Chapter 3
Mechanical Systems

Course Instructor

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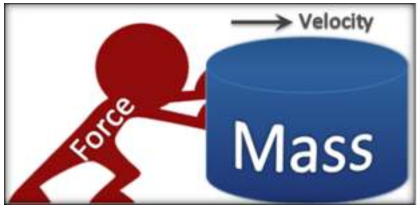
A mathematical model of any mechanical system can be developed by applying Newton's laws to the system. In this part, we shall deal with the problem of deriving mathematical models of simple mechanical systems

Newton's laws: three well-known laws called *Newton's laws*

1- Newton's first law



concerns the conservation of momentum, states that the total momentum of a mechanical system is constant in the absence of external forces.

- ***For translational* or linear motion, momentum is the product of mass *m* *and velocity v, or mv,***

- **For rotational motion, momentum is the product of moment of inertia *J* *and angular velocity ω, or $J\,\omega$, and is called angular momentum.***

Newton's laws: three well-known laws called *Newton's laws*

2- Newton's second law

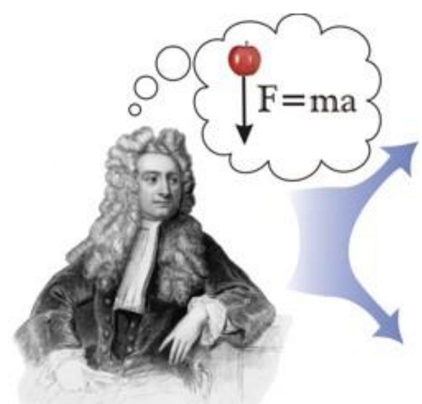
gives the force-acceleration relationship of a rigid translating body or the torque-angular acceleration relationship of a rigid rotating body

- **For translational motion: the acceleration of the rigid body in the direction of applied force is directly proportional to the force acting on it and is inversely proportional to the mass of the body.**

If ΣF is the sum of all* forces acting on mass *m through the center of mass in a given direction, then

$$m.a = \Sigma F$$

a is acceleration = Force/mass



Portrait source: U.S. Centennial of Flight Commission
http://www.centennialofflight.gov/essay/Dictionary/newton/DI36.htm

- **<u>for rotational motion</u>.**

For a rigid body in pure rotation about a fixed axis, Newton's second law states that;

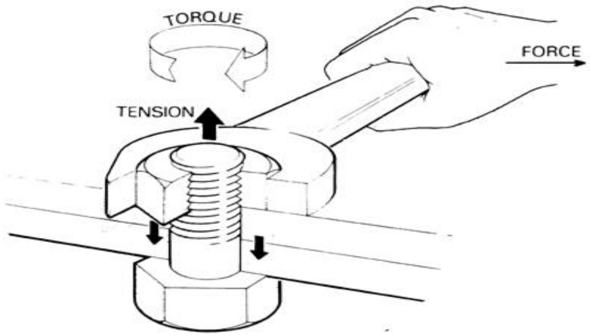
(moment of inertia) (angular acceleration) = torque

Or

$$\mathbf{J}.\alpha = \Sigma T$$

where
ΣT is the sum of all torques acting about a given axis,
***J is the moment of inertia* of a body about that axis, and**
a is the angular acceleration of the body.

Definitions



- **Torque or moment of force.**

Defined as any cause that tends to produce a <u>change in the rotational motion</u> of a body on which it acts.
Torque is the <u>product of a force and the perpendicular</u> distance from a point of rotation to the line of action of the force.

- **Moments of inertia. The *moment of inertia J of a rigid body about an axis* is defined by**

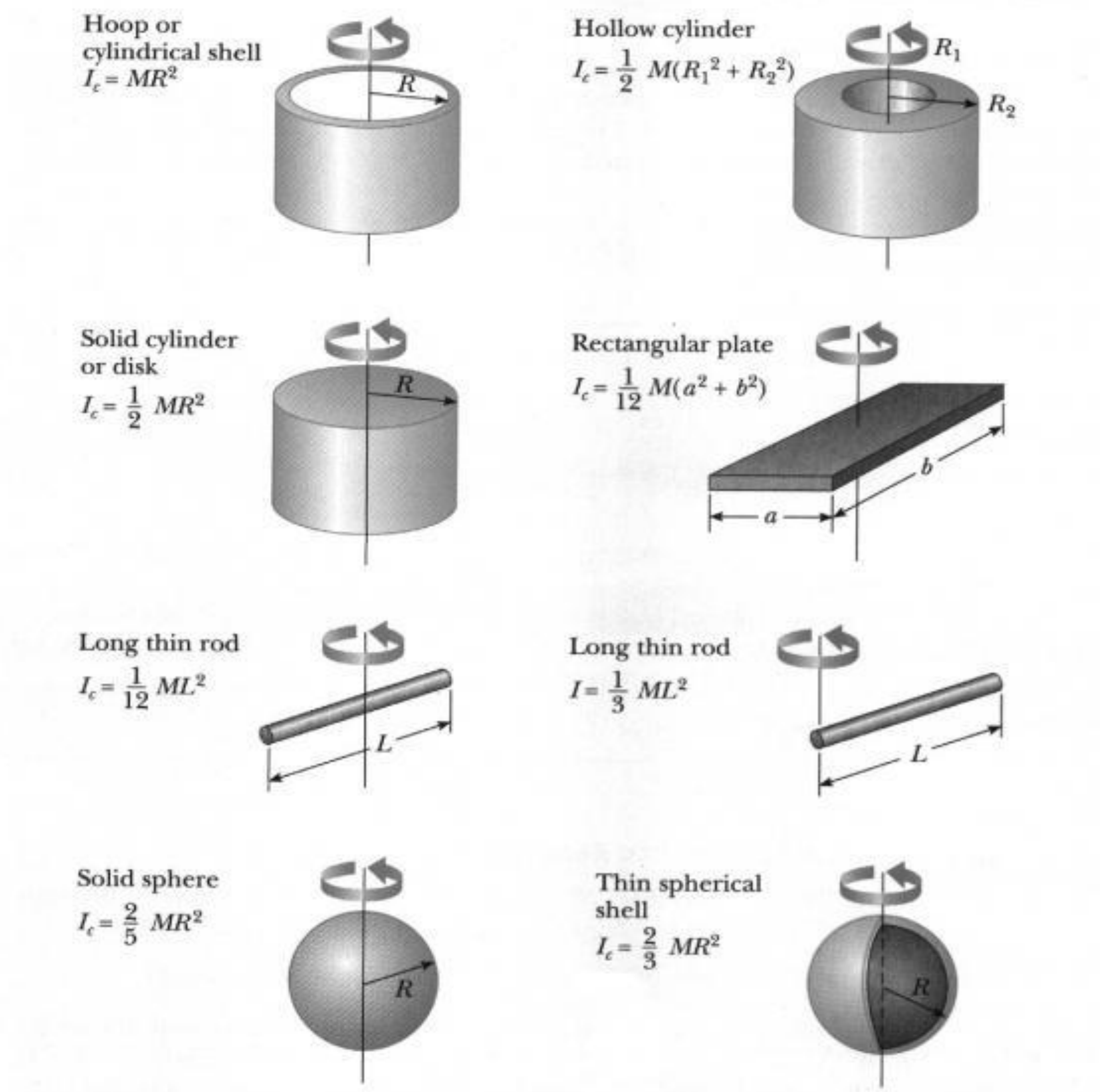
$$J = \int r^2 . dm$$

where

dm is an element of mass, r is distance from the axis to dm, and integration is **performed over the body.**

