_1^2 dU = \int_1^2 mv dv \quad \longrightarrow \quad U_{1-2} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 = T_2 - T_1$$

where $(T = \frac{1}{2} m v^2)$ is the particle kinetic energy.

- The kinetic energy (KE) of a particle is defined as the total work done on a particle to bring it from a state of rest to that of velocity v .
- KE has same units as work which is (J) and is also scalar.
- KE always positive.

Principle of Work and Kinetic Energy

- The work-energy relation can be written as:

$$T_1 + U_{1-2} = T_2$$

When a particle is acted upon by an active force, the final kinetic energy of the particle will be the sum of the particle initial kinetic energy and the amount of work produced by the force.

Power

- Is the capacity of a machine ...
- The power P is the time rate of doing work.
- The power P developed by a force \mathbf{F} doing work U is given as

$$P = \frac{dU}{dt} = \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} \quad \longrightarrow \quad P = \mathbf{F} \cdot \mathbf{v}$$

- The SI units for power are $\text{N}\cdot\text{m}/\text{s} = \text{J}/\text{s} = \text{watt}$ (W).
- The US units are horsepower (hp) = 550 ft-lb/s
- 1 hp = 746 W = 0.746 KW.

Efficiency

- The ratio of the work done by the machine to that done on the machine during the same time interval is called the mechanical efficiency, e_m .
- Due to losses, the efficiency is always less than unity.
- Losses are due to friction, thermal losses to the environment, machine backlashes, electrical losses and others.
- The mechanical efficiency is given by:
$$e_m = \frac{P_{output}}{P_{input}}$$
- The system overall efficiency can be written as:

$$e_T = e_m \cdot e_e \cdot e_{th}$$

where e_e and e_{th} are electrical and thermal, respectively.

The Value of Human Work

Among the quaint units still in engineering use is the "horsepower":

$$1 \text{ hp} = 550 \frac{\text{ft}\cdot\text{lb}}{\text{sec}} \text{ (rate of doing work)}$$

Then

$$\begin{aligned} 1 \text{ hp} &= \left(735 \frac{\text{ft}\cdot\text{lb}}{\text{sec}} \right) \left(\frac{4.448 \text{ N}}{\text{lb}} \right) \left(\frac{1 \text{ m}}{3.281 \text{ ft}} \right) \\ &= 745.6 \text{ N}\cdot\text{m/s} \\ &= 745.6 \text{ watts} = 0.746 \text{ kw} \end{aligned}$$

A human can (over an extended period) do work at the rate of about 1/7 hp or about 0.1 kw. Since horses now spend most of their time in pastures, let us consider a new unit of power -- the personpower (pp) !! An obvious definition is

$$1 \text{ pp} = 0.1 \text{ kw}$$

In an 8-hour work day, a person can do $(0.1 \text{ kw})(8 \text{ h}) = 0.8 \text{ kw-h}$ of work, which is worth about 5¢ if you were to buy it from your local power company.

One gallon of gasoline (current worth about \$4.00) can release about 140,000 BTU when burned:

$$(140,000 \text{ BTU}) \left(\frac{1 \text{ kw-h}}{3413 \text{ BTU}} \right) = 41 \text{ kw-h,}$$

or about as much work as you could do in 51 8-hour days !

Reference: Harold S. Morton, Jr. (late).

Topics

- Potential Energy: Definition
- Elastic Potential Energy.
- Gravitational Potential Energy.
- Work Energy Equation.
- Conservative Force Fields.

Conservative vs. Non-conservative force

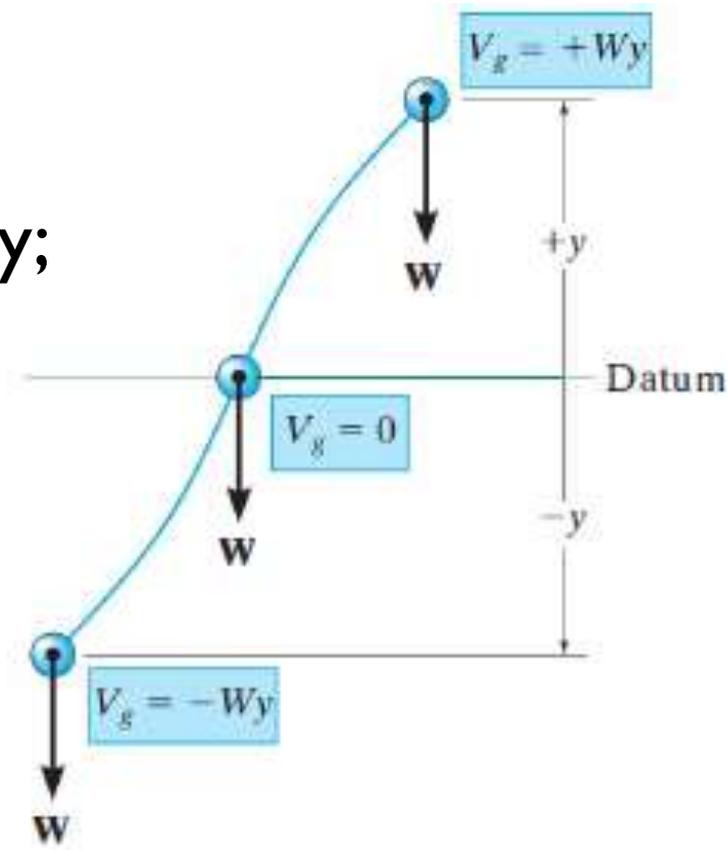
- The work of a conservative force is independent of the path and depends only on the force's initial and final positions , e.g. Weight, Spring Force.
 - Work done by the weight depends only on the height.
 - Work done by a spring force depends only on the spring's elongation or compression.
 - The work done by a friction force depends on the path-the longer the path, the greater the work.
- ➡ Friction force is non-conservative. The work is dissipated from the body in the form of heat.

Energy Definition

- Energy: Capacity for doing work
- kinetic energy: A measure of the particle's capacity to do work, which is associated with the motion of the particle.
- Potential energy: A measure of the amount of work a conservative force will do when it moves from a given position to the datum or reference plane

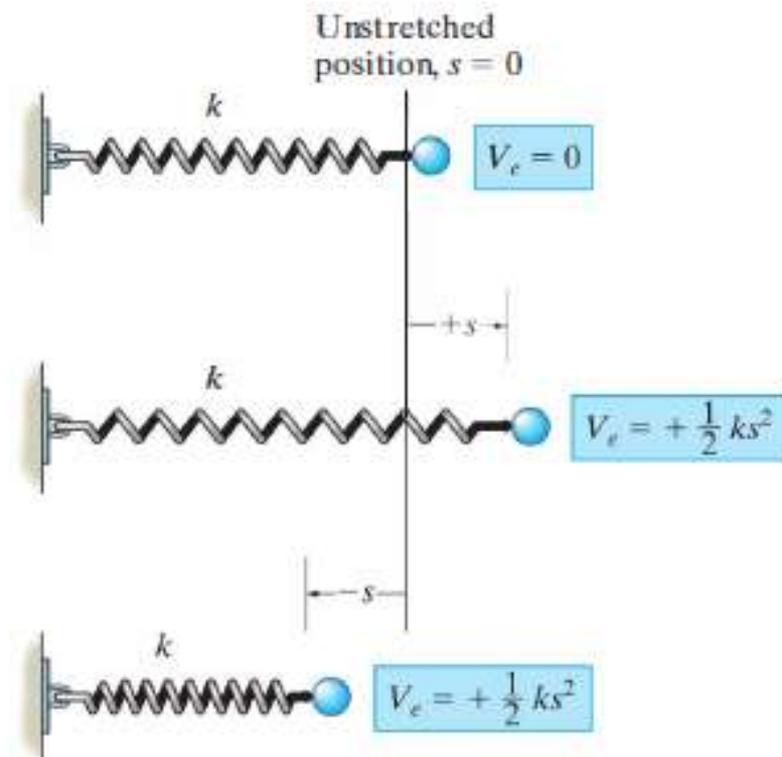
Gravitational Potential Energy

- A particle above datum has positive gravitational potential energy; W has capacity of doing positive work when the particle is moved back down to the datum.
- A particle below datum, has negative potential energy; The weight does negative work when the particle is moved back up to the datum.



Elastic Potential Energy

- Elastic potential energy always positive.
- Spring always has capacity of doing positive work when returned to its unstretched position.



Potential Energy: Definition

- When a body is given potential energy, it will have the ability to acquire kinetic energy.
- **Potential Energy**: is the work done on a body (particle) against some other force, so that when the body is then released, that other force gives the body kinetic energy.

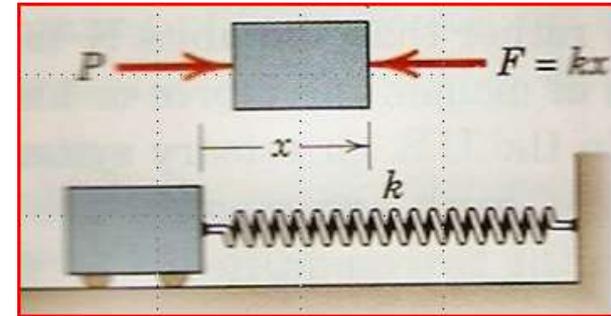
$$T_1 + U_{1-2} = T_2$$

- Potential energy can be associated with conservative forces: Gravitational force, Elastic spring force.
- Non-Conservative forces (energy consuming) have no potential energy but we can calculate its work, e.g. Friction forces.

Work Associated with Spring Force

The work done **by** a spring **on** a mass is

$$U_{1-2} = \frac{1}{2}k(x_1^2 - x_2^2)$$



The work done **on** the spring **by** the mass is:

$$-\frac{1}{2}k(x_1^2 - x_2^2)$$

The change in potential energy is this work done against the spring:

$$\Delta V = V_2 - V_1 = -\frac{1}{2}k(x_1^2 - x_2^2)$$

In General:

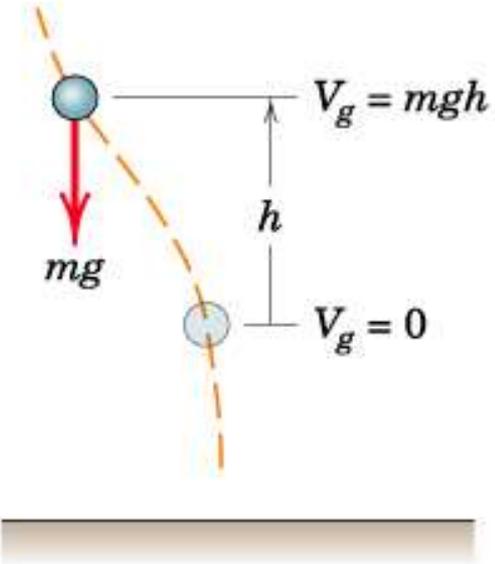
$$V = \frac{1}{2}kx^2$$

Gravitational Potential Energy

In lifting an object from position y_1 to position y_2 , the work done by lifting force against the gravity is $mg(y_2 - y_1)$.

The change in gravitational potential energy V_g is:

$$\Delta V_g = V_2 - V_1 = mg(y_2 - y_1) \quad \text{In General:} \quad V_g = mgh$$



Note: Raising the body increases its gravitational potential energy.

Work Energy Equation

- In general, $T_1 + U_{1-2} = T_2$, where U_{1-2} is the total work done on the body.
- We have seen that change in the potential energy is the negative of the work done on the body by the force:

$$\Delta V = - U_{1-2}$$

- The total work U_{1-2} can be split into work $(U_{1-2})_c$ associated with conservative forces and work U'_{1-2} associated with non-conservative forces, we can write:

$$T_1 + (U_{1-2})_c + U'_{1-2} = T_2$$

- Also, we can replace $(U_{1-2})_c$ with $-\Delta V = -(V_2 - V_1)$

Work Energy Equation

- We have the following:

$$T_1 + [- (V_2 - V_1)] + U_{1-2} = T_2$$

$$T_1 + V_1 + U_{1-2} = T_2 + V_2$$

- This result can be written as :

$$U_{1-2} = \Delta V + \Delta T$$

Where Δ represents the change in the energy (final minus initial).

