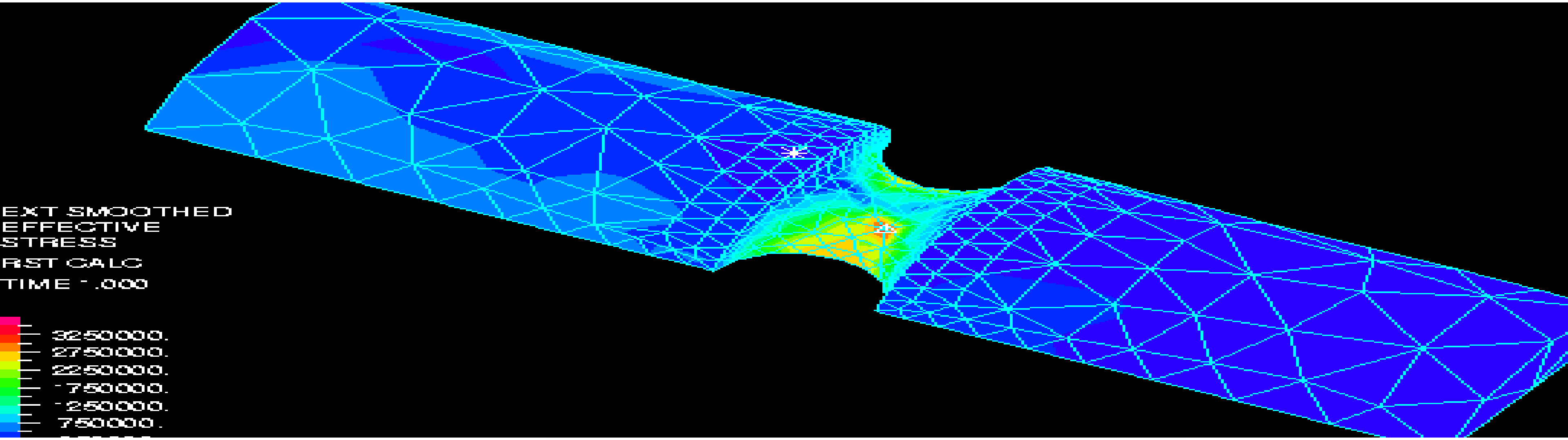
جامعة فهد بن سلطان
Fahad Bin Sultan University

# COLLEGE OF ENGINEERING
# DEPARTMENT OF CIVIL ENGINEERING



# CIVE 517 : Finite Element Analysis

## Shape functions in 1D

**Prof. Khaldoon Bani-Hani**
Fall, 2021

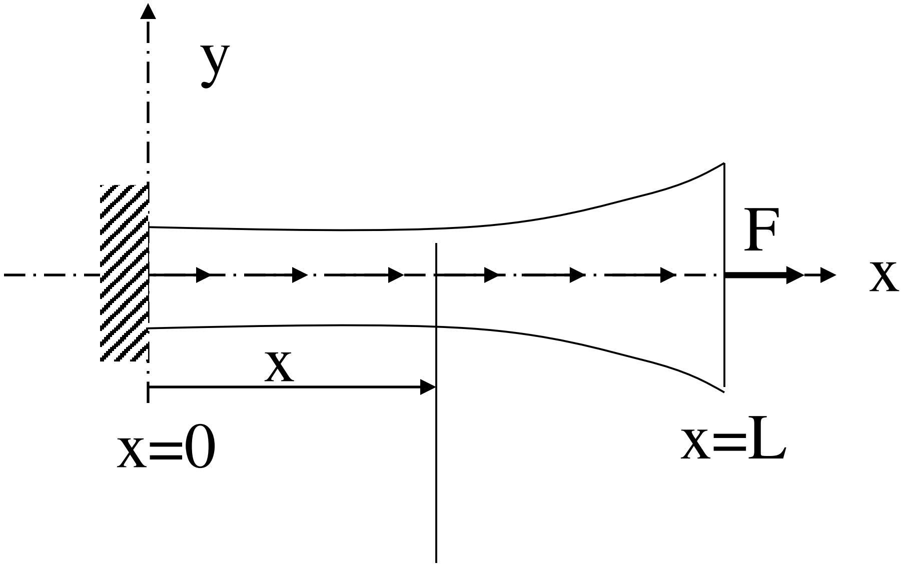
**Reading assignment:**

**Lecture notes, Logan 2.2, 3.1**

**Summary:**

- **Linear shape functions in 1D**
- **Quadratic and higher order shape functions**
- **Approximation of strains and stresses in an element**