3.1 MATHEMATICAL MODELING OF SIMPLE MECHANICAL SYSTEMS

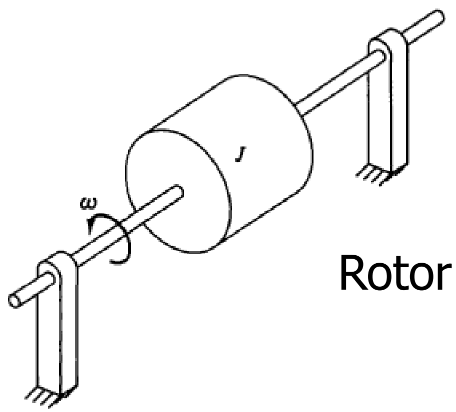
Examples of



Example 1: Rotational system

The moment of inertia of the rotor about the axis of rotation is J*. Let us assume that at t = 0 the rotor is rotating at the angular velocity w(0) = w_0*

We also assume that the friction in the bearings is viscous friction and that no external torque is applied to the rotor. Then the only torque acting on the rotor is the friction torque *bw in the bearings.*



Rotor mounted in bearings.

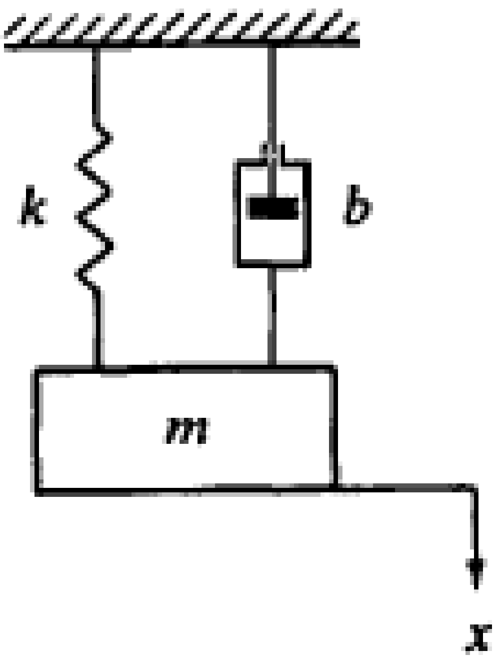
1-By Applying Newton's second law find the mathematical model of the system (equation of motion)

2- identify and find the output of the system using Laplace transform

Example 2: Spring-mass-damper system.

Most physical systems involve some type of damping-viscous damping, magnetic damping, and so on. Such damping not only slows the motion of (a part of) the system, but also causes the motion to stop eventually.

Figure below is a schematic diagram of a spring-mass-damper system. Suppose that the mass is pulled downward and then released



1-By Applying Newton's second law find the mathematical model of the system (equation of motion)

2- identify and find the output of the system using Laplace transform for particular case ***m = 0.1 slug, b = 0.4 lbf-s/ft, and k = 4 lbfl/ft.***

$X(0) = X_0$; $\dot{X}(0) = 0$

WORK, ENERGY, AND POWER

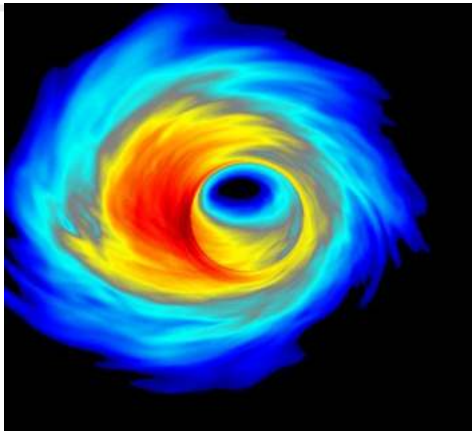
Work.

The *work done in a mechanical system is the product of a force and* the distance (or a torque and the angular displacement) through which the force is exerted, with both force and distance measured in the same direction.

For instance, if a body is pushed with a horizontal force of *F along a horizontal floor for* a distance of *x , the work done in pushing the body is*

$$W = F.x \quad (N\text{-}m)$$

Energy

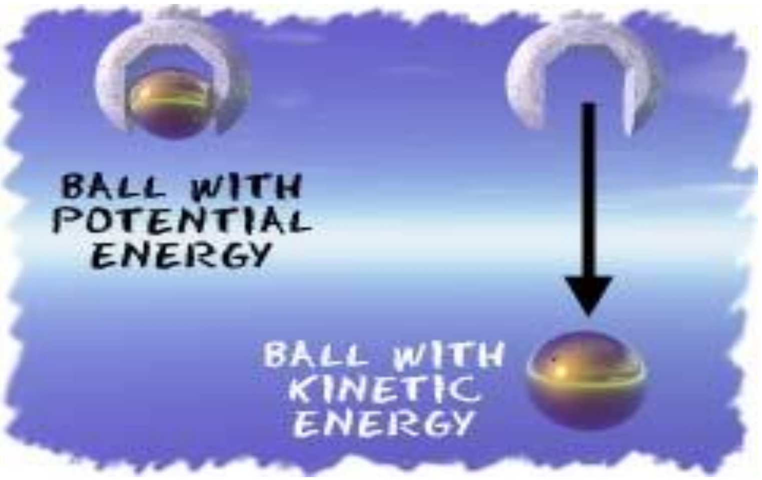


- In a general way, *energy can be defined as the capacity or ability to do* work. Energy is found in many different forms and can be converted from one form into another (electrical energy into mechanical..).
- When a system does mechanical work, the system's energy decreases by the amount equal to the energy required for the work done.
- According to the law of conservation of energy, energy can be neither created nor destroyed. This means that the increase in the total energy within a system is equal to the net energy input to the system. So if there is no energy input, there is no change in total energy of the system.

Potential energy.

In a mechanical system, only mass and spring elements can store potential energy. For a body of mass *m* in the gravitational field of the earth, the potential energy *U measured from some* reference level is *mg times the altitude h measured from the same reference level, or*

$$U = \int_0^h mg\,dx = mgh$$



Potential energy.