Work Energy Equation

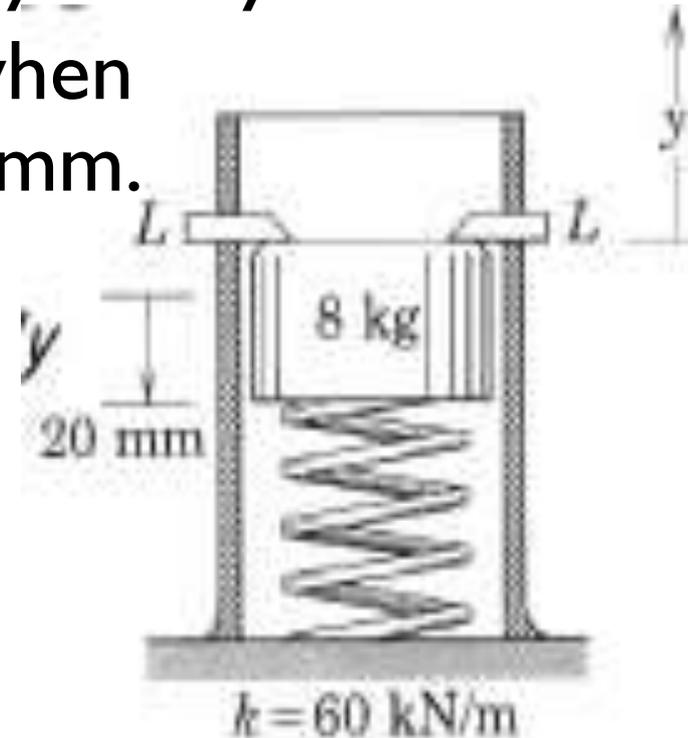
- Note that U_{1-2} is the work done by force not treated as potential energy.
- If $U_{1-2} = 0$ then we get:

$$T_1 + V_1 = T_2 + V_2$$

- In the case we can say that the mechanical energy is conserved: $E = T + V$.

Example

- The 8-kg cylinder is latched in place with the 60kN/m spring compressed a distance 20mm. If the two latches are simultaneously removed, determine:
 1. The maximum height reached by the cylinder
 2. The velocity v of the cylinder when the spring has moved up by 12 mm.The cylinder is not attached to the spring.



Example

- For the system

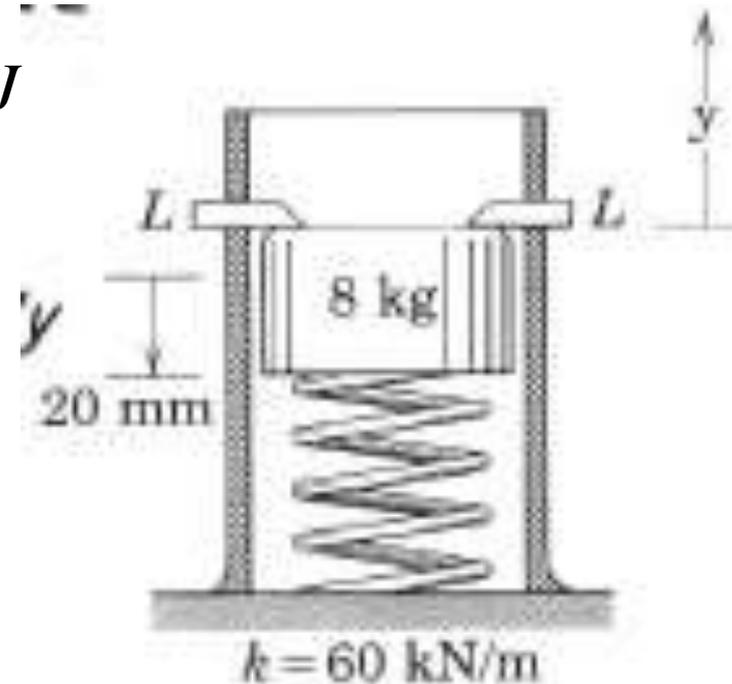
$$U_{1-2} = \Delta T + \Delta V_g + \Delta V_e$$

$$U_{1-2} = 0, \quad \Delta T = 0, \quad \Delta V_g = mgh$$

$$\Delta V_e = 0 - \frac{1}{2}(60000)(0.02)^2 = -12J$$

$$0 = 0 + 8(9.81)y - 12$$

$$y = 0.1529m$$



Example

- B-

$$U_{1-2} = \Delta T + \Delta V_g + \Delta V_e$$

$$U_{1-2} = 0$$

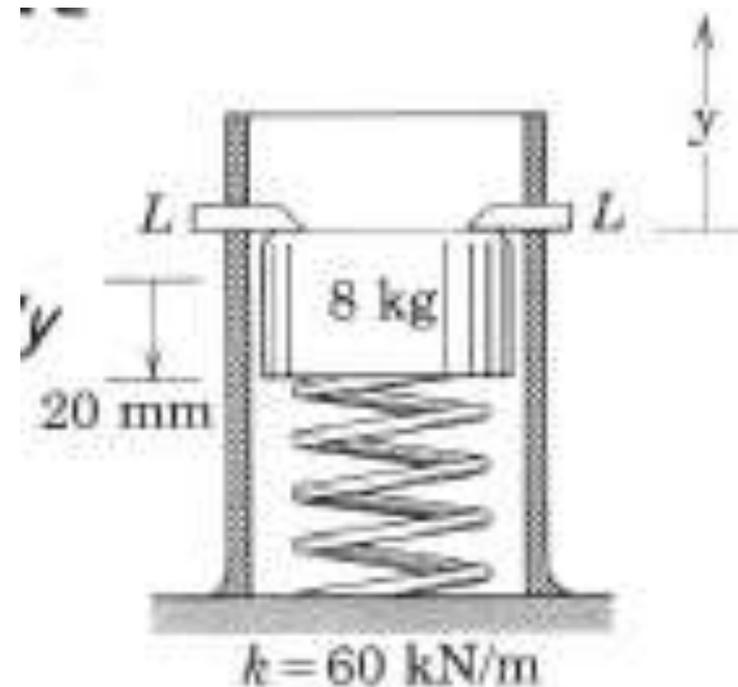
$$\Delta T = \frac{1}{2}(8)v^2 = 4\dot{y}^2$$

$$\Delta V_g = 8(9.81)(0.012) = 0.942$$

$$\begin{aligned}\Delta V_e &= \frac{1}{2}(0.008^2 - 0.02^2) \\ &= -10.08J\end{aligned}$$

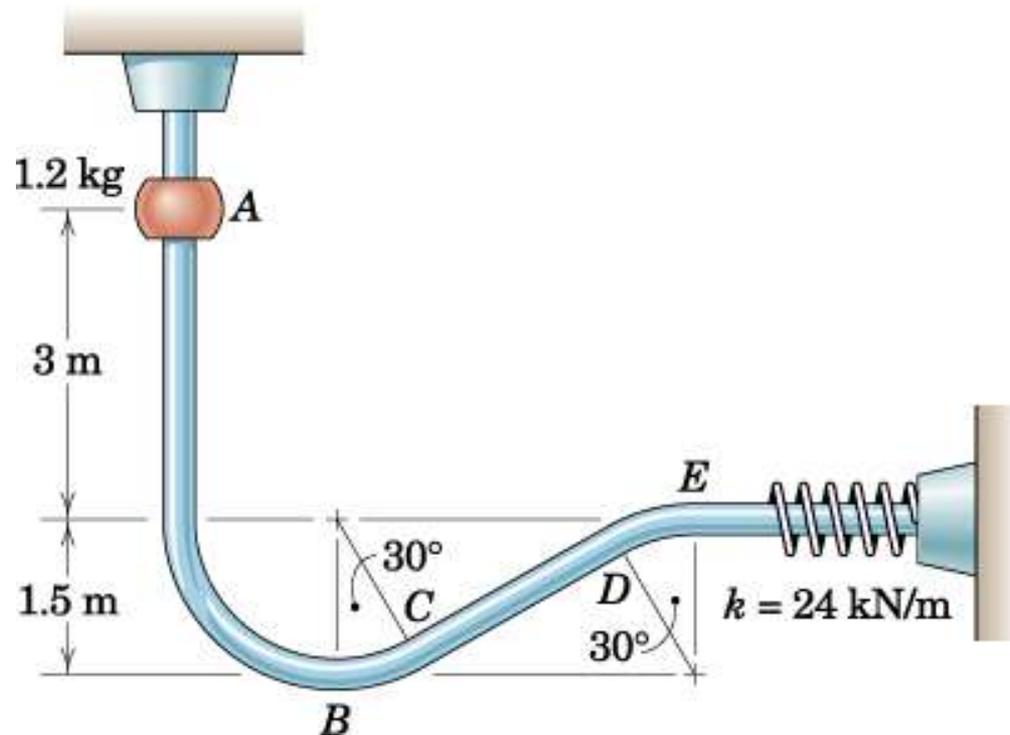
$$0 = 4\dot{y}^2 + 0.942 - 10.08$$

$$\dot{y}^2 = 2.28 \Rightarrow \dot{y} = 1.511m/s$$



Example:

A 1.2kg slider is released from rest in position *A* and slides without friction along the vertical plane guide as shown. Determine the maximum deflection δ of the spring by the potential energy method.



Solution

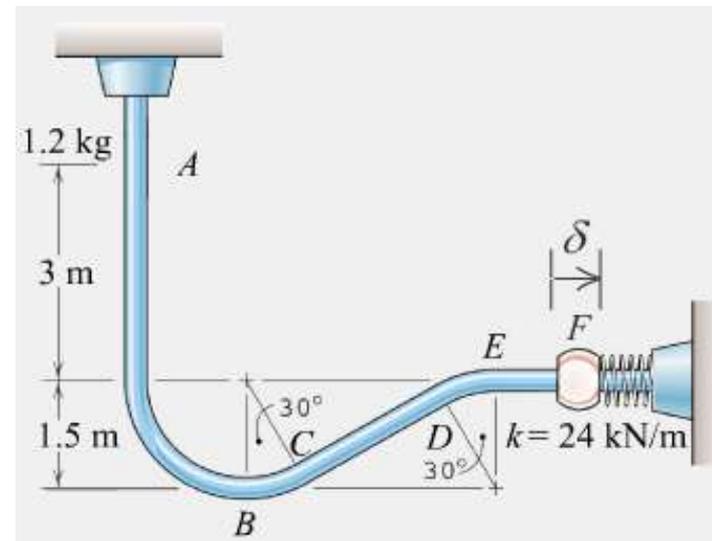
- Assume that state at F to be the state of maximum spring deflection. We can write:

$$T_A + V_A + U'_{A-F} = T_F + V_F$$

- There's no non-conservative forces $U'_{A-F} = 0$;
- We consider the datum to be at state A. Thus

$$0 + 0 + 0 = 0 + \frac{1}{2}k\delta^2 + mgh_{A-F}$$

$$\delta = \sqrt{\frac{2mgh_{A-F}}{k}} = \sqrt{\frac{2(1.2)(9.81)(3)}{24000}} \\ = 0.0542 \text{ m}$$



Topics

- Linear Impulse and Linear Momentum
- Conservation of Linear Momentum
- Angular Impulse and Angular Momentum
- Conservation of Angular Momentum
- Impact
- Example

Linear Momentum

- Linear momentum is defined as: $\mathbf{G} = m \mathbf{v}$
- Notice the linear momentum is a *vector quantity*.
- Recalling Newton's Second Law: $\Sigma \mathbf{F} = m \mathbf{a}$, substituting the acceleration: $\mathbf{a} = d \mathbf{v} / dt$ and the momentum into Newton's Second Law:

$$\Sigma \mathbf{F} = \frac{d \mathbf{G}}{dt} = \dot{\mathbf{G}}$$

- Multiplying both sides with dt yields: $\Sigma \mathbf{F} dt = d \mathbf{G}$
- Integrating both sides gives:

$$\int_{t_1}^{t_2} \Sigma \mathbf{F} dt = \int_{\mathbf{G}_1}^{\mathbf{G}_2} d \mathbf{G} = \mathbf{G}_2 - \mathbf{G}_1$$

Linear Momentum

- or , this can be written as follows:

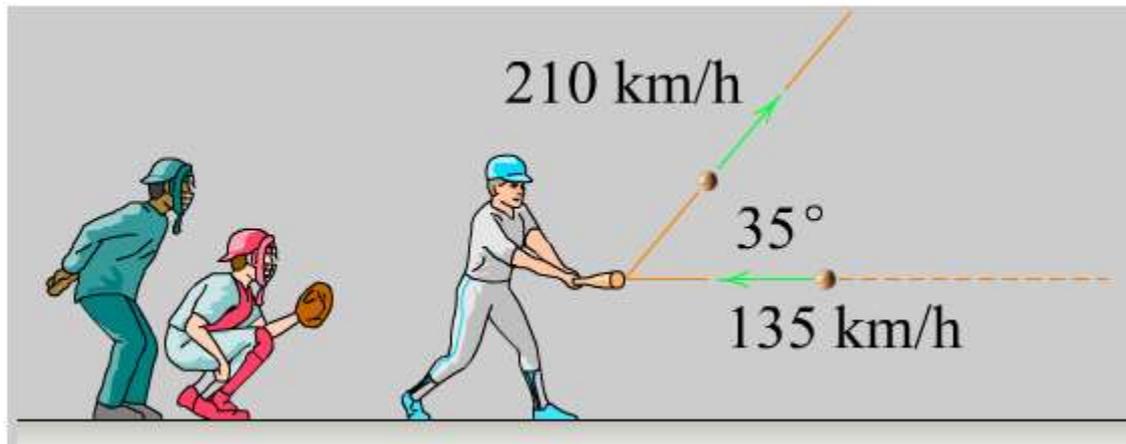
$$\mathbf{G}_1 + \int_{t_1}^{t_2} \Sigma \mathbf{F} dt = \mathbf{G}_2$$