# Axially loaded elastic bar



A(x) = cross section at x
b(x) = body force distribution
(force per unit length)
E(x) = Young's modulus

**Potential energy** of the axially loaded bar corresponding to the exact solution u(x)

$$\boxed{\Pi(u) = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx - \int_0^L bu\ dx - Fu(x = L)}$$

Finite element formulation, takes as its starting point, not the strong formulation, but the **Principle of Minimum Potential Energy**.

**Task is to find the function 'w' that minimizes the potential energy of the system**

$$\boxed{\Pi(w) = \frac{1}{2}\int_0^L EA\left(\frac{dw}{dx}\right)^2 dx - \int_0^L bw\ dx - Fw(x = L)}$$

**From the Principle of Minimum Potential Energy, that function 'w' is the exact solution.**

# Rayleigh-Ritz Principle

**Step 1.** Assume a solution

$$w(x) = a_0\varphi_o(x) + a_1\varphi_1(x) + a_2\varphi_2(x) + \ldots$$

Where $\varphi_o(x)$, $\varphi_1(x)$,… are "admissible" functions and $a_o$, $a_1$, etc are constants to be determined.

**Step 2.** Plug the approximate solution into the potential energy

$$\Pi(\mathrm{w}) = \frac{1}{2}\int_0^L \mathrm{EA}\left(\frac{\mathrm{dw}}{\mathrm{dx}}\right)^2 \mathrm{dx} - \int_0^L \mathrm{bw}\ \mathrm{dx} - \mathrm{Fw}(\mathrm{x} = \mathrm{L})$$

**Step 3.** Obtain the coefficients $a_o$, $a_1$, etc by setting

$$\frac{\partial \Pi(\mathrm{w})}{\partial a_i} = 0, \quad i = 0,1,2,\ldots$$

The approximate solution is

$$u(x) = a_0\varphi_o(x) + a_1\varphi_1(x) + a_2\varphi_2(x) + \ldots$$

Where the coefficients have been obtained from step 3

Need to find a systematic way of choosing the approximation functions.

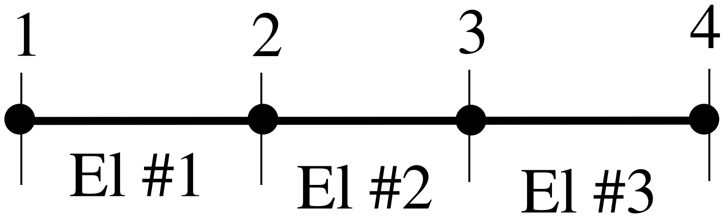
One idea: Choose polynomials!

$w(x) = a_0$ Is this good? (Is ‘1’ an “admissible” function?)

$w(x) = a_1 x$ Is this good? (Is ‘x’ an “admissible” function?)

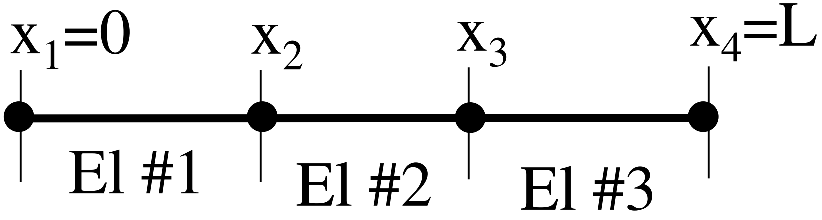
# Finite element idea:

**Step 1:** Divide the truss into **finite elements** connected to each other through special points ("**nodes**")



Total potential energy=sum of potential energies of the elements

$$\Pi(\mathrm{w}) = \frac{1}{2}\int_0^L \mathrm{EA}\left(\frac{\mathrm{dw}}{\mathrm{dx}}\right)^2 \mathrm{dx} - \int_0^L \mathrm{bw}\ \mathrm{dx} - \mathrm{Fw}(\mathrm{x} = \mathrm{L})$$