▪ **For a translational spring**, **the potential energy *U is equal to the net work done* on the spring by the forces acting on its ends as it is compressed or stretched.**

Since the spring force *F is equal to k.x, where x is the net displacement of the ends of the* spring, the total energy stored is

$$U = \int_0^x F\,dx = \int_0^x kx\,dx = \frac{1}{2}kx^2$$

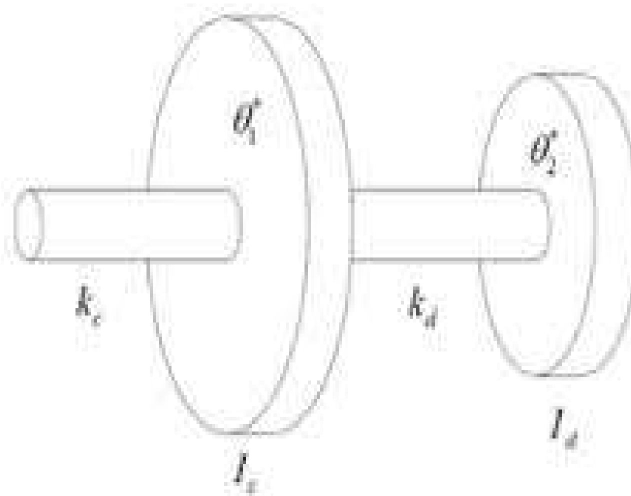
If the initial and final values of *x are X_1 and X_2, respectively, then* change in potential energy

$$\Delta U = \int_{x_1}^{x_2} F\,dx = \int_{x_1}^{x_2} kx\,dx = \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2$$

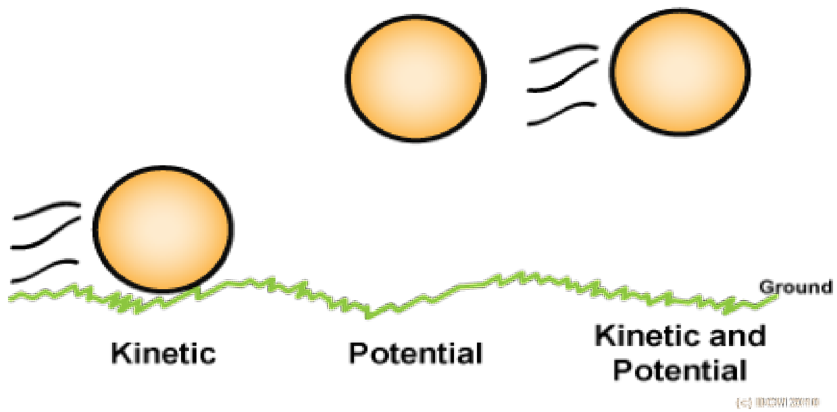
Potential energy.

Similarly, for a torsional spring, change in potential energy

$$\Delta U = \int_{\theta_1}^{\theta_2} T\, d\theta = \int_{\theta_1}^{\theta_2} k\theta\, d\theta = \frac{1}{2}k\theta_2^2 - \frac{1}{2}k\theta_1^2$$



Kinetic energy.



◆A mass *m in pure translation with velocity v has kinetic energy,*

$$T = \frac{1}{2}mv^2,$$

◆ *A moment of inertia J in pure rotation with angular velocity* has kinetic energy

$$T = \frac{1}{2}J\dot{\theta}^2.$$

◆ Thus, the change in kinetic energy *T of a mass m moving in a straight line is*

$$\Delta T = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

◆ The change in kinetic energy of a moment of inertia in pure rotation at angular velocity 8 is

$$\Delta T = \frac{1}{2}J\dot{\theta}_2^2 - \frac{1}{2}J\dot{\theta}_1^2$$

Power.

Power is the time rate of doing work. That is,

$$\text{power} = P = \frac{dW}{dt}$$

Where *dW denotes work done during time interval dt.*

The average power during a* duration of *t_2 - t_1 seconds can be determined by measuring the work done in t_2 - t_1 seconds, or

$$\text{average power} = \frac{\text{work done in } (t_2 - t_1) \text{ seconds}}{(t_2 - t_1) \text{ seconds}}$$

Energy method for deriving equations of motion

- **Newton's method are used for deriving equations of motion of mechanical systems. Several other approaches for obtaining equations of motion are available, one of which is based on the law of conservation of energy.**

- **we can derive such equations from the fact that the total energy of a system remains the same if no energy enters or leaves the system.**

- **In mechanical systems, friction dissipates energy as heat. Systems that do not involve friction are called *conservative* systems.**

Energy method for deriving equations of motion

Consider a conservative system in which the energy is in the form of kinetic or potential energy (or both). Since energy enters and leaves the conservative system in the form of mechanical work, we obtain

$$\Delta(T + U) = \Delta W$$

where $\Delta(T + U)$ is the change in the total energy and ΔW is the net work done on the system by an external force. If no external energy enters the system, then