- This is the principle of ***linear impulse momentum***.
- This principle is useful when forces are constant or functions of time.
- The ***linear impulse momentum*** can be calculated from the applied force as follows:

$$\int_{t_1}^{t_2} \Sigma \mathbf{F} dt$$

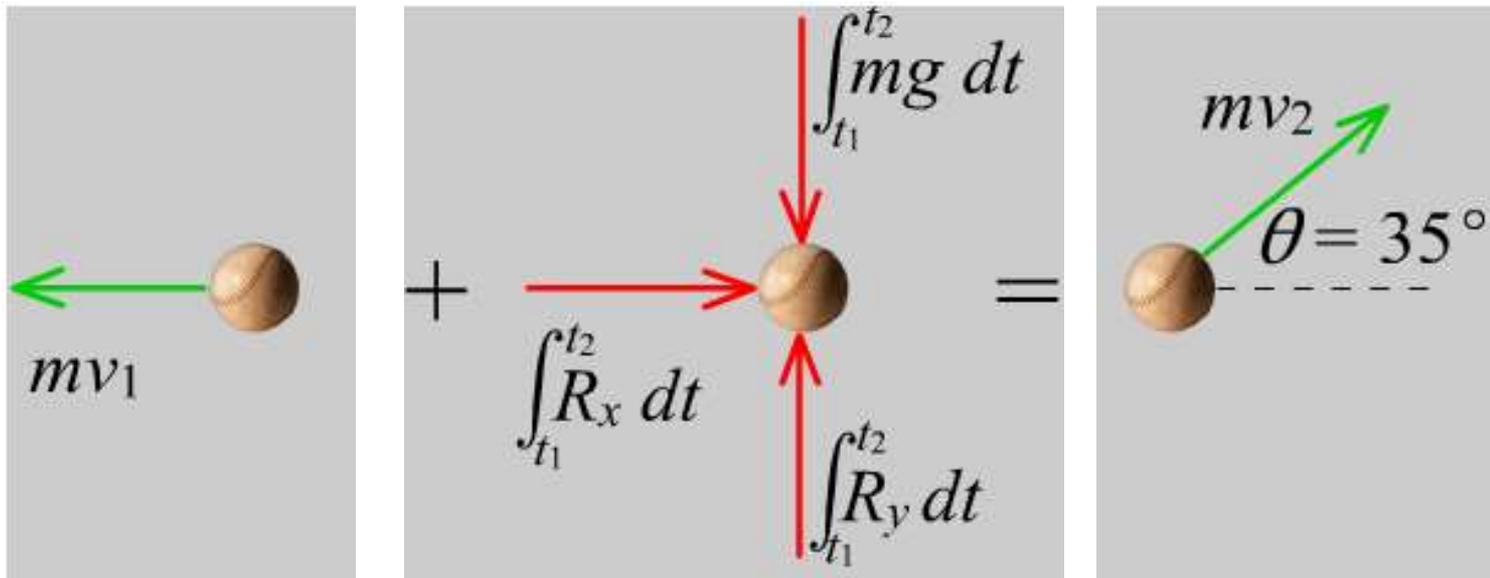
Example

Consider a baseball being struck by a bat. The incoming horizontal velocity is 135 km/h, while the outgoing velocity is 210 km/h at 35° angle shown. If the collision duration time is $\Delta t = 0.005$ s, determine the average force magnitude exerted by the bat on the ball. A baseball has a mass of 146-g.



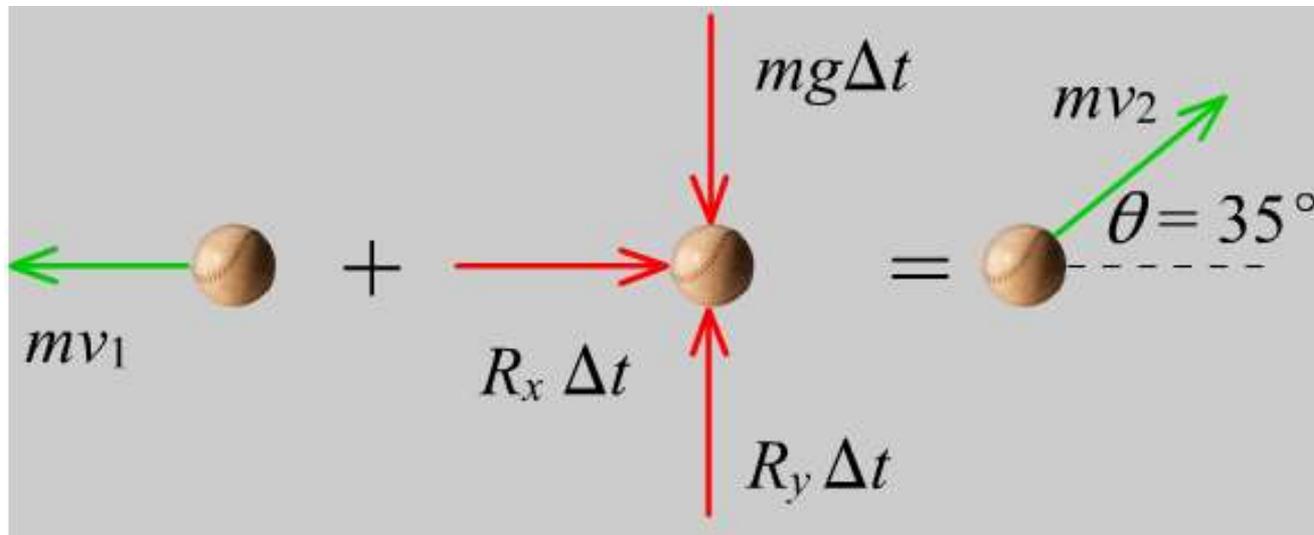
Solution

- We construct the “three-picture approach”, which shows:
 - the initial momenta in the first illustration
 - all the external impulses in the second illustration
 - the final momenta in the third illustration.



Solution: cont

- We move all the constant forces to the outside of the integral signs.
- The force mg is constant, and R_x and R_y become the average impact force components.
- The $\int_{t_1}^{t_2} dt$ becomes $t_2 - t_1 = \Delta t$.



Solution: cont

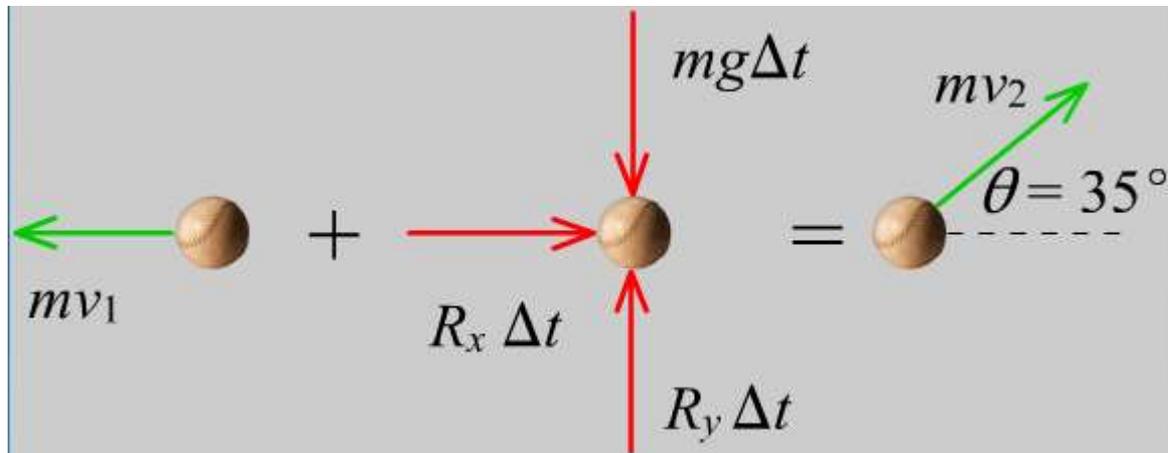
- In the x-direction:

$$mv_{1x} + \int_{t_1}^{t_2} \sum F_x dt = mv_{2x} : -mv_1 + R_x \Delta t = mv_2 \cos \theta$$

- In the y-direction:

$$mv_{1y} + \int_{t_1}^{t_2} \sum F_y dt = mv_{2y} : 0 + R_y \Delta t - mg \Delta t = mv_2 \sin \theta$$

- We assume that mg is non-impulsive when compared with R_x and R_y .



Solution: cont

- Substituting the numbers and solving,

$$-mv_1 + R_x \Delta t = mv_2 \cos \theta$$

$$-0.146 \left(\frac{135}{3.6} \right) + R_x (0.005) = 0.146 \left(\frac{210}{3.6} \right) \cos 35^\circ$$

$$R_x = 2490N$$

$$R_y \Delta t - \cancel{mg \Delta t}^0 = mv_2 \sin \theta$$

$$R_y (0.005) - \cancel{mg \Delta t}^0 = 0.146 \left(\frac{210}{3.6} \right) \sin 35^\circ$$

$$R_y = 977N$$

Solution: cont

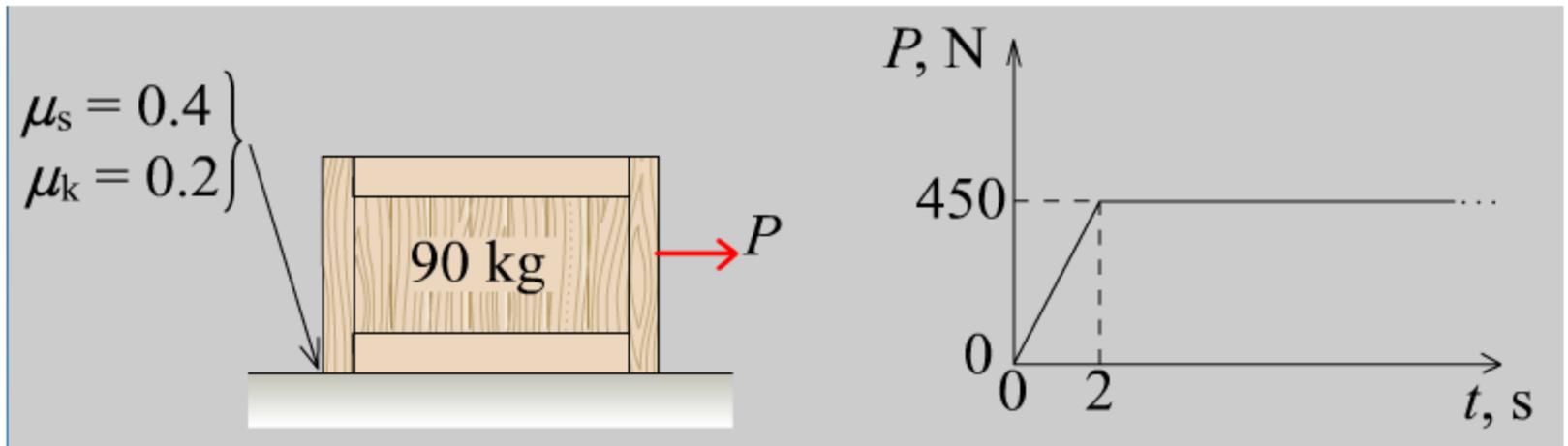
$$R_x = 2490N \quad R_y = 977N$$

$$R = \sqrt{(R_x)^2 + (R_y)^2} = \sqrt{(2490)^2 + (977)^2} = 2670N$$

Because R_x and R_y turned out to be much greater than mg , our assumption of neglecting mg is certainly justified.

Example

A 90-kg block is at rest on the surface shown when a force, P , is applied to the block. This force varies with time as shown in the graph. Determine the velocity of the block at $t=5\text{s}$.