**Total potential energy**

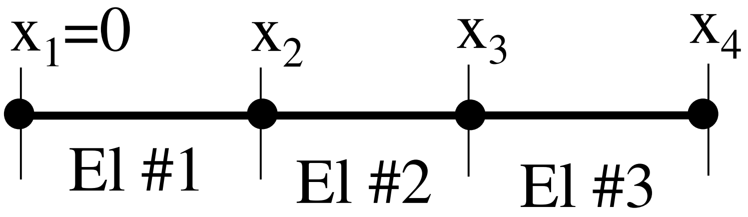
$$\Pi(w) = \frac{1}{2}\int_0^L EA\left(\frac{dw}{dx}\right)^2 dx - \int_0^L bw\ dx - Fw(x=L)$$

**Potential energy of element 1:**

$$\Pi_1(w) = \frac{1}{2}\int_{x_1}^{x_2} EA\left(\frac{dw}{dx}\right)^2 dx - \int_{x_1}^{x_2} bw\ dx$$

**Potential energy of element 2:**

$$\Pi_2(w) = \frac{1}{2}\int_{x_2}^{x_3} EA\left(\frac{dw}{dx}\right)^2 dx - \int_{x_2}^{x_3} bw\ dx$$



**Potential energy of element 3:**

$$\Pi_3(w) = \frac{1}{2}\int_{x_3}^{x_4} EA\left(\frac{dw}{dx}\right)^2 dx - \int_{x_3}^{x_4} bw\ dx - Fw(x = L)$$

Total potential energy=sum of potential energies of the elements

$$\Pi(w) = \Pi_1(w) + \Pi_2(w) + \Pi_3(w)$$

**Step 2:** Describe the behavior of each element

Recall that in the "**direct stiffness**" approach for a bar element, we derived the stiffness matrix of each element directly (See lecture on Trusses) using the following steps:

**TASK 1:** Approximate the displacement within each bar as a straight line

**TASK 2:** Approximate the strains and stresses and realize that a bar (with the approximation stated in Task 1) is exactly like a spring with k=EA/L

**TASK 3:** Use the principle of **force equilibrium** to generate the stiffness matrix

Now, we will show you a systematic way of deriving the stiffness matrix (sections 2.2 and 3.1 of Logan).

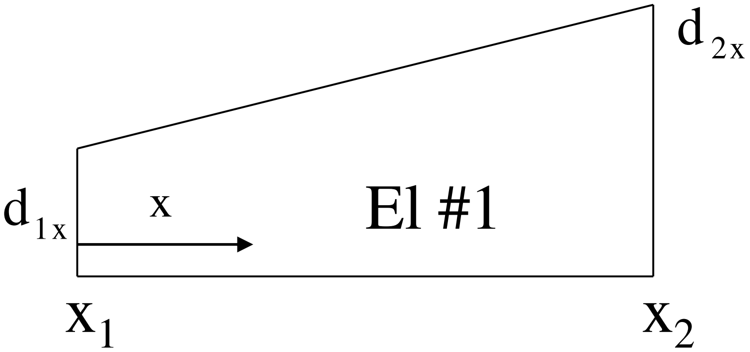
**TASK 1:** APPROXIMATE THE DISPLACEMENT WITHIN EACH ELEMENT
**TASK 2:** APPROXIMATE THE STRAIN and STRESS WITHIN EACH ELEMENT
**TASK 3:** DERIVE THE STIFFNESS MATRIX OF EACH ELEMENT (next class) USING THE PRINCIPLE OF MIN. POT ENERGY

Notice that the first two tasks are similar in the two methods. The only difference is that now we are going to use the principle of minimum potential energy, rather than force equilibrium, to derive the stiffness matrix.