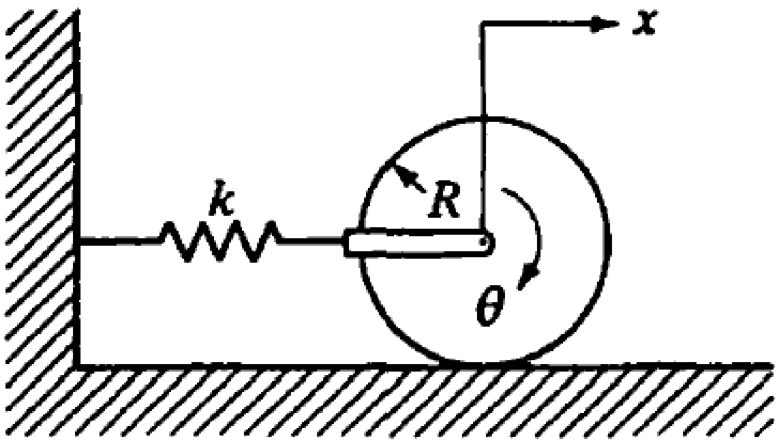
$$\Delta(T + U) = 0$$

which results in

$$T + U = \text{constant}$$

Energy method for deriving equations of motion

Example 1

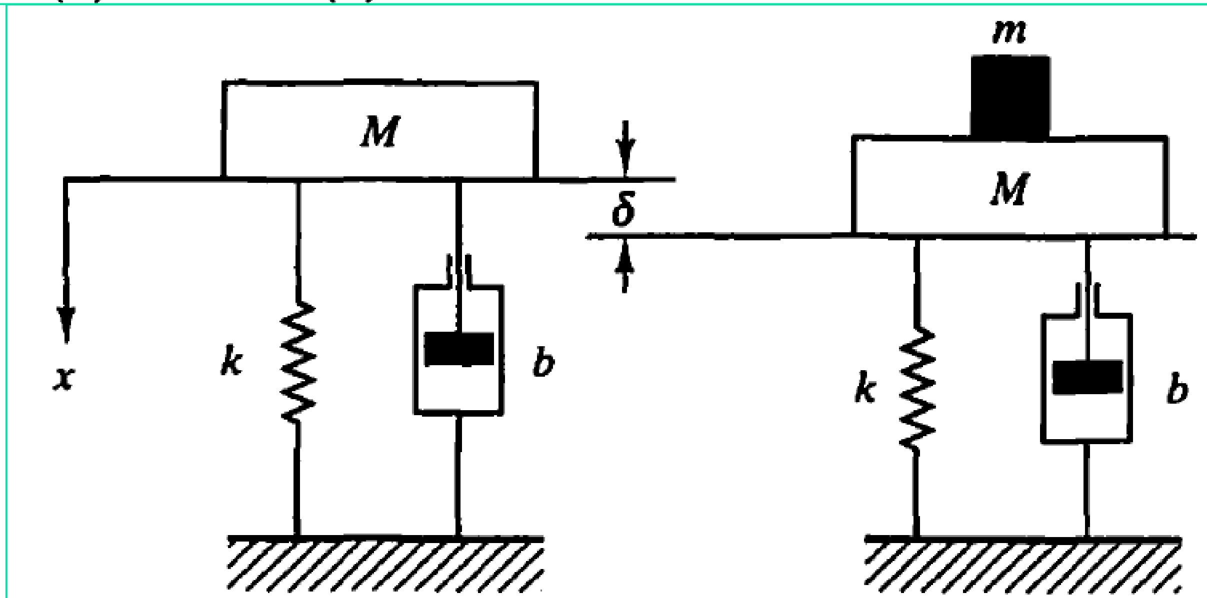


The figure shows a homogeneous cylinder of radius *R* and mass *m* that is free to rotate about its axis of rotation and that is connected to the wall through a spring.

Assuming that the cylinder rolls on a rough <u>surface without sliding</u>, obtain the <u>kinetic energy and potential energy of the system</u>. Then derive the equations of motion from the fact that the total <u>energy is constant</u>.

Example 2

A mass $M = 8$ kg is supported by a spring with spring constant $k = 400$ N/m and a damper with $b = 40$ N-s/m, as shown in Figure 3–31. When a mass $m = 2$ kg is gently placed on the top of mass M, the system exhibits vibrations. Assuming that the displacement x of the masses is measured from the equilibrium position before mass m is placed on mass M, determine the response $x(t)$ of the system. Determine also the static deflection δ—the deflection of the spring when the transient response died out. Assume that $x(0) = 0$ and $\dot{x}(0) = 0$.



Introduction

In the field of system dynamics, transfer functions are frequently used to characterize the input--output relationships of components or systems that can be described by linear, time-invariant differential equations.

Transfer Function.
Definition

The *transfer function of a linear, time-invariant differential-*equation system is defined as the ratio of the Laplace transform of the output (response function) to the Laplace transform of the input (driving function) under the assumption that all initial conditions are zero.

Transfer-Function: Approach to Modeling Dynamic Systems

Consider the linear time-invariant system defined by the differential equation

$$a_0 \overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 \overset{(m)}{x} + b_1 \overset{(m-1)}{x} + \cdots + b_{m-1}\dot{x} + b_m x \qquad (n \geq m)$$

Where: Y is the output of the system
x is the input of the system

The transfer function of this system is

$$\text{Transfer function} = G(s) = \left.\frac{\mathscr{L}[\text{output}]}{\mathscr{L}[\text{input}]}\right|_{\text{zero initial conditions}}$$

$$= \frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

Transfer-Function: Approach to Modeling Dynamic Systems

By using the concept of a transfer function, it is possible to represent system dynamics by algebraic equations in *s*. *If the highest power of s in the denominator of* the transfer function is equal to *n*, *the system is called an n^{th}-order system*.

Comments on the Transfer Function

- **The transfer function of a system is a mathematical model of that system, in that it is an operational method of expressing the differential equation that relates the output variable to the input variable.**

- **The transfer function is a property of a system itself, unrelated to the magnitude and nature of the input or driving function.**

Transfer-Function: Approach to Modeling Dynamic Systems

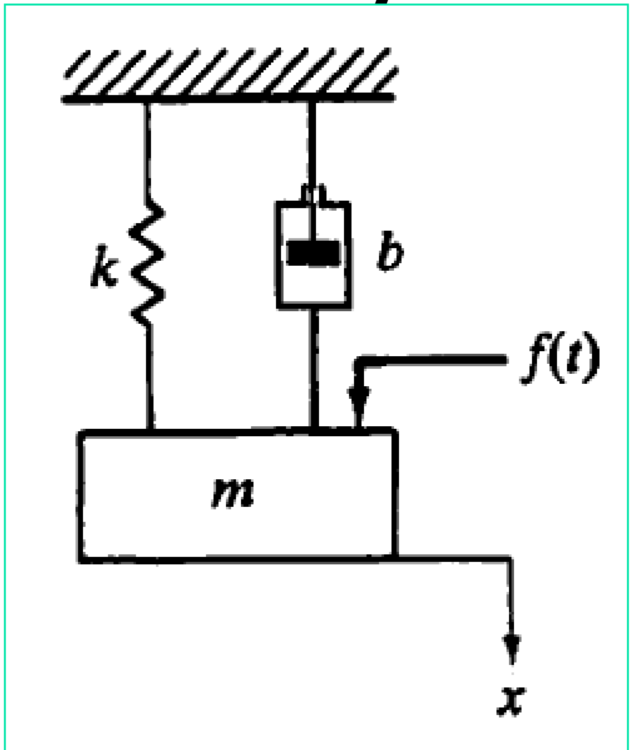
- The transfer function does not provide any information concerning the physical structure of the system. (The transfer functions of many physically different systems can be identical.)

- If the transfer function of a system is known, the output or response can be studied for various forms of inputs with a view toward understanding the nature of the system.

- If the transfer function of a system is unknown, it may be established experimentally by introducing known inputs and studying the output of the system.

Example

Consider the mechanical system shown in Figure below. The displacement *x of the mass m* is measured from the equilibrium position. In this system, the external force *f(t) is the* input and *x is the output*. *Find the* Transfer Function of the system



Impulse-Response Function

The transfer function of a linear, time-invariant system is

$$G(s) = \frac{Y(s)}{X(s)}$$

Where

X(s) is the Laplace transform of the input
Y(s) is the Laplace transform **of the output**
(we assume that all initial conditions involved are zero).

the output ***Y(s) can be written as :***

$$Y(s) = G(s)X(s)$$

Impulse-Response Function

If we consider the output (response) of the system to a <u>unit-impulse input</u> ($\delta(t)$). *The Laplace transform of the output of the system is*

$$Y(s) = G(s)$$

The inverse Laplace transform of the <u>output</u> yields the impulse response of the system. The inverse Laplace transform of *G(s)* is called the *<u>impulse-response function, or the weighting function</u>, of the system.*

$$\mathcal{L}^{-1}[G(s)] = g(t)$$

Impulse-Response Function

- **The impulse-response function *g(t) is thus the response of a linear system to a* unit-impulse input when the initial conditions are zero.**