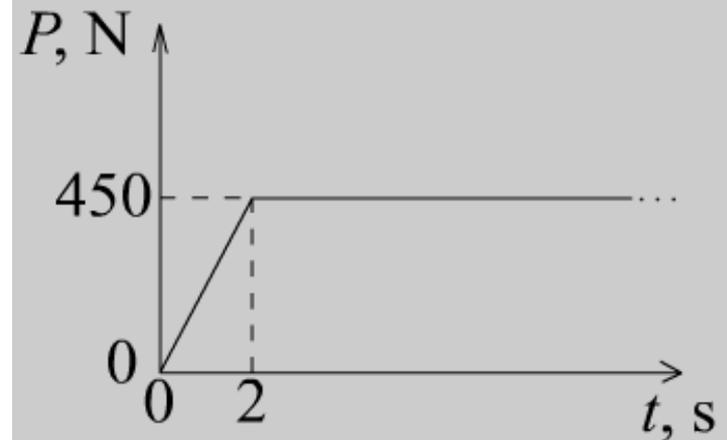
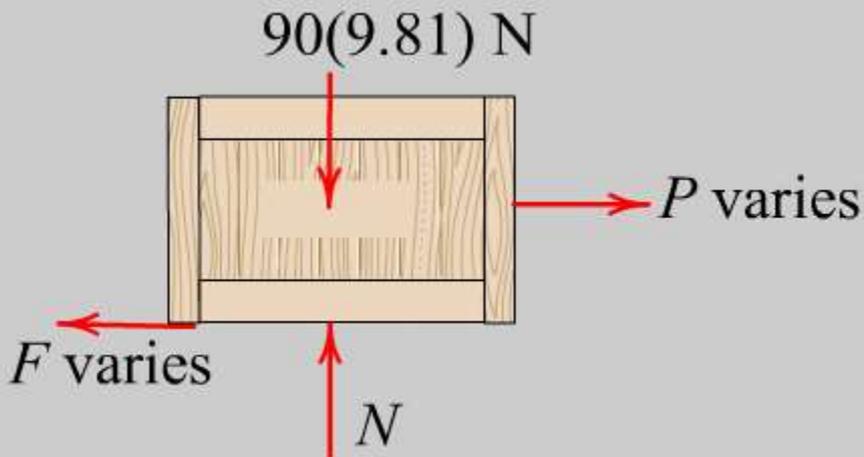
Solution:

- The box will begin to move only when the value of the applied force P exceeds the maximum static friction force, when

$$P = F_{max} = \mu_s N$$

$$225t = 0.4(90)(9.81)$$

- It is convenient, therefore, to begin the motion analysis at $t = 1.57$ s. Prior to that time $\Sigma F_x = 0$.

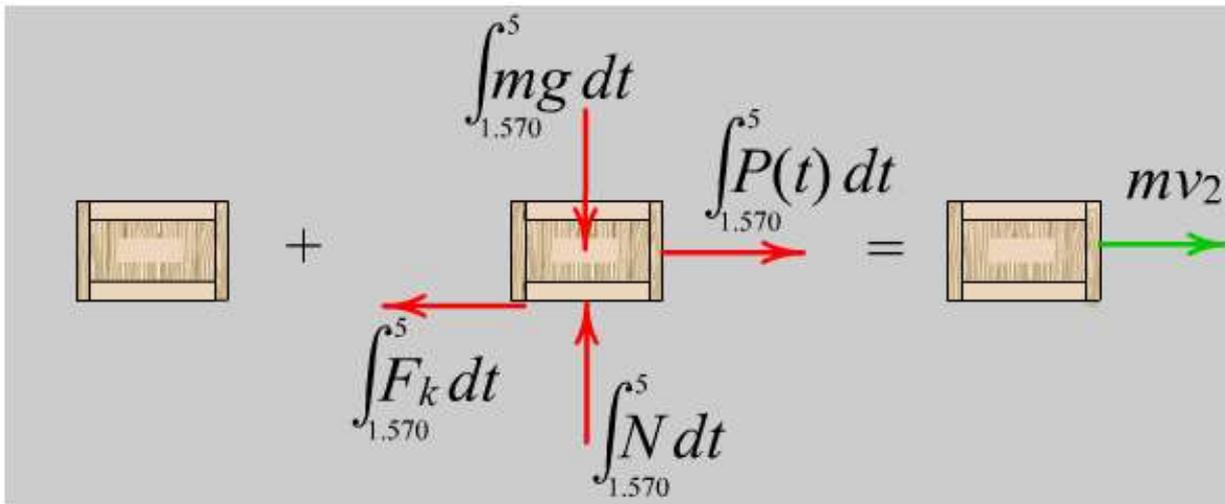


• Three-Picture Approach: $\mathbf{G}_1 + \int_{t_1}^{t_2} \Sigma \mathbf{F} dt = \mathbf{G}_2$

$$m\mathbf{v}_1 + \int_{t_1}^{t_2} \Sigma \mathbf{F} dt = m\mathbf{v}_2$$

$$mv_{1x} + \int_{t_1}^{t_2} \Sigma F_x dt = mv_{2x}$$

$$mv_{1y} + \int_{t_1}^{t_2} \Sigma F_y dt = mv_{2y}$$

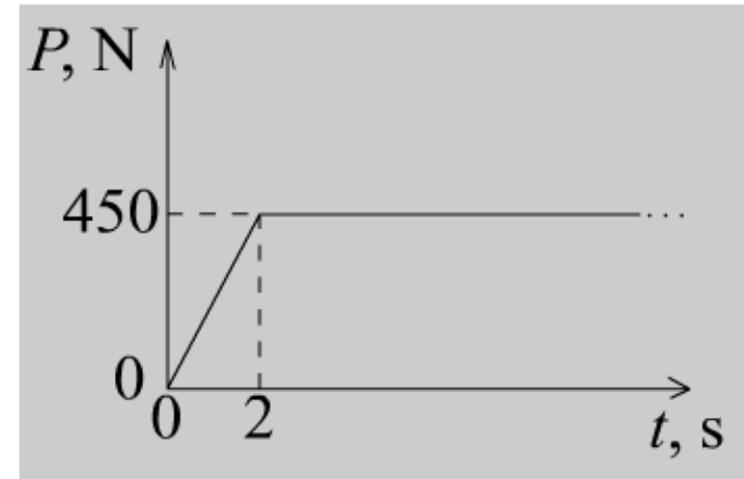


$$0 + \int_{t_1}^{t_2} \left[P(t) - 0.2(90)(9.81) \right] dt = 90v_2$$

Solution: cont

- The quantity $0 + \int_{t_1}^{t_2} [P(t) - 0.2(90)(9.81)] dt = 90v_2$ can be evaluated as follows:

$$\begin{aligned} & \int_{1.57}^5 P(t) dt - \int_{1.57}^5 176.6 dt \\ &= \int_{1.57}^2 225t dt + \int_2^5 450 dt - \int_{1.57}^5 176.6 dt \end{aligned}$$



$$\begin{aligned} &= \frac{225}{2} t^2 \Big|_{1.57}^2 + 450t \Big|_2^5 - 176.6t \Big|_{1.57}^5 \\ &= 172.8 + 1350 - 606 \\ &= 917 \text{ N} \cdot \text{s} \end{aligned}$$



Therefore:

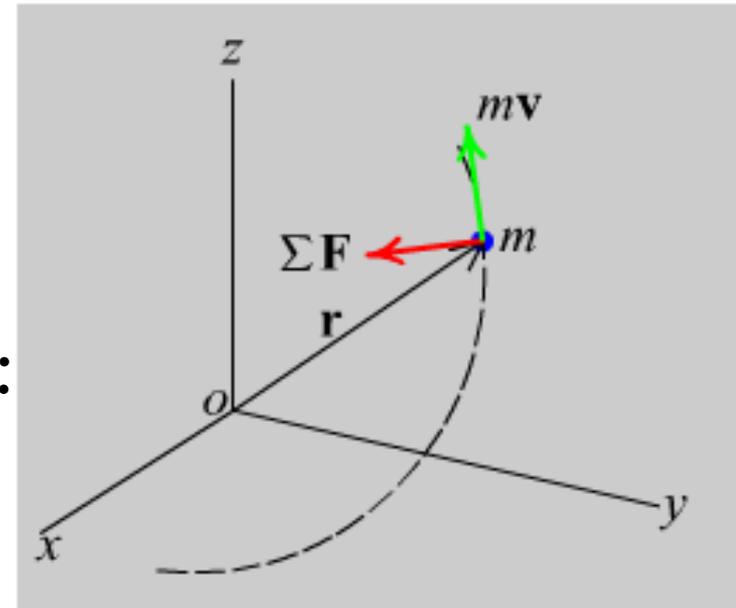
$$\begin{aligned} 917 &= 90v_2 \\ v_2 &= 10.19 \text{ m/s} \end{aligned}$$

Angular Impulse and Momentum

By definition, the angular momentum of a particle of mass m about an origin O is the moment of the linear momentum about this origin:

$$\mathbf{H}_o = \mathbf{r} \times m\mathbf{v}$$

- Units of the angular momentum:
 - SI: $\text{kg} \cdot \text{m}^2/\text{s} = \text{N} \cdot \text{m} \cdot \text{s}$



British: $\text{ft} \cdot \text{lb} \cdot \text{sec}$

Angular Impulse and Momentum

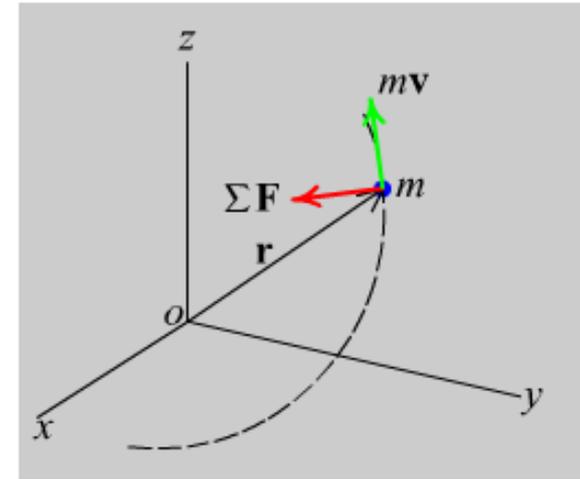
$$\mathbf{H}_o = \mathbf{r} \times m\mathbf{v}$$

- Differentiating with respect to time:

$$\dot{\mathbf{H}}_o = \dot{\mathbf{r}} \times m\mathbf{v} + \mathbf{r} \times m\dot{\mathbf{v}} = \mathbf{r} \times m\dot{\mathbf{v}}$$

- From Newton's law:

$$\dot{\mathbf{H}}_o = \mathbf{r} \times m\mathbf{a} = \mathbf{r} \times \sum \mathbf{F} \quad \longrightarrow \quad \dot{\mathbf{H}}_o = \sum \mathbf{M}_o$$



Angular Impulse and Momentum

Rewriting the angular momentum expression in the form:

$$\frac{d\mathbf{H}_o}{dt} = \sum \mathbf{M}_o \quad \longrightarrow \quad d\mathbf{H}_o = \sum \mathbf{M}_o dt$$

Integrating the differential form of the momentum:

$$\int_{(\mathbf{H}_o)_1}^{(\mathbf{H}_o)_2} d\mathbf{H}_o = \int_{t_1}^{t_2} \sum \mathbf{M}_o dt$$

$$(\mathbf{H}_o)_2 - (\mathbf{H}_o)_1 = \int_{t_1}^{t_2} \sum \mathbf{M}_o dt$$

• This is the principle of **angular impulse momentum**. The

angular impulse is defined as:

$$\int_{t_1}^{t_2} \sum \mathbf{M}_o dt$$

The angular impulse may be regarded as the area under the moment time curve.

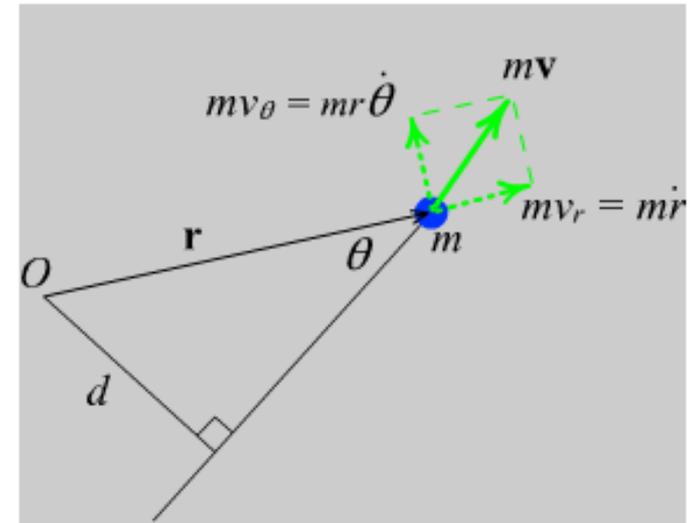
Angular Impulse and Momentum

- If there are no external impulses, the angular momentum of the system under consideration is conserved.

- Note: this will be true also if $\Sigma \mathbf{F}$ is pointed towards or away from origin O .