# **TASK 1:** APPROXIMATE THE DISPLACEMENT WITHIN EACH ELEMENT

Simplest assumption: displacement varying linearly inside each bar

$$w(x) = a_0 + a_1 x$$



How to obtain $a_0$ and $a_1$*?*

$$w(x_1) = a_0 + a_1 x_1 = d_{1x}$$
$$w(x_2) = a_0 + a_1 x_2 = d_{2x}$$

$$w(x_1) = a_0 + a_1 x_1 = d_{1x}$$
$$w(x_2) = a_0 + a_1 x_2 = d_{2x}$$

Solve simultaneously

$$a_0 = \frac{x_2}{x_2 - x_1} d_{1x} - \frac{x_1}{x_2 - x_1} d_{2x}$$
$$a_1 = -\frac{1}{x_2 - x_1} d_{1x} + \frac{1}{x_2 - x_1} d_{2x}$$

Hence

$$w(x) = a_0 + a_1 x = \underbrace{\frac{x_2 - x}{x_2 - x_1}}_{N_1(x)} d_{1x} + \underbrace{\frac{x - x_1}{x_2 - x_1}}_{N_2(x)} d_{2x} = N_1(x) d_{1x} + N_2(x) d_{2x}$$

“**Shape functions**” $N_1(x)$ and $N_2(x)$

In matrix notation, we write

$$\boxed{w(x) = \underline{N}\ \underline{d}} \qquad (1)$$

Vector of nodal shape functions

$$\underline{N} = \left[ N_1(x) \quad N_2(x) \right] = \left[ \frac{x_2 - x}{x_2 - x_1} \quad \frac{x - x_1}{x_2 - x_1} \right]$$

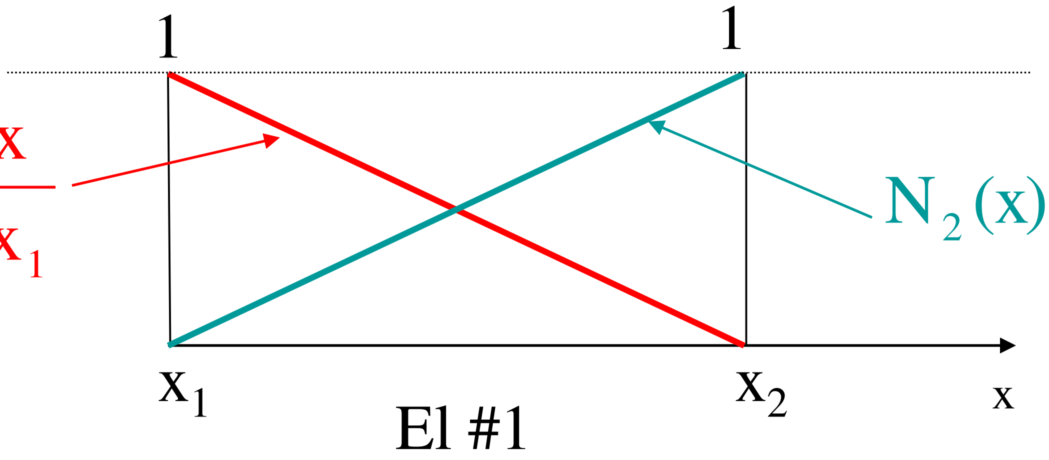
Vector of nodal displacements

$$\underline{d} = \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix}$$

# NOTES: PROPERTIES OF THE SHAPE FUNCTIONS

**1. <u>Kronecker delta property</u>: The shape function at any node has a value of 1 at that node and a value of zero at ALL other nodes.**