- **The Laplace transform of *g(t)* gives the transfer function. Therefore, the transfer function and impulse-response function of a linear, time-invariant system contain the same information about the system dynamics.**

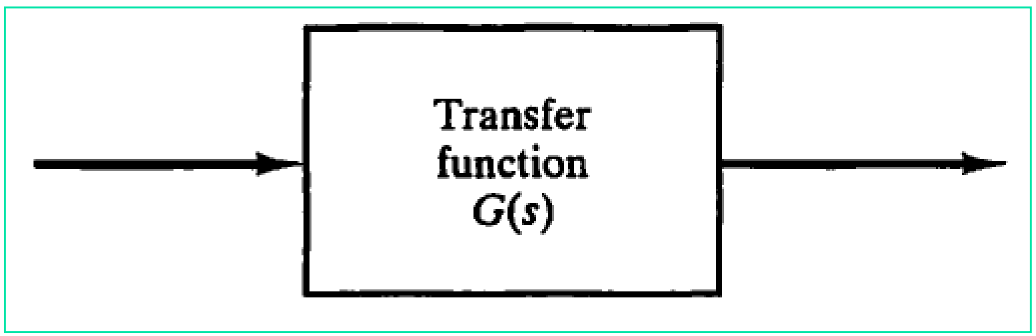
- **It is hence possible to obtain complete information about the dynamic characteristics of a system by exciting it with an impulse input and measuring the response.**

Block diagrams of dynamic systems

- is a pictorial representation of the functions performed by each component of the system and of the flow of signals within the system.
- Such a diagram depicts the interrelationships that exist among the various components.
- In a block diagram, all system variables are linked to each other through functional blocks.
- The *functional block, or simply block, is a symbol for the mathematical* operation on the input signal to the block that produces the output.
- The transfer functions of the components are usually entered in the corresponding blocks, which are connected by arrows to indicate the direction of the flow of signals.

BLOCK DIAGRAMS

- Figure below shows an <u>element of a block diagram</u>. <u>The arrowhead</u> pointing toward the block indicates the <u>input to the block</u>, and the arrowhead leading away from the block represents the <u>output of the block</u>. (such arrows represent signals).



Element of a block diagram

- the <u>dimension</u> of the output signal from a block is the dimension of the input signal multiplied by the dimension of the transfer function in the block.

BLOCK DIAGRAMS

- it is easy to form the <u>overall block diagram</u> for the entire system by <u>connecting the blocks</u> of the components according to the signal flow

- A block diagram contains information concerning <u>dynamic behavior</u>, but it does not include any information about the <u>physical construction of the system</u>.

    - Consequently, many <u>dissimilar systems</u> can be represented by the same block diagram.

- in a block diagram the main source of energy is <u>not explicitly shown</u> and that the block diagram of a given system is not unique.

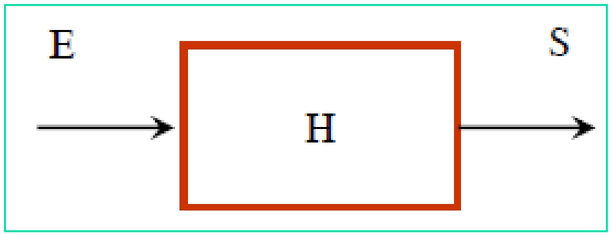
Formalism

A block diagram is constituted by an <u>assembly of four elements types</u>

- **Blocks (rectangles)**
- **comparators And Summing points**
- **Branch point or derivations Points**
- **Arrows (the flow of signals)**

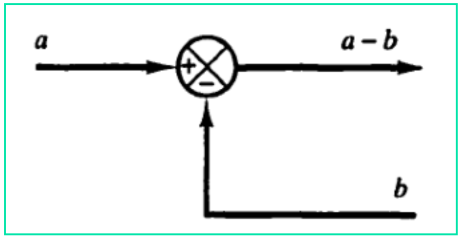
Formalism

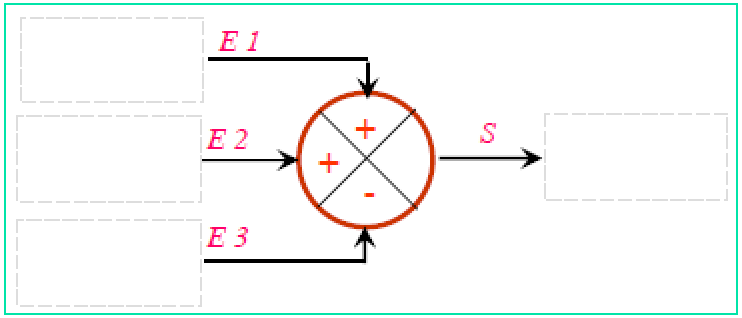
Block: **The block has <u>an input</u> E and <u>an output S</u>. The function transfer of the block H is obtained from operating equations**



$$S(p) = E(p) \times H(p)$$

Summing points: **The <u>plus or minus</u> sign at each arrowhead indicates whether the associated signal is to <u>be added or subtracted</u>. It is important that the quantities being added or subtracted have the <u>same dimensions and the same units</u>.**



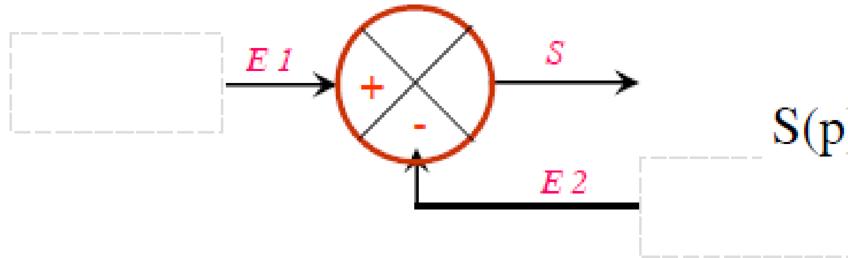


$$S(p) = E_1(p) + E_2(p) - E_3(p)$$

BLOCK DIAGRAMS

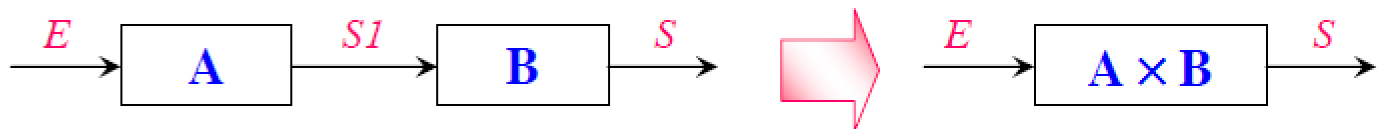
Formalism

Comparator:



$$S(p) = E_1(p) - E_2(p)$$

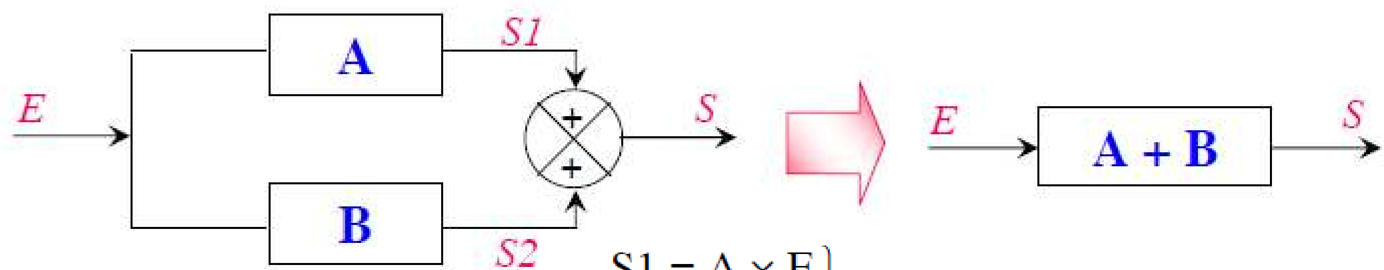
Product:



$$S1 = A \times E$$

$$S = B \times S1 \quad \Rightarrow S = [A \times B] \times E$$

Summation:

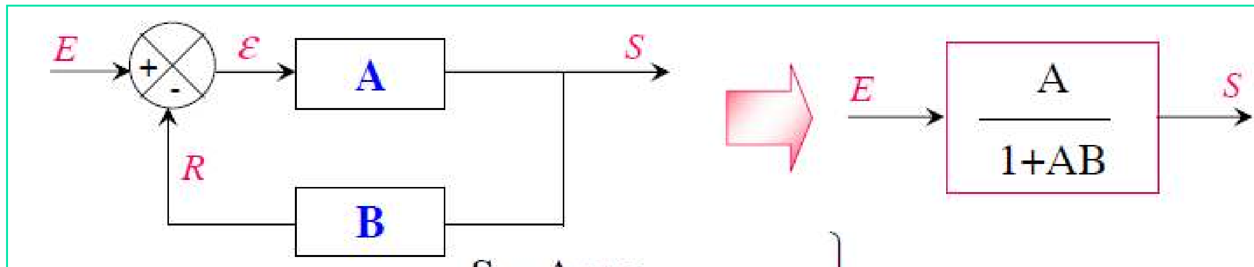


$$\left.\begin{aligned} S1 &= A \times E \\ S2 &= B \times E \\ S &= S1 + S2 \end{aligned}\right\} \Rightarrow S = [A + B] \times E$$

BLOCK DIAGRAMS

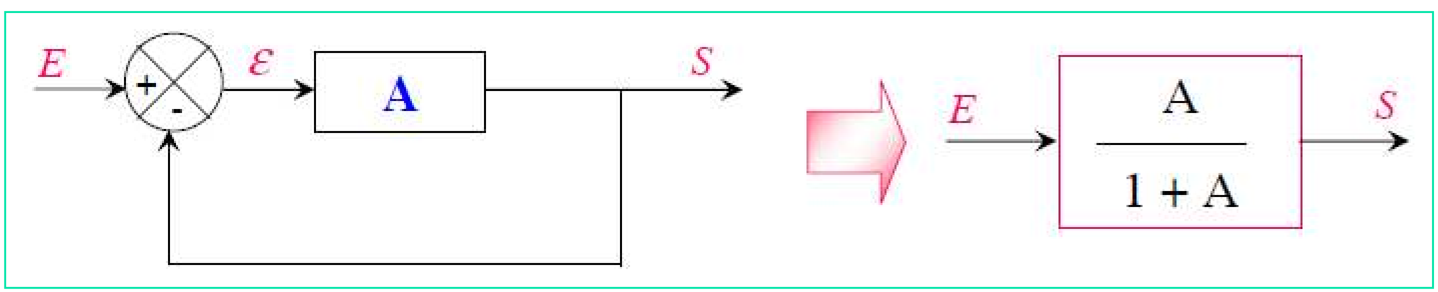
Formalism

Anti reaction:



$$\left.\begin{aligned} S &= A \times \varepsilon \\ S &= A \times (E - R) \\ S &= A \times [E - (B \times S)] \end{aligned}\right\} \Rightarrow S = \frac{A}{1+AB} \times E$$

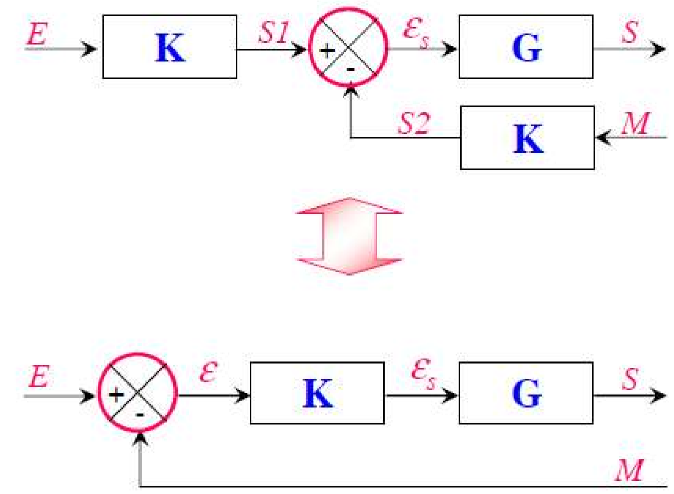
Anti reaction with unit back:



$$\left.\begin{aligned} S &= A \times \varepsilon \\ S &= A \times (E - S) \end{aligned}\right\} \Rightarrow S = \frac{A}{1 + A} \times E$$

BLOCK DIAGRAMS

Moving a summation:



$$S = G \times \varepsilon_S$$

$$S = G\times(S1 - S2)$$

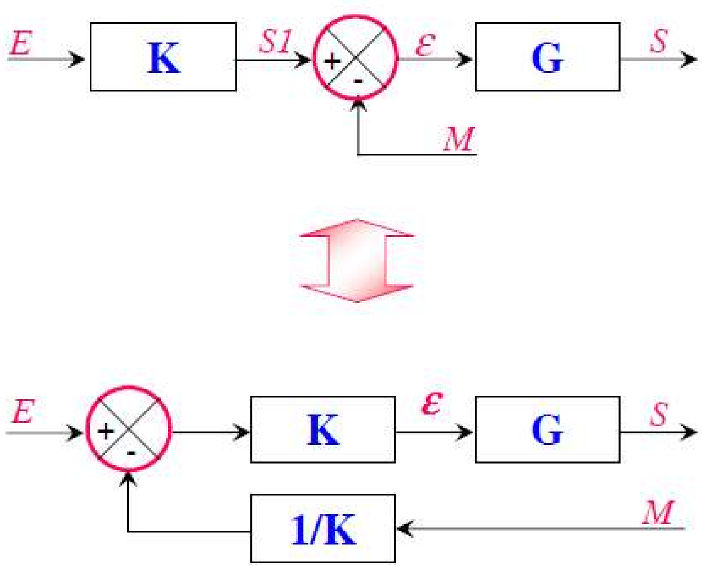
$$S = G\times\left[(E\times K)-(M\times K)\right]$$