- The angular momentum magnitude is: *the linear momentum ($m \mathbf{v}$) **times** the arm d which runs from O to the line of action of $m \mathbf{v}$.*

$$(\mathbf{H}_o)_1 + \int_{t_1}^{t_2} \sum \mathbf{M}_o dt = (\mathbf{H}_o)_2$$

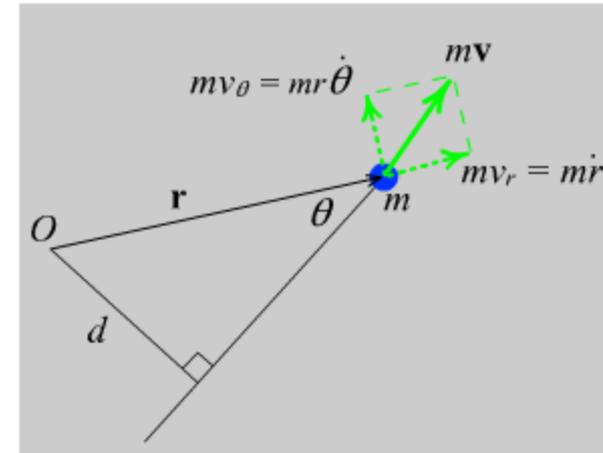


$$\begin{aligned} \mathbf{H}_o &= \mathbf{r} \times m\mathbf{v} = rmv \sin \theta \\ &= mv(r \sin \theta) = mvd \end{aligned}$$

Angular Impulse and Momentum

- The moment may be viewed as the moment of mv_θ plus the moment of mv_r .
- The moment of the latter is zero and therefore:

$$H_o = rmv_\theta = rm r\dot{\theta} = mr^2\dot{\theta}$$



Example

- Determine the angular momentum of the 2-kg sphere about point O in five different ways.

A. Vector Approach

$$\mathbf{H}_o = \mathbf{r} \times m\mathbf{v}$$

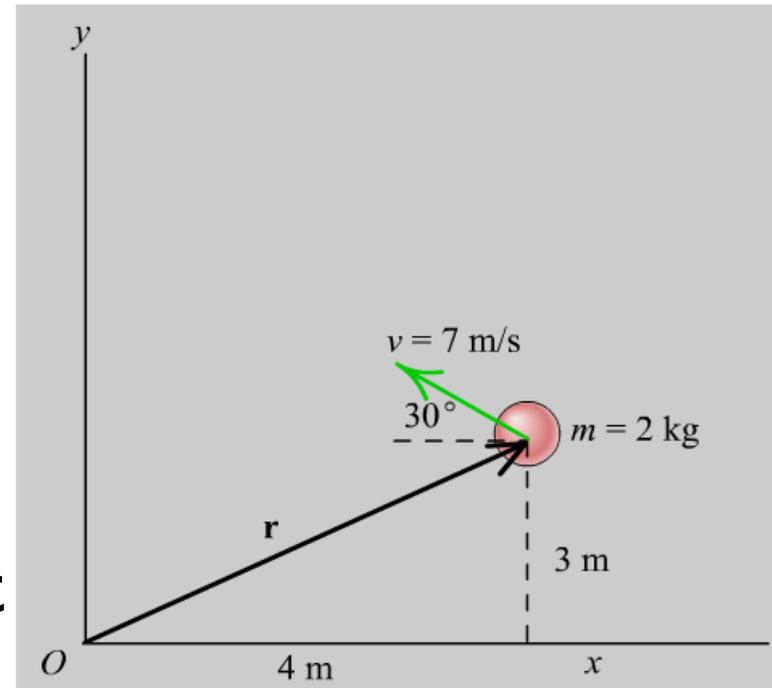
$$\mathbf{r} = 4\mathbf{i} + 3\mathbf{j}$$

$$m\mathbf{v} = 2(-7 \cos 30^\circ \mathbf{i} + 7 \sin 30^\circ \mathbf{j})$$

$$= -12.12\mathbf{i} + 7\mathbf{j}$$

Carrying out the cross product we obtain:

$$\mathbf{H}_o = 64.4\mathbf{k} \text{ kg}\cdot\text{m}^2 / \text{s}$$

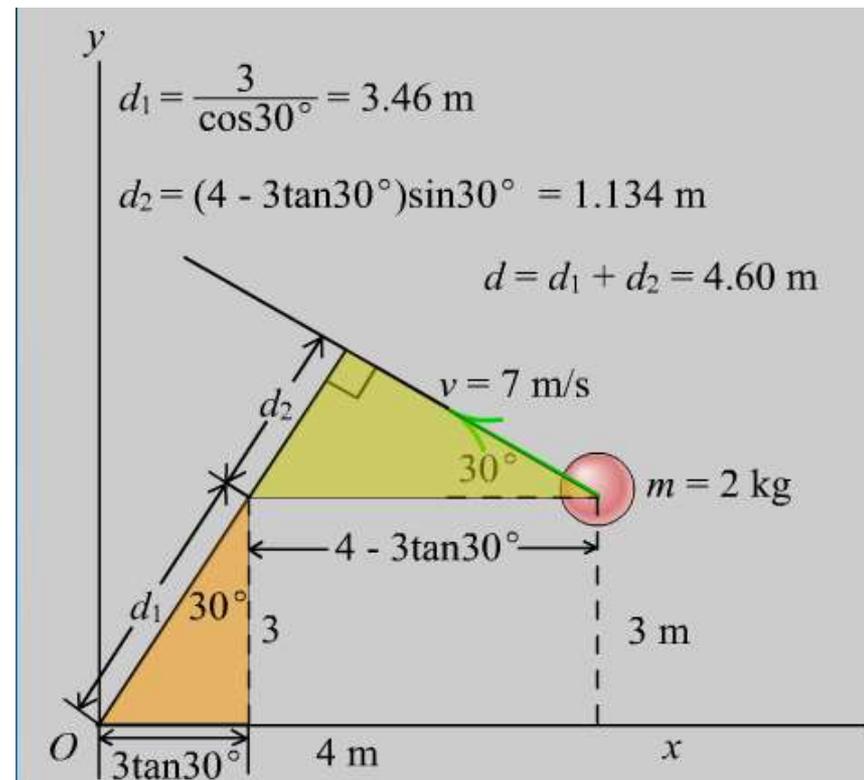


Example

- Determine the angular momentum of the 2-kg sphere about point O in five different ways.

B. Moment arm approach

$$\begin{aligned} \mathbf{H}_O &= mvd \\ &= 2(7)(4.6) \\ &= 64.4 \text{ kg}\cdot\text{m}^2 / \text{s} \\ &\text{counterclockwise} \end{aligned}$$



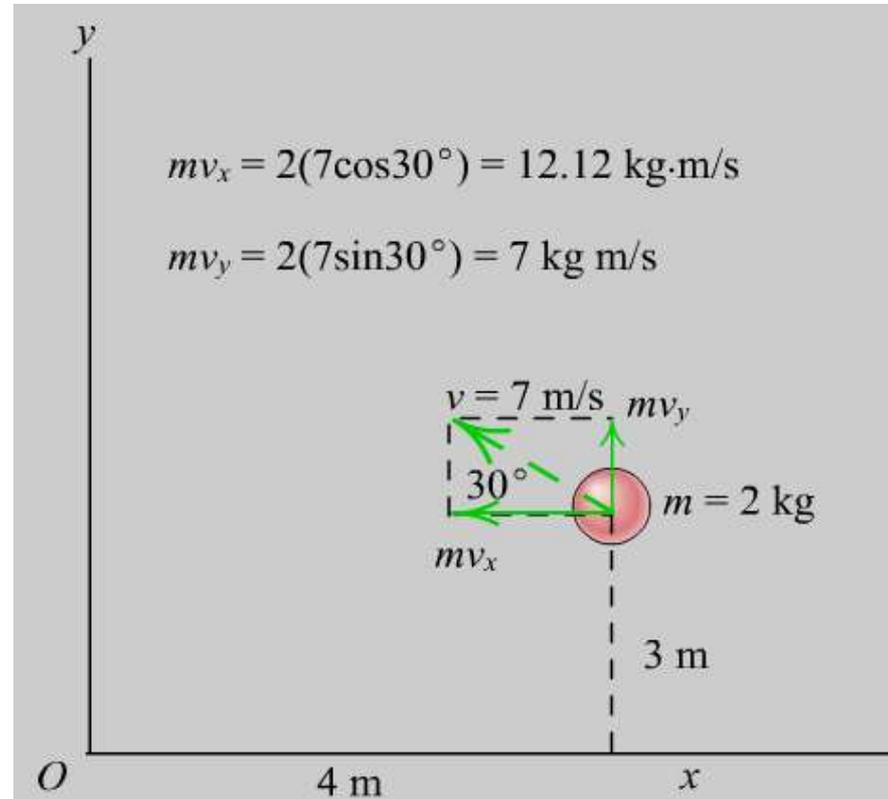
Example

- Determine the angular momentum of the 2-kg sphere about point O in five different ways.

C. Principle of moments

$$\begin{aligned}H_o &= mv_x (3) + mv_y (4) \\ &= 12.2(3) + 7(4) \\ &= 64.4 \text{ kg}\cdot\text{m}^2 / \text{s}\end{aligned}$$

counterclockwise

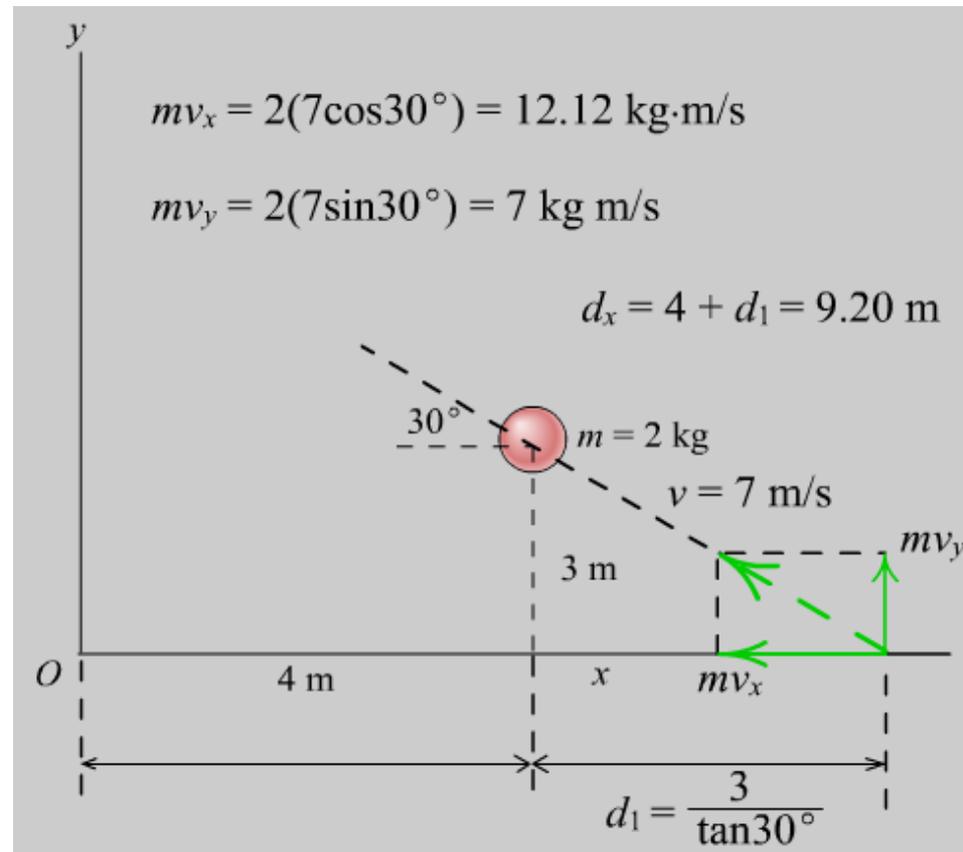


Example

- Determine the angular momentum of the 2-kg sphere about point O in five different ways.