$$N_1(x) = \frac{x_2 - x}{x_2 - x_1}$$



$$N_2(x) = \frac{x - x_1}{x_2 - x_1}$$

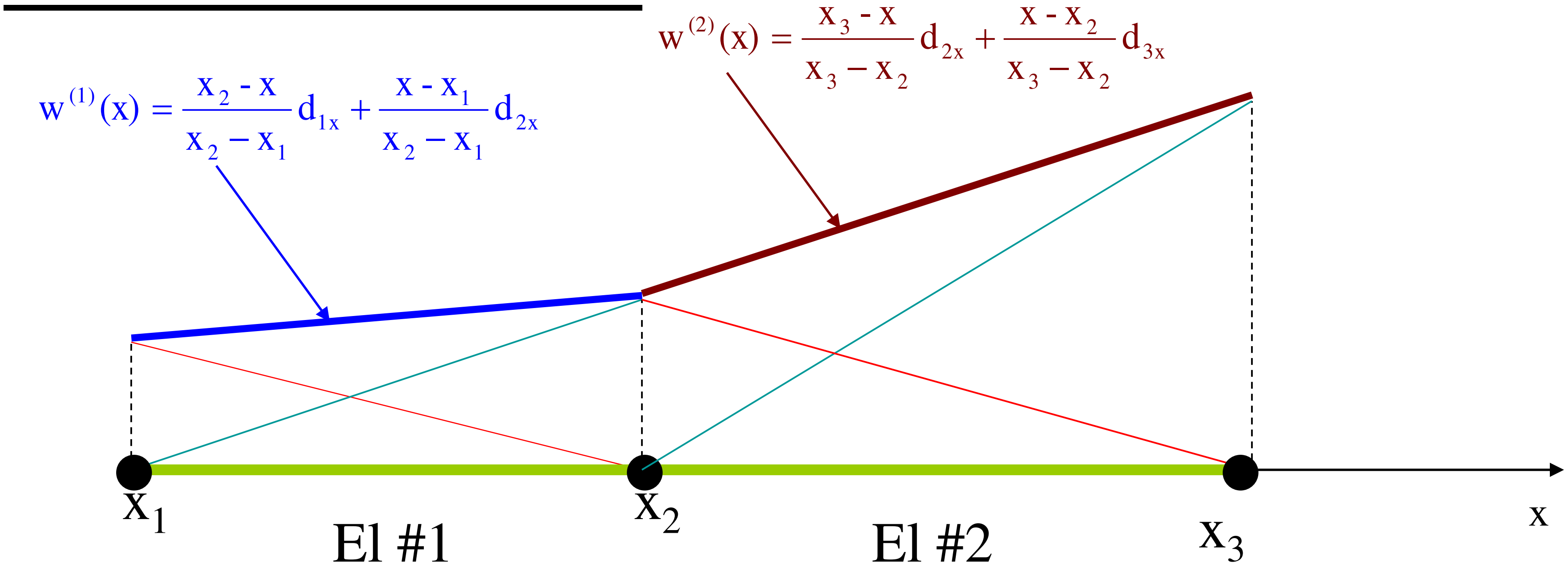
Check

$$N_1(x) = \frac{x_2 - x}{x_2 - x_1}$$

$$\Rightarrow N_1(x = x_1) = \frac{x_2 - x_1}{x_2 - x_1} = 1$$

$$\text{and } N_1(x = x_2) = \frac{x_2 - x_2}{x_2 - x_1} = 0$$

## 2. **Compatibility**: The displacement approximation is **continuous across element boundaries**



$$w^{(1)}(x) = \frac{x_2 - x}{x_2 - x_1} d_{1x} + \frac{x - x_1}{x_2 - x_1} d_{2x}$$

$$w^{(2)}(x) = \frac{x_3 - x}{x_3 - x_2} d_{2x} + \frac{x - x_2}{x_3 - x_2} d_{3x}$$

At $x=x_2$

$$w^{(1)}(x = x_2) = \frac{x_2 - x_2}{x_2 - x_1} d_{1x} + \frac{x_2 - x_1}{x_2 - x_1} d_{2x} = d_{2x}$$

$$w^{(2)}(x = x_2) = \frac{x_3 - x_2}{x_3 - x_2} d_{2x} + \frac{x_2 - x_2}{x_3 - x_2} d_{3x} = d_{2x}$$

Hence the displacement approximation is continuous across elements

## 3. Completeness

$$N_1(x) + N_2(x) = 1 \quad \text{for all } x$$
$$N_1(x)x_1 + N_2(x)x_2 = x \quad \text{for all } x$$

Use the expressions $N_1(x) = \dfrac{x_2 - x}{x_2 - x_1}$;

$$N_2(x) = \frac{x - x_1}{x_2 - x_1}$$

And check

$$N_1(x) + N_2(x) = \frac{x_2 - x}{x_2 - x_1} + \frac{x - x_1}{x_2 - x_1} = 1$$

and $N_1(x)\, x_1 + N_2(x)\, x_2 = \dfrac{x_2 - x}{x_2 - x_1} x_1 + \dfrac{x - x_1}{x_2 - x_1} x_2 = x$

## Rigid body mode

$$\boxed{N_1(x) + N_2(x) = 1 \qquad \text{for all } x}$$

What do we mean by "**rigid body modes**"?