$$\Rightarrow S = G\times K(E - M)$$

$$S = G \times \varepsilon_S$$

$$S = G\times(K\times\varepsilon)$$

$$\Rightarrow S = G\times K(E - M)$$



$$S = G \times \varepsilon$$

$$S = G\times(S1 - M)$$

$$S = G\times\left[(E\times K)-M\right]$$

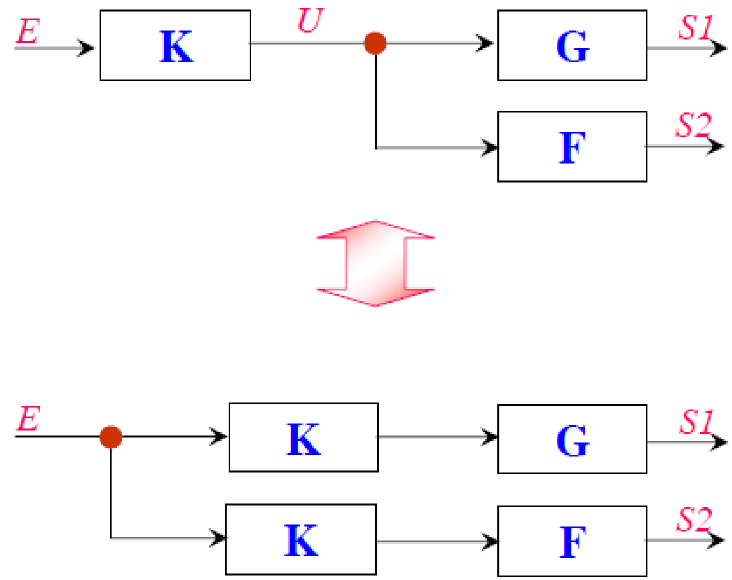
$$\Rightarrow S = G\times E\times K - G\times M$$

$$S = G\times K\times(E - \frac{1}{K}M)$$

$$\Rightarrow S = G\times K\times E - G\times M$$

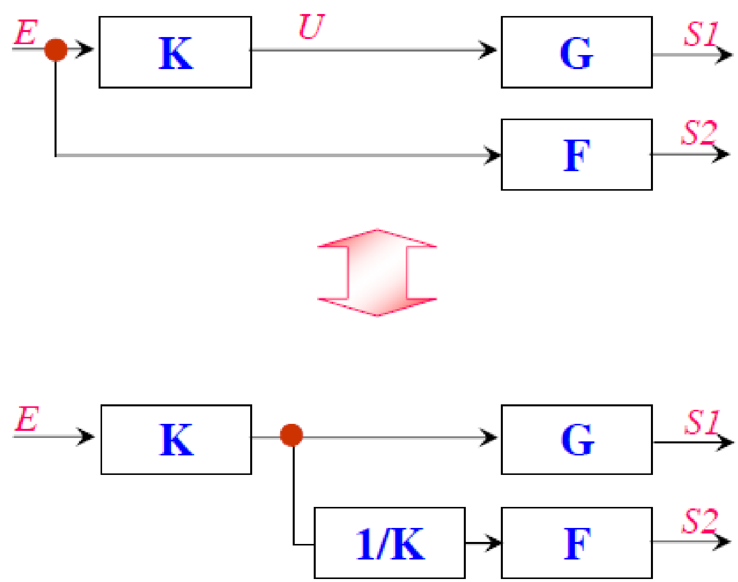
BLOCK DIAGRAMS

Moving a junction:



$$S1 = [G \times K] \times E$$

$$S2 = [F \times K] \times E$$

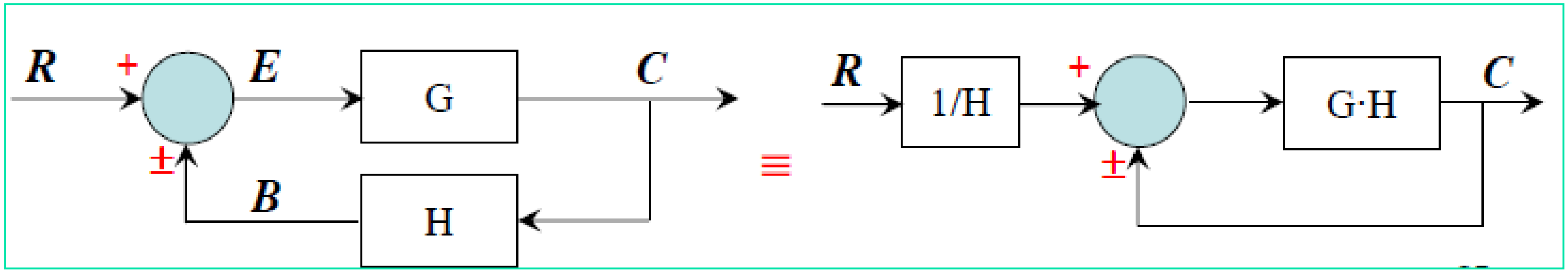


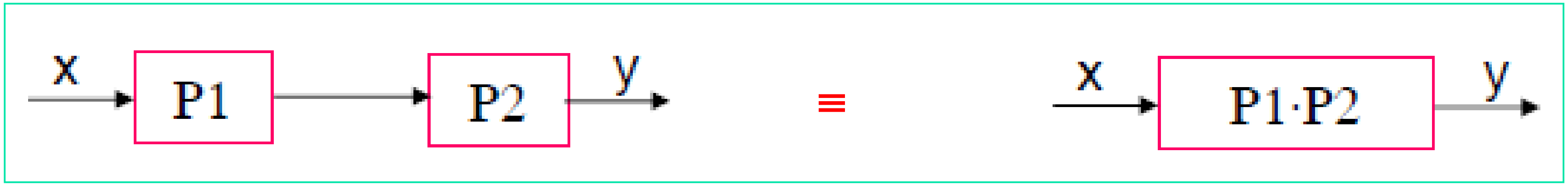
$$S1 = [G \times K] \times E$$

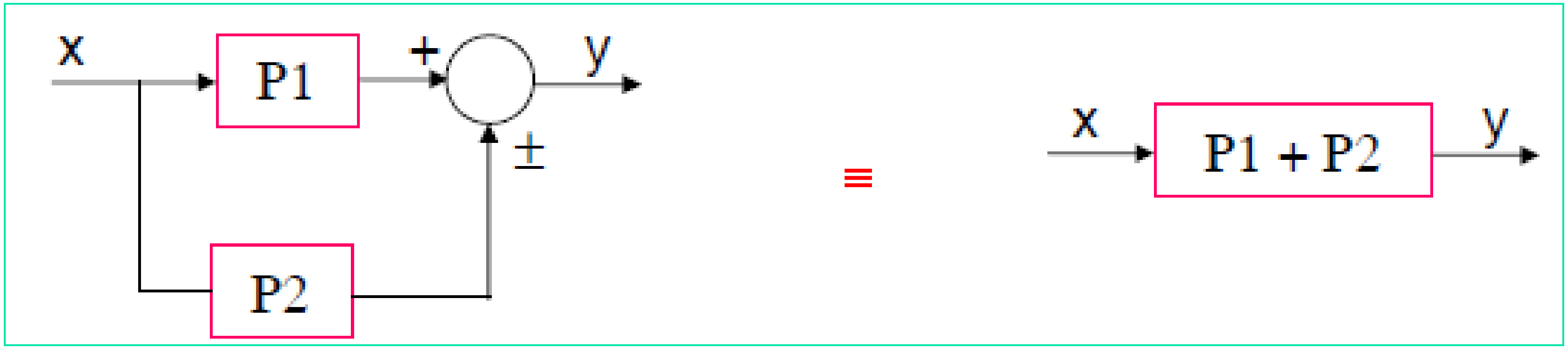
$$S2 = F \times E$$

BLOCK DIAGRAMS

Transformation rules (see documents):







Block diagram of a closed-loop system (Figure 4.4):

-The output C(s) is fed back to the summing point, where it is compared with the input R(s).

- The output C(s) of the block is obtained by multiplying the transfer function G(s) by the input to the block, E(s).

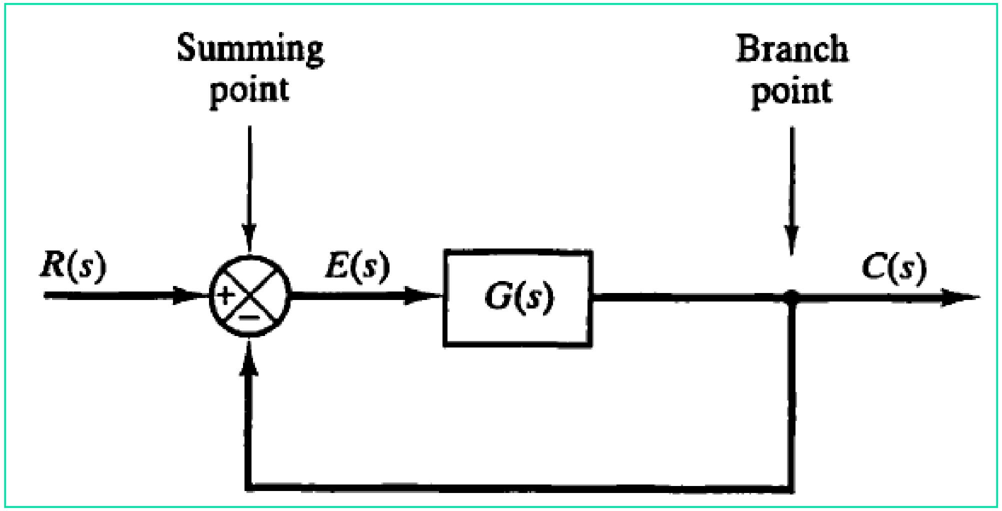


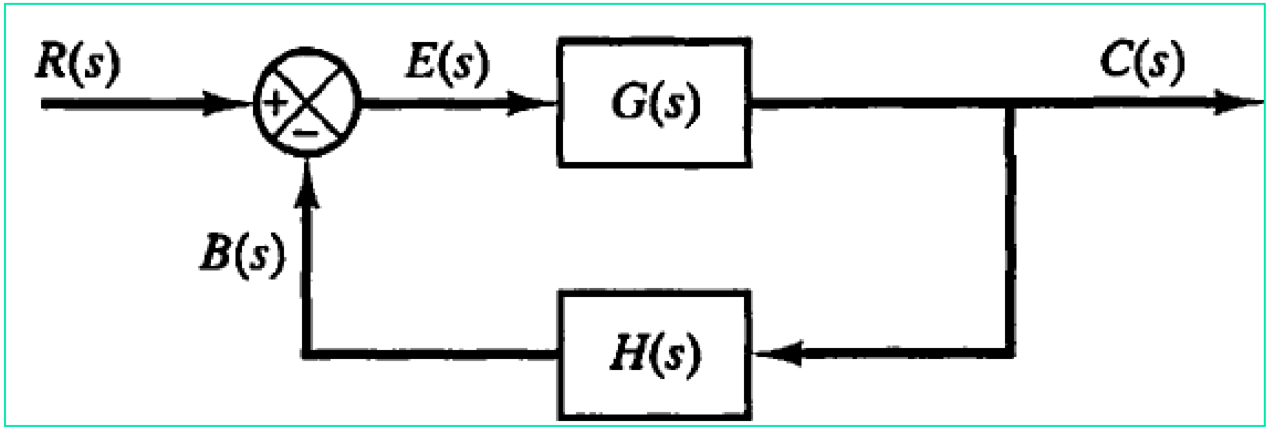
Fig. 4.4 Block diagram of a closed loop system.

BLOCK DIAGRAMS

Block diagram of a closed-loop system:

When the output is fed back to the summing point <u>for comparison with the input</u>, it is necessary <u>to convert the form of the output</u> signal to that <u>of the input signal</u>. This conversion is accomplished by the feedback element whose <u>transfer function is *H(s)*</u>*, as shown in Figure below.*

- The feedback element <u>modify the output</u> before it is compared with the input. (B(s) = H(s)C(s)).

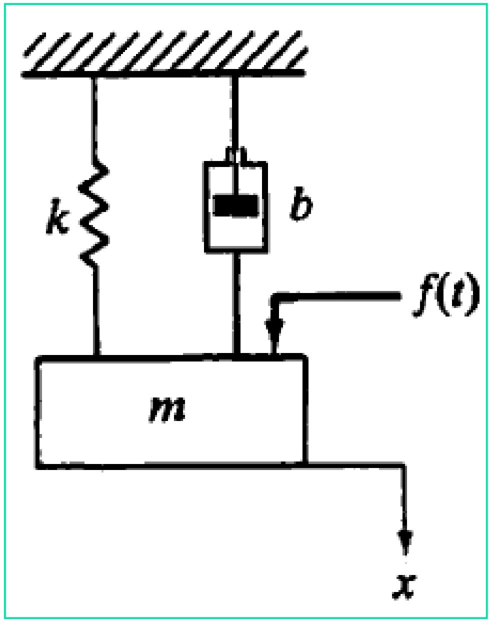


Block diagram of a closed-loop system with feedback element

Example:

We Consider again the mechanical system shown below. The transfer function of this system is

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$



Show the block diagrams of the system