D. Sliding mv to x-axis

$$\begin{aligned}H_o &= mv_y d_x \\ &= 7(9.2) \\ &= 64.4 \text{ kg}\cdot\text{m}^2 / \text{s} \\ &\text{counterclockwise}\end{aligned}$$



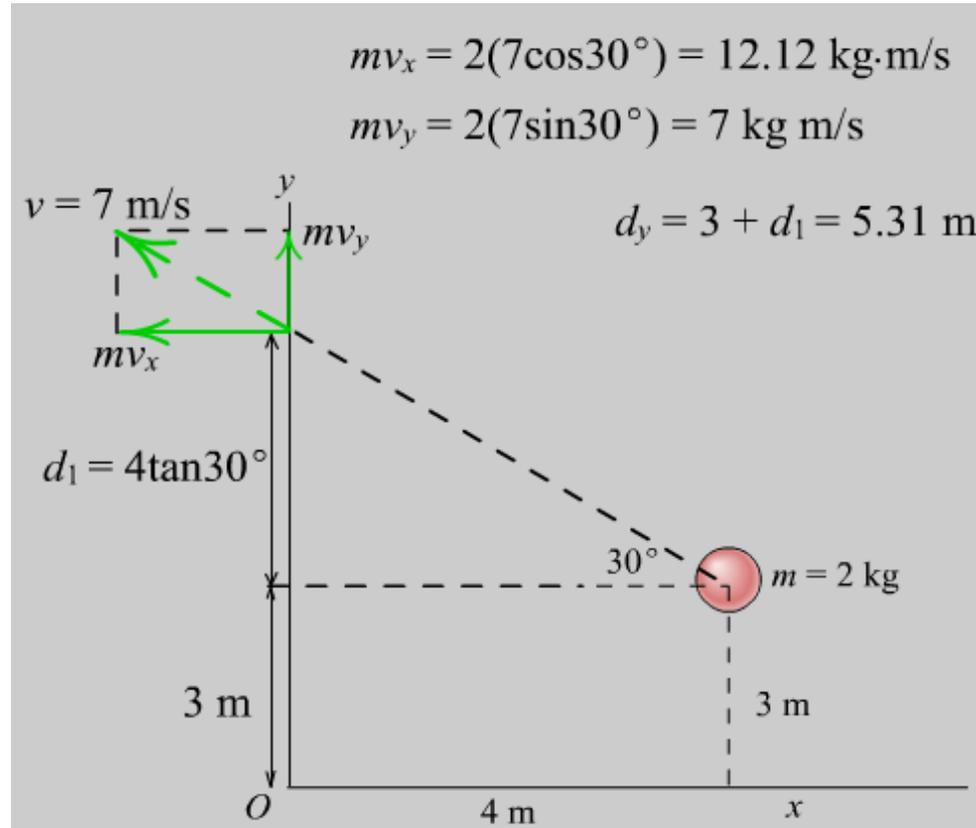
Example

- Determine the angular momentum of the 2-kg sphere about point O in five different ways.

E. Sliding mv to y -axis

$$\begin{aligned}H_o &= mv_x d_y \\ &= 12.12(5.31) \\ &= 64.4 \text{ kg}\cdot\text{m}^2 / \text{s}\end{aligned}$$

counterclockwise



Example 2

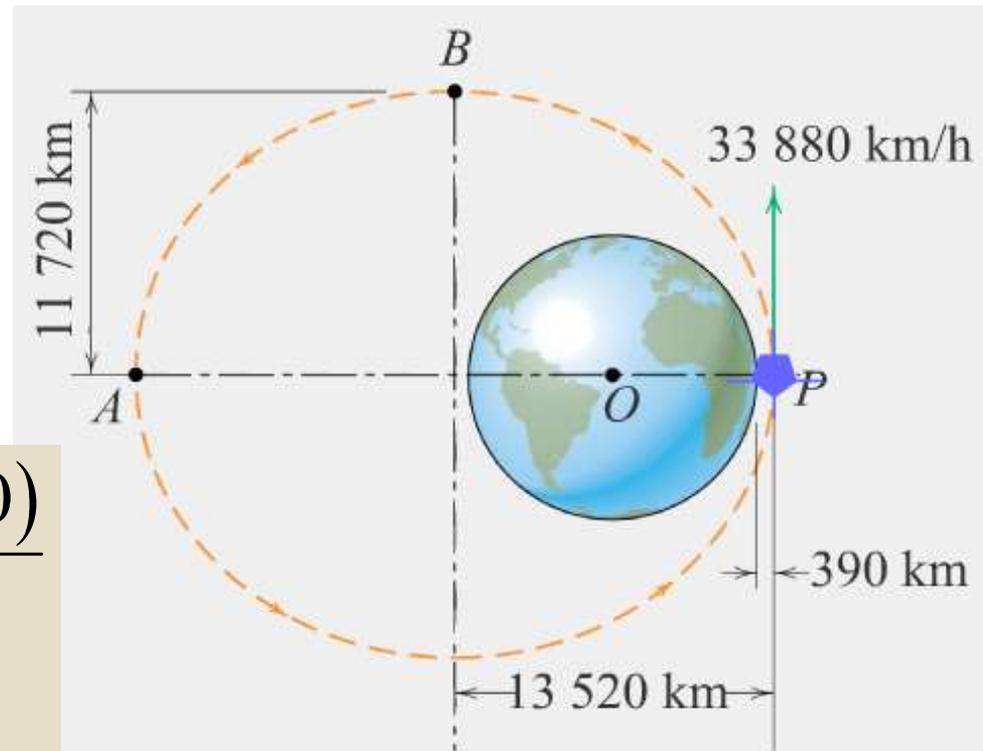
- The spacecraft has a speed of 33880 km/h as it passes point P of its elliptical orbit. Determine its speed as it passes point B and as it passes point A . The radius of the earth is $R=6371$ Km.

Solution:

- Conservation of angular momentum about O :

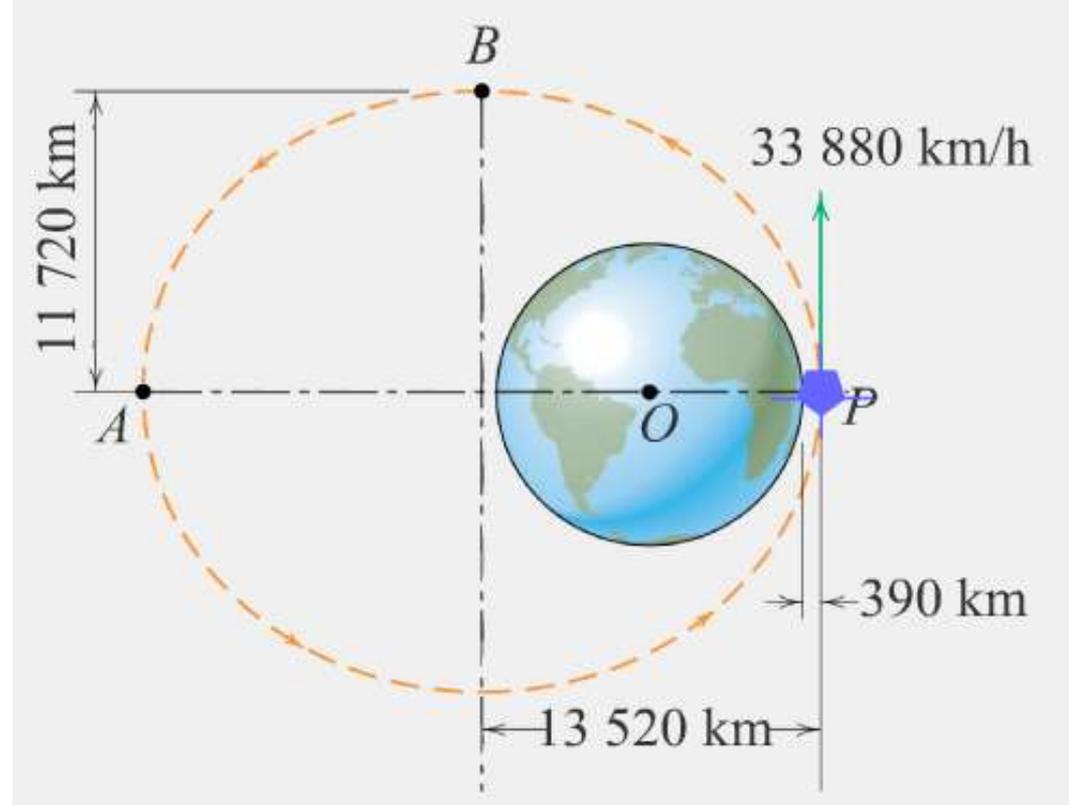
$$mv_P d_P = mv_B d_B$$

$$\begin{aligned} v_B &= \frac{v_P d_P}{d_B} = \frac{33880(6371 + 390)}{11720} \\ &= 19540 \text{ km/h} \end{aligned}$$



Example 2

$$mv_P d_P = mv_A d_A$$

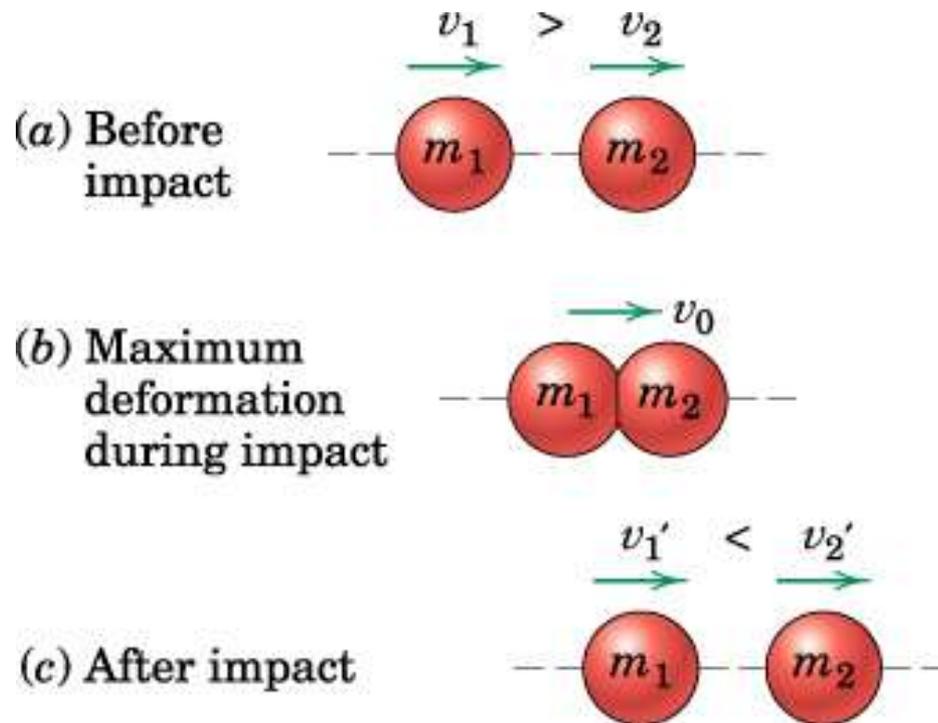


$$\begin{aligned} v_A &= \frac{v_P d_P}{d_A} = \frac{33880(6371 + 390)}{2(13520) - 6371 - 390} \\ &= 11300 \text{ km/h} \end{aligned}$$

Impact

- **Impact:**

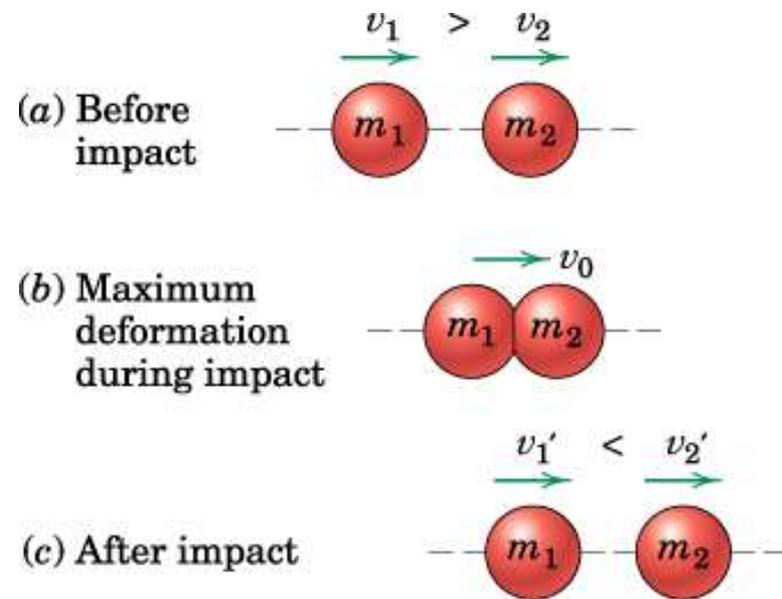
- Refers to the collision of two bodies
- Characterized by the generation of relatively large contact forces which act over a short interval of time.



Impact

- Following initial contact:
 - Deformation occurs till max deformation is reached.
 - A period of restoration occurs and bodies start moving and contact area decreases.
 - When contact area becomes zero, each body will have different velocity.
 - The magnitude and direction of the velocity depends on all the system parameters and initial state.
 - All bodies are assumed to be perfectly elastic.
- From conservation of linear momentum we have:

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$



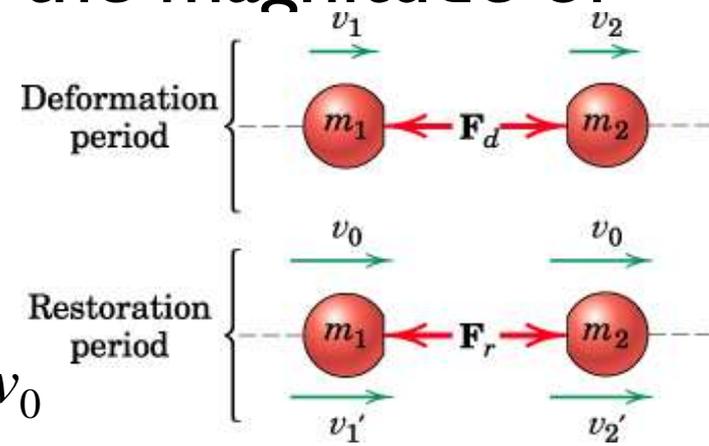
Impact

Coefficient of Restitution: The magnitude of the restoration impulse F_r to the magnitude of the deformation impulse F_d .