Assume that $d_{1x}=d_{2x}=1$, this means that the element should translate in the positive x direction by 1. Hence ***ANY point*** (x) on the bar should have unit displacement. Let us see whether the displacement approximation allows this.

$$w(x) = N_1(x)d_{1x} + N_2(x)d_{2x} = N_1(x) + N_2(x) = 1$$

YES!

## Constant strain states

$$\boxed{N_1(x)x_1 + N_2(x)x_2 = x \ \text{ at all } x}$$

What do we mean by "**constant strain states**"?

Assume that $d_{1x}=x_1$ and $d_{2x}=x_2$. The strain at ***ANY point*** (x) within the bar is

$$\varepsilon(x) = \frac{d_{2x} - d_{1x}}{x_2 - x_1} = \frac{x_2 - x_1}{x_2 - x_1} = 1$$

Let us see whether the displacement approximation allows this.

$$w(x) = N_1(x)d_{1x} + N_2(x)d_{2x} = N_1(x)x_1 + N_2(x)x_2 = x$$

$$\text{Hence, } \varepsilon(x) = \frac{dw(x)}{dx} = 1$$

YES!

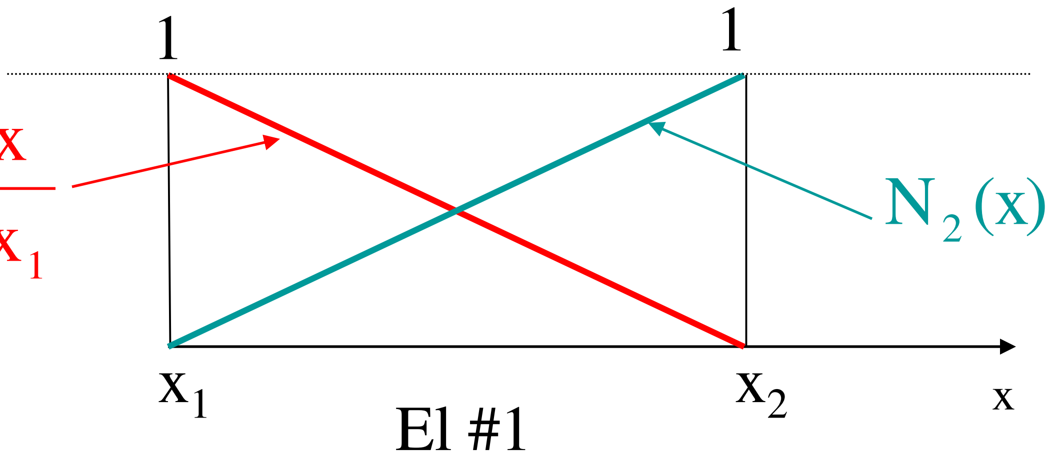
Completeness = Rigid body modes + Constant Strain states

**Compatibility + Completeness $\Rightarrow$ Convergence**

Ensure that the solution gets better as more elements are introduced and, in the limit, approaches the exact answer.

# 4. How to write the expressions for the shape functions easily (without having to derive them each time):

Start with the **Kronecker delta property** (the shape function at any node has value of 1 at that node and a value of zero at all other nodes)



$$N_1(x) = \frac{x_2 - x}{x_2 - x_1}$$

$$N_2(x) = \frac{x - x_1}{x_2 - x_1}$$

Node at which $N_1$ is 0

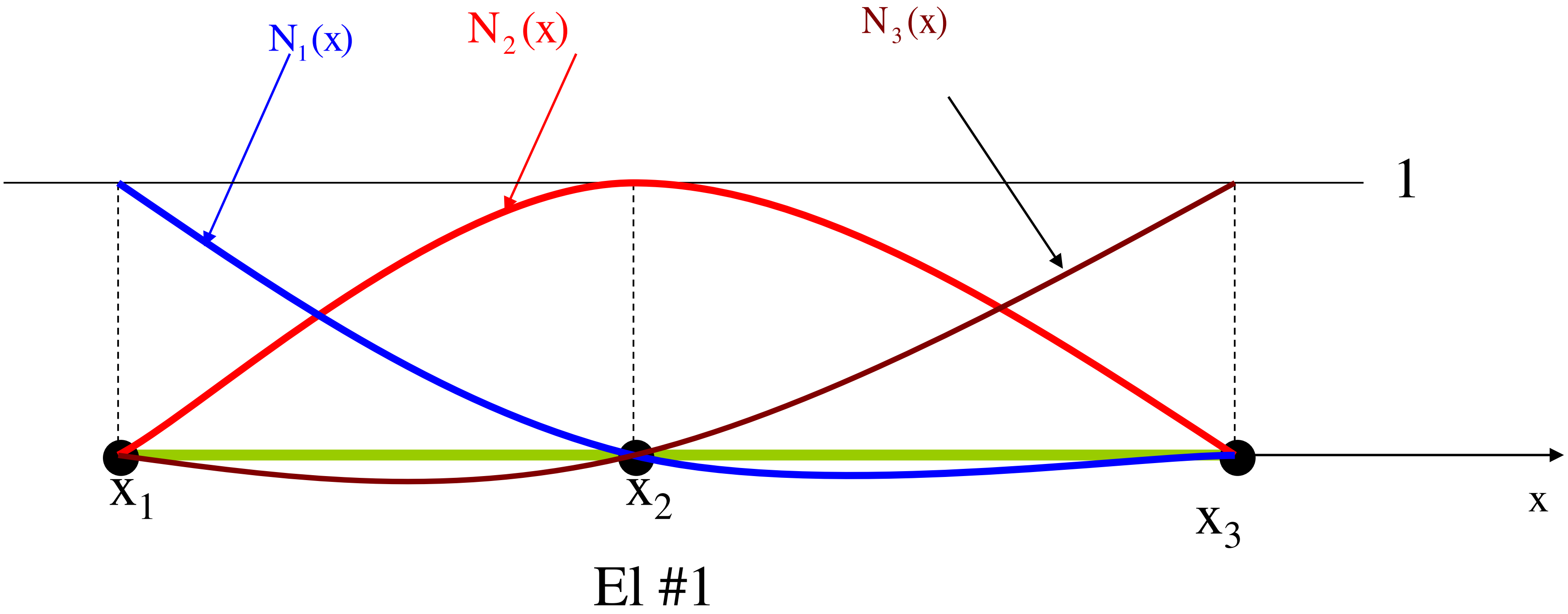
$$N_1(x) = \frac{(x_2 - x)}{(x_2 - x_1)}$$

Notice that the length of the element = $x_2$-$x_1$

$$N_2(x) = \frac{(x_1 - x)}{(x_1 - x_2)} = \frac{(x - x_1)}{(x_2 - x_1)}$$

The denominator is
the numerator evaluated at
the node itself

# A slightly fancier assumption:
# displacement varying **quadratically** inside each bar



$$N_1(x) = \frac{(x_2 - x)(x_3 - x)}{(x_2 - x_1)(x_3 - x_1)}$$

$$N_2(x) = \frac{(x_1 - x)(x_3 - x)}{(x_1 - x_2)(x_3 - x_2)}$$

$$N_3(x) = \frac{(x_1 - x)(x_2 - x)}{(x_1 - x_3)(x_2 - x_3)}$$

$$\boxed{w(x) = N_1(x)d_{1x} + N_2(x)d_{2x} + N_3(x)d_{3x}}$$

This is a **<u>quadratic finite element</u>** in 1D and it has three nodes and three associated shape functions per element.