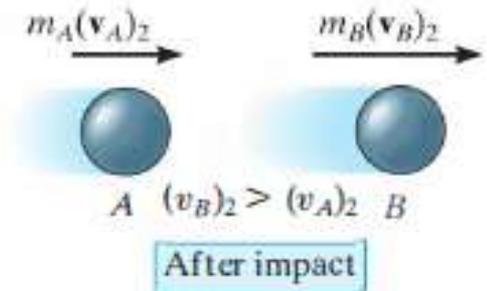
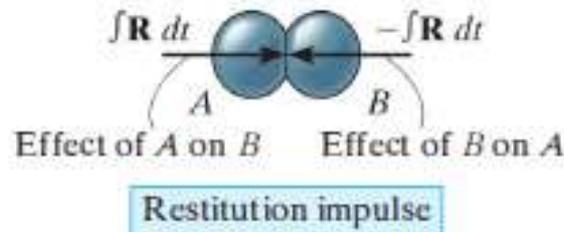
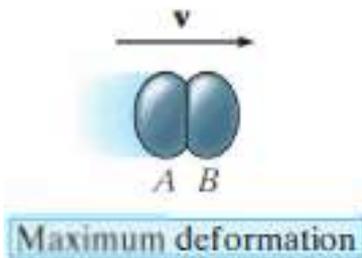
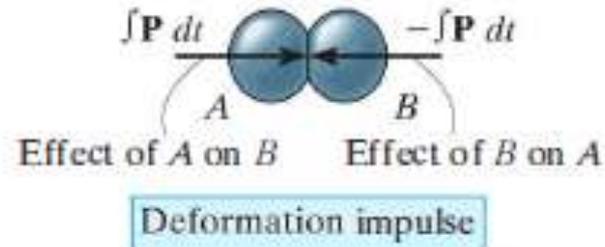
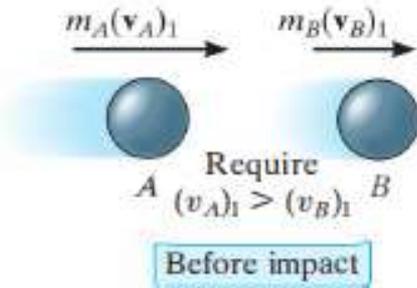
$$e = \frac{\int_{t_0}^t F_r dt}{\int_0^t F_d dt} = \frac{m_1(v_0 - v_1')}{m_1(v_1 - v_0)} \quad \text{and} \quad e = \frac{\int_{t_0}^t F_r dt}{\int_0^t F_d dt} = \frac{m_2(v_0 - v_2')}{m_2(v_2 - v_0)}$$

where e is the coefficient of restoration, v_0 is the velocity during the deformation period, and t_0 is the time taken during deformation. We can eliminate v_0 and get:

$$e = \frac{v_2' - v_1'}{v_1 - v_2} = \frac{|\text{relative velocity of separation}|}{|\text{relative velocity of approach}|}$$



Central Impact



$$m_A(v_A)_1 + m_B(v_B)_1 = m_A(v_A)_2 + m_B(v_B)_2$$

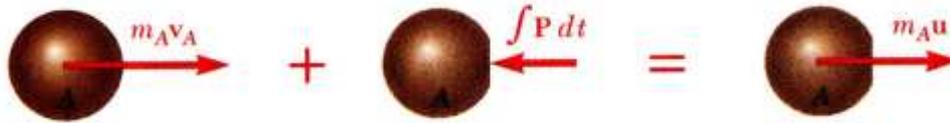
$$m_A(v_A)_1 - \int P dt = m_A v \quad m_A v - \int R dt = m_A(v_A)_2$$

$$e = \frac{\int R dt}{\int P dt} = \frac{v - (v_A)_2}{(v_A)_1 - v}$$

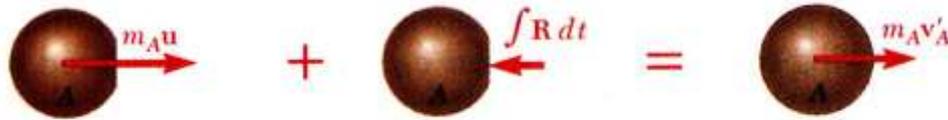
$$e = \frac{\int R dt}{\int P dt} = \frac{(v_B)_2 - v}{v - (v_B)_1}$$

$$e = \frac{(v_B)_2 - (v_A)_2}{(v_A)_1 - (v_B)_1}$$

Direct Central Impact



- Period of deformation: $m_A v_A - \int P dt = m_A u$



- Period of restitution: $m_A u - \int R dt = m_A v'_A$

- A similar analysis of particle B yields

- Combining the relations leads to the desired second relation between the final velocities.

- *Perfectly plastic impact*, $e = 0$: $v'_B = v'_A = v'$

- *Perfectly elastic impact*, $e = 1$:
Total energy and total momentum conserved.

$e = \text{coefficient of restitution}$

$$e = \frac{\int R dt}{\int P dt} = \frac{u - v'_A}{v_A - u}$$

$$0 \leq e \leq 1$$

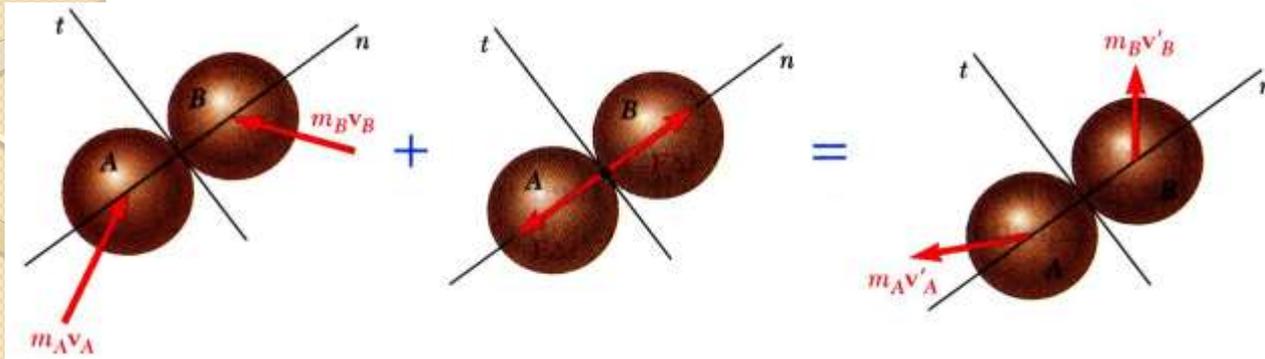
$$e = \frac{v'_B - u}{u - v_B}$$

$$v'_B - v'_A = e(v_A - v_B)$$

$$m_A v_A + m_B v_B = (m_A + m_B) v'$$

$$v'_B - v'_A = v_A - v_B$$

Oblique Central Impact



- Final velocities are unknown in magnitude and direction. Four equations are required.

- No tangential impulse component; tangential component of momentum for each particle is conserved.

$$(v_A)_t = (v'_A)_t \quad (v_B)_t = (v'_B)_t$$

- Normal component of total momentum of the two particles is conserved.

$$m_A (v_A)_n + m_B (v_B)_n = m_A (v'_A)_n + m_B (v'_B)_n$$

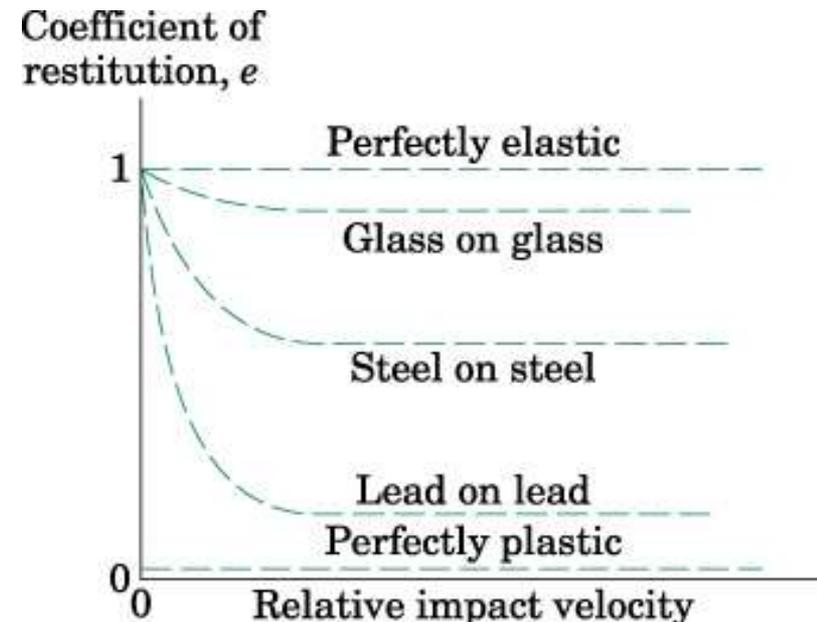
- Normal components of relative velocities before and after impact are related by the coefficient of restitution.

$$(v'_B)_n - (v'_A)_n = e[(v_A)_n - (v_B)_n]$$

Impact

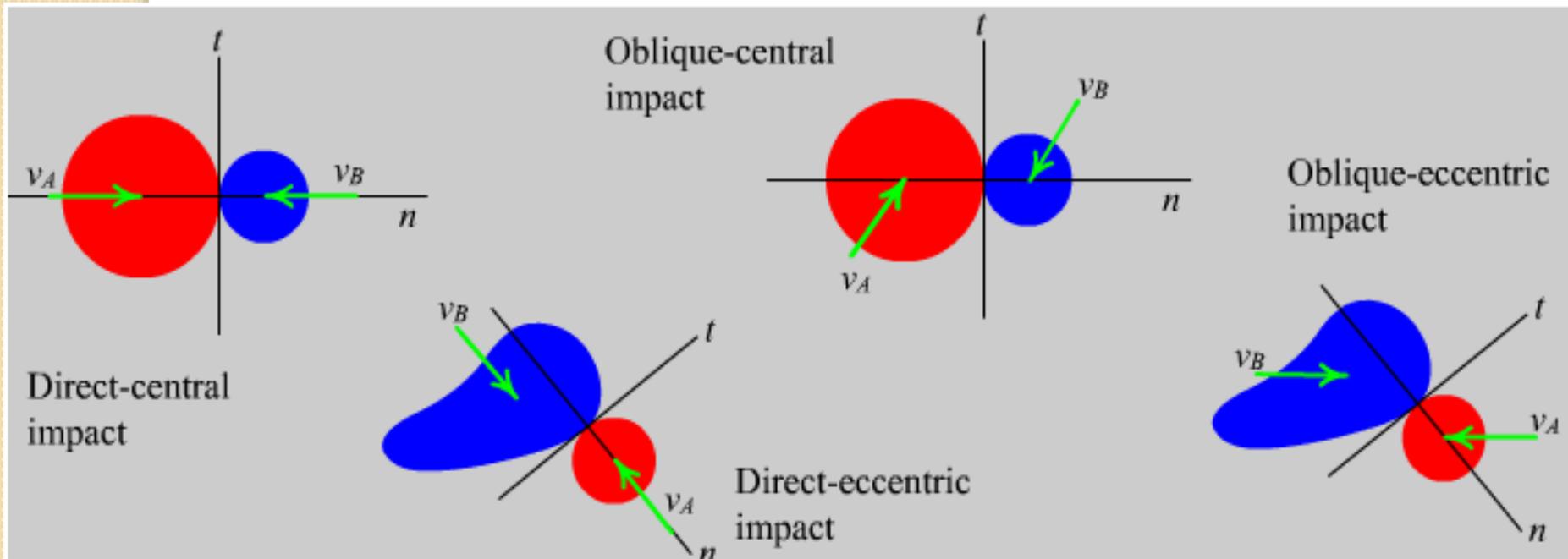
Energy Loss During Impact:

- Energy is lost during impact as: heat during localized inelastic deformation, generation and dissipation of stress waves, generation of sound energy.
- Ideal impact will assume $e = 1$, no energy loss or perfectly elastic impact.
- However, different materials have different values for e as shown.