## **TASK 2:** APPROXIMATE THE STRAIN and STRESS WITHIN EACH ELEMENT

From equation (1), the displacement within each element

$$w(x) = \underline{N}\,\underline{d}$$

Recall that the **strain** in the bar $\varepsilon = \dfrac{dw}{dx}$

Hence

$$\boxed{\varepsilon = \left[\frac{d\underline{N}}{dx}\right]\underline{d} = \underline{B}\,\underline{d}} \qquad (2)$$

The matrix $\underline{B}$ is known as the "**strain-displacement matrix**"

$$\underline{B} = \left[\frac{d\underline{N}}{dx}\right]$$

For a linear finite element

$$\underline{N} = \left[ N_1(x) \quad N_2(x) \right] = \left[ \frac{x_2 - x}{x_2 - x_1} \quad \frac{x - x_1}{x_2 - x_1} \right]$$

Hence

$$\underline{B} = \left[ \frac{-1}{x_2 - x_1} \quad \frac{1}{x_2 - x_1} \right] = \frac{1}{x_2 - x_1} \left[ -1 \quad 1 \right]$$

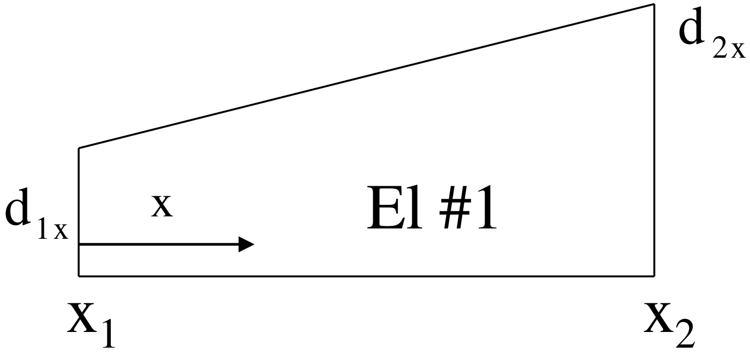
$$\varepsilon = \underline{B}\ \underline{d} = \left[ \frac{-1}{x_2 - x_1} \quad \frac{1}{x_2 - x_1} \right] \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix}$$

$$= \frac{d_{2x} - d_{1x}}{x_2 - x_1}$$

Hence, strain is a **constant** within each element (only for a linear element)!

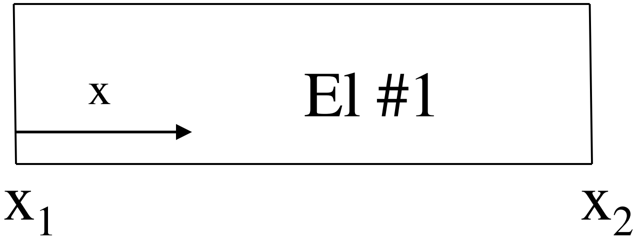
## Displacement is linear

$$w(x) = a_0 + a_1 x$$



## Strain is constant

$$\varepsilon = \frac{d_{2x} - d_{1x}}{x_2 - x_1}$$



Recall that the **stress** in the bar $\sigma = \mathrm{E}\varepsilon = \mathrm{E}\frac{\mathrm{du}}{\mathrm{dx}}$

Hence, inside the element, the approximate stress is

$$\boxed{\sigma = \mathrm{E}\underline{\mathrm{B}}\ \underline{\mathrm{d}}} \qquad (3)$$

For a linear element the stress is also constant inside each element. This has the implication that the stress (and strain) is **discontinuous across element boundaries** in general.

# Summary

Inside an element, the three most important approximations **in terms of the nodal displacements** ($\underline{d}$) are:

**Displacement approximation** in terms of shape functions

$$\boxed{u(x) = \underline{N}\ \underline{d}} \qquad (1)$$

**Strain approximation** in terms of strain-displacement matrix

$$\boxed{\varepsilon(x) = \underline{B}\ \underline{d}} \qquad (2)$$

**Stress approximation** in terms of strain-displacement matrix and Young's modulus

$$\boxed{\sigma = E\underline{B}\ \underline{d}} \qquad (3)$$

# Summary

For a **linear element**

**Displacement approximation** in terms of shape functions

$$\boxed{u(x) = \begin{bmatrix} \dfrac{x_2 - x}{x_2 - x_1} & \dfrac{x - x_1}{x_2 - x_1} \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix}}$$

**Strain approximation**

$$\boxed{\varepsilon = \frac{1}{x_2 - x_1} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix}}$$

**Stress approximation**

$$\boxed{\sigma = \frac{E}{x_2 - x_1} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix}}$$