Impact

- Types of Impact:
 - Direct-Central Impact, Direct-Eccentric Impact,
 - Oblique-Central Impact, Oblique-Eccentric Impact



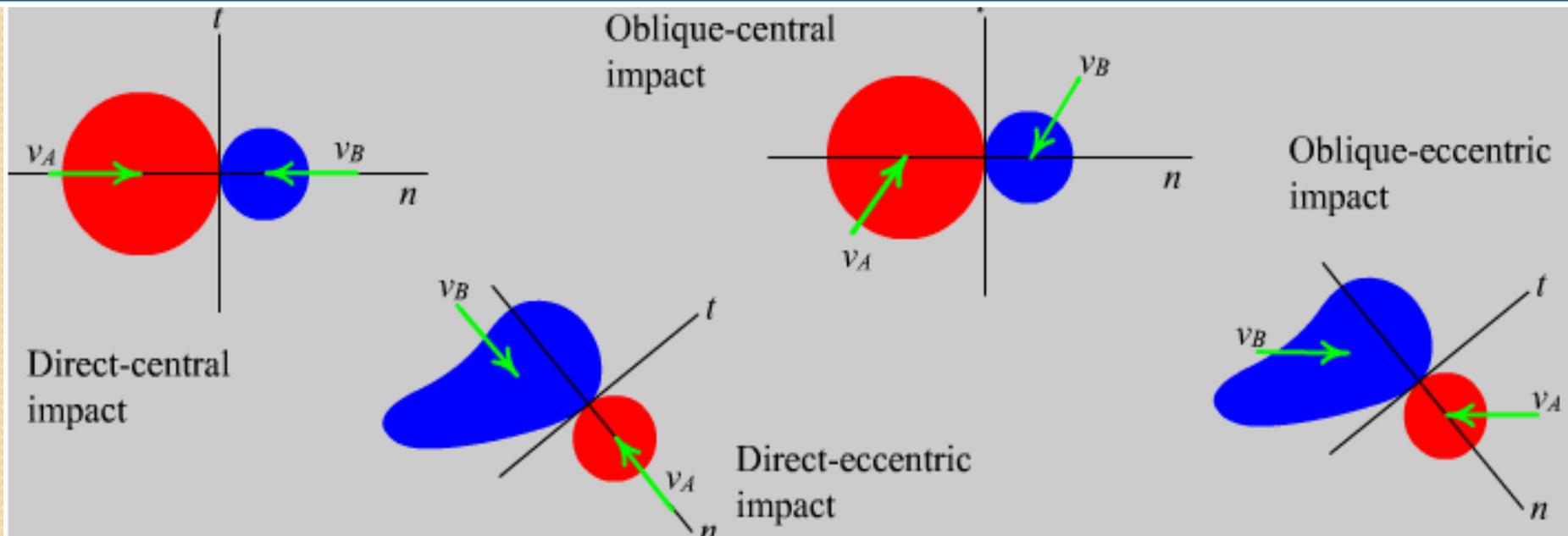
Impact

(1) If both mass centers are on the n -axis: central impact

(2) If either mass center is off the n -axis: eccentric impact

(3) If \mathbf{v}_A and \mathbf{v}_B are parallel to the n -axis: direct impact

(4) If either \mathbf{v}_A or \mathbf{v}_B is not parallel to the n -axis: oblique impact



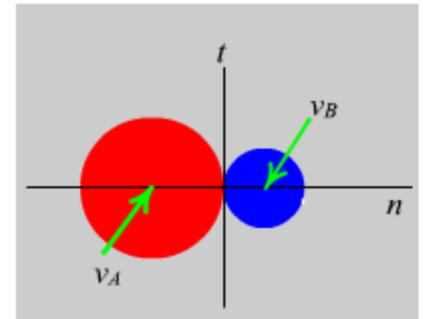
Oblique Central Impact

Momentum in the n -direction is conserved for both masses

$$m_A (v_A)_n + m_B (v_B)_n = m_A (v'_A)_n + m_B (v'_B)_n$$

Restitution relationship holds in the n -direction:

$$e = \frac{(v'_B)_n - (v'_A)_n}{(v_A)_n - (v_B)_n}$$



Momentum in the t -directions of sphere A is conserved

$$m_A (v_A)_t = m_A (v'_A)_t \Rightarrow (v_A)_t = (v'_A)_t$$

Momentum in the t -directions of sphere B is conserved

$$m_B (v_B)_t = m_B (v'_B)_t \Rightarrow (v_B)_t = (v'_B)_t$$

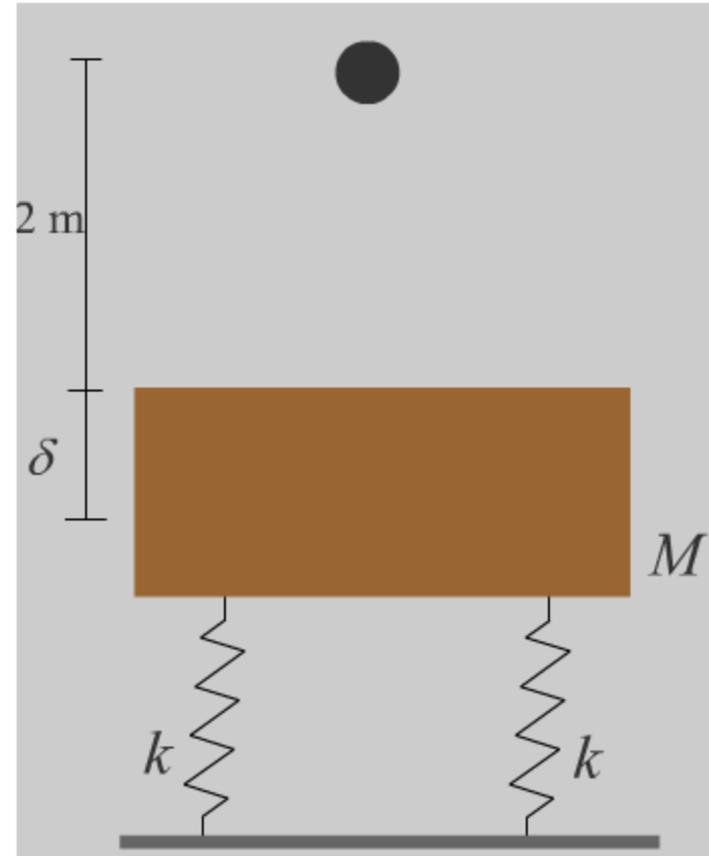
Combination (Chain Reaction) Problem

- For this type of problems, clearly divide the overall sequence into individual problems which treat each sub-event.
- Assign Roman numeral to each sub-event, and assign numbers to each state.
- For example, sub-event I goes from state 1 to state 2; sub-event II goes from state 2 to state 3, etc
- The output of each state becomes the input for the next sub-event.

Example:

The 2-kg piece of putty (m) is dropped 2 m onto the 18-kg block (M) initially at rest on the two springs, each with a stiffness $k=1.2$ kN/m.

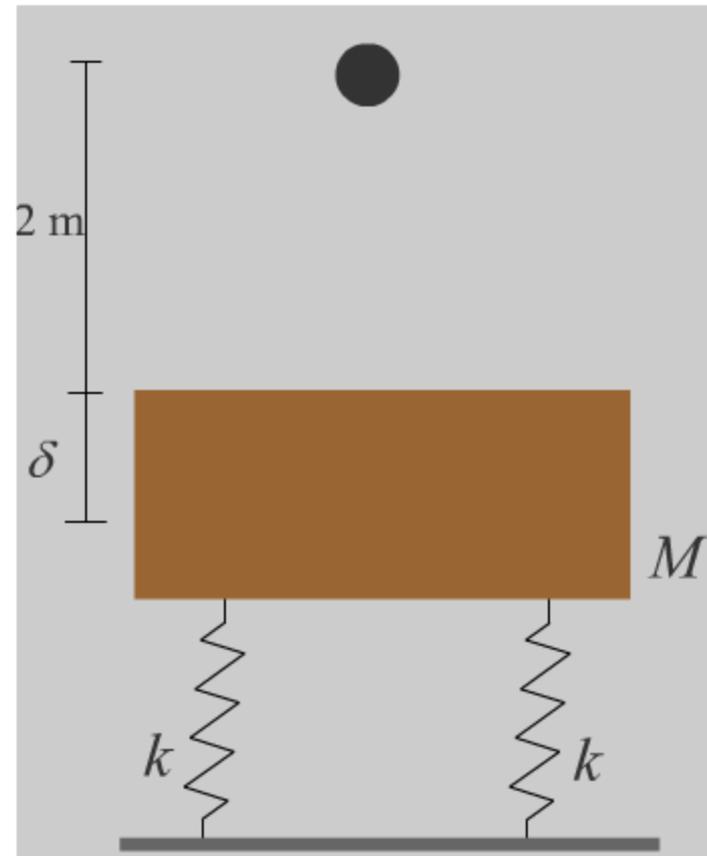
- Calculate the additional deflection δ due to the impact of the putty which adheres to the block upon contact.



Solution:

The sub-events are:

- I. Drop of the putty (State 1 \rightarrow State 2)
- II. Collision of the putty with the block (State 2 \rightarrow State 3)
- III. Subsequent deflection of the block with the adhered putty. (State 3 \rightarrow State 4)



Solution:

I. Drop of putty: (State 1 \rightarrow State 2)

Work-Energy: $T_1 + U_{1-2} = T_2$

$$\frac{1}{2}mv_1^2 + mgh = \frac{1}{2}mv_2^2$$

$$v_2 = \sqrt{2gh} = \sqrt{2(9.81)(2)}$$

$$v_2 = 6.26 \text{ m/s}$$

Solution:

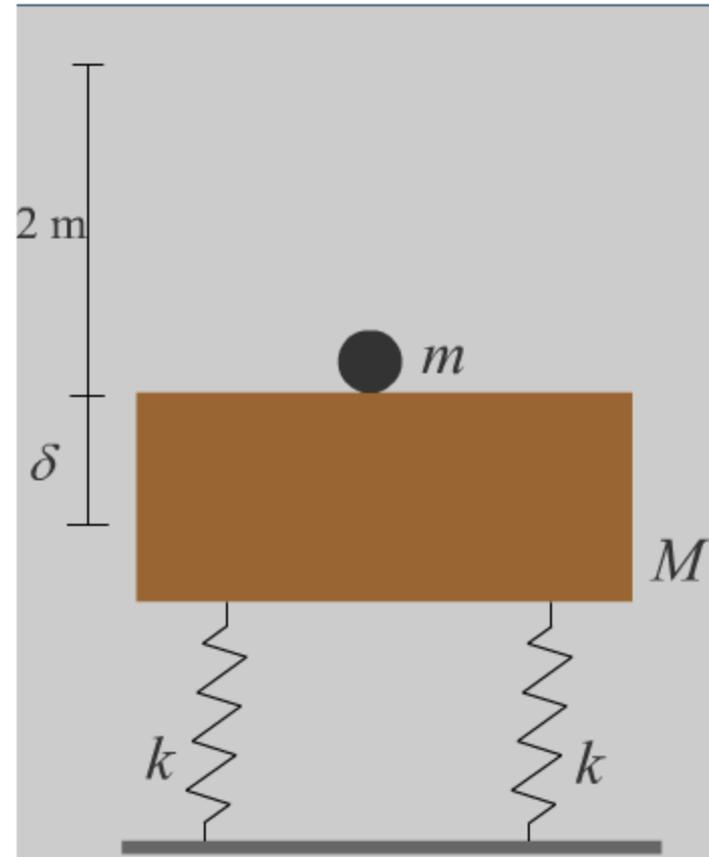
- II. Collision of the putty with the block (State 2 → State 3)

Linear momentum of the system of putty and the block is conserved.

$$mv_2 = (m + M)v_3$$

$$v_3 = \frac{mv_2}{(m + M)} = \frac{2}{2 + 18} \quad (6.26)$$

$$v_3 = 0.626 \text{ m/s}$$



Solution:

III. Subsequent deflection of the block with the adhered putty. (State 3 \rightarrow State 4)

Work-Energy: $T_3 + U_{3-4} = T_4$

$$\frac{1}{2}(m+M)v_3^2 + (m+M)gh + \frac{1}{2}k(x_3^2 - x_4^2) = 0$$

$(m+M) = 20 \text{ kg}$, $h = \delta \text{ m}$, $g = 9.81 \text{ m/s}^2$, $v_3 = 0.626 \text{ m/s}$,

$k = 1200 \text{ N/m}$, $x_3 = \delta_{st} \text{ m}$, $x_4 = \delta_{st} + \delta \text{ m}$

where δ_{st} = initial deflection of the springs due to the putty weight.

$$\delta_{st} = \frac{Mg}{2k} = \frac{18(9.81)}{2(1200)} = 0.0736 \text{ m}$$

Solution:

$$\frac{1}{2}(m + M)v_3^2 + (m + M)gh + \frac{1}{2}k(x_3^2 - x_4^2) = 0$$

Solving the above work-energy equation we can obtain:

$$\delta = 0.0659 \text{ m}$$

$$\delta = 65.9 \text{ mm}$$

or

Solution:

- Assume that no displacement of the system occurs during the collision, so no change in the net spring force occur.
- Thus we consider no impulse due to the weight and the spring forces (which sum to zero while the system is at one static equilibrium position)
- Then we can state that the system's linear momentum is conserved in the vertical direction.



Questions and Comments