Worksheet N° 1

Topics:	Error Definitions, Taylor series

Exercise 1. Determine the five-digit (a) chopping and (b) rounding values of the irrational number π.

Exercise 2. (Error Definitions) Suppose that you have to measure the lengths of a bridge and a rivet and come up with 9999 and 9 cm, respectively. If the true values are 10000 and 10 cm, respectively, compute

(a) the true error for each case. (True Error E_t=True value - Approximation)
(b) the relative error in each case. (Relative Error $=\frac{\text{True value - Approximation}}{\text{True value}}$)
(c) the percent relative error for each case. (percent relative error $\epsilon_t = \frac{\text{True value - Approximation}}{\text{True value}} \times 100\%$)

Exercise 3. Determine the true, relative and percent relative errors when approximating p by p^* when

(a) $p = 0.3000 \times 10^1$ and $p^* = 0.3100 \times 10^1$;
(b) $p = 0.3000 \times 10^{-3}$ and $p^* = 0.3100 \times 10^{-3}$;
(c) $p = 0.3000 \times 10^4$ and $p^* = 0.3100 \times 10^4$.

Exercise 4. (Finite-Digit Arithmetic) Suppose that $x = \frac{5}{7}$ and $y = \frac{1}{3}$. Use five-digit chopping for calculating $x + y$, $x - y$, $x \times y$, and $x \div y$. For each operation determine the true, relative and percent relative errors.

Exercise 5. Suppose that to $x = \frac{5}{7}$, $y = \frac{1}{3}$, $u = 0.714251$, $v = 98765.9$, and $w = 0.111111 \times 10^{-4}$.
Determine the five-digit chopping values of $x - u$, $(x - u) \div w$, $(x - u) \times v$ and $u + v$. For each operation determine the true, relative and percent relative errors.

Exercise 6. (Accuracy of an approximation) **Definition:** The number p^* is said to approximate p to n significant digits (or figures) if n is the largest nonnegative integer for which $\left|\frac{p-p^*}{p}\right| \times 100\% < \epsilon_s$ where

$$\epsilon_s = (0.5 \times 10^{2-n})\%$$

Let $p = 0.54617$ and $q = 0.54601$. Use four-digit arithmetic to approximate $p - q$ and determine the true, relative and percent relative errors using rounding and chopping. In each case, determine the number of significant digits of accuracy.

Exercise 7. The Maclaurin series of e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} + ... \text{ for } -\infty < x < +\infty.$$

Approximate $e^{0.5} = \sqrt{e}$ using:

(a) a single term of the Maclaurin series of e^x.

(b) the first 2 terms of the Maclaurin series of e^x.

(c) the first 3 terms of the Maclaurin series of e^x.

(d) the first 4 terms of the Maclaurin series of e^x.

(e) the first 5 terms of the Maclaurin series of e^x.

(f) the first 6 terms of the Maclaurin series of e^x.

For each approximation, find the percent approximate relative error and state the least number of significant figures that we can trust in the result.

Exercise 8. (HW) The Maclaurin series of $\sqrt{1+x}$ is

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \ldots \text{ for } |x| < 1.$$

Approximate $\sqrt{1.15}$ and $\sqrt{1.1}$ using:

(a) a single term of the Maclaurin series of $\sqrt{1+x}$.

(b) the first 2 terms of the Maclaurin series of $\sqrt{1+x}$.

(c) the first 3 terms of the Maclaurin series of $\sqrt{1+x}$.

(d) the first 4 terms of the Maclaurin series of $\sqrt{1+x}$.

For each approximation, find the percent approximate relative error and state the least number of significant figures that we can trust in the result.

Worksheet N$^\circ$ 2

Topics:	Finding roots of equations: Bisection method, False-Position method, Newton-Raphson method, Fixed point method, Secant method

Exercise 1. Show that

(a) the equation $x^5 + x - 3 = 0$ has a root in the interval $[1, 2]$.
(b) the equation $x^3 - 2x + 5 = 0$ has a root in the interval $[-3, -2]$.
(c) the equation $e^{-x} - x = 0$ has a root in the interval $[0, 1]$.
(d) the equation $\cos(x) - x = 0$ has a root in the interval $[0, 1]$.
(e) the equation $\cos(x) - 3x + 1 = 0$ has a root in the interval $\left[0, \frac{\pi}{2}\right]$.
(f) the equation $e^x - x - 2 = 0$ has a root in the interval $[1, 2]$.
(g) the equation $x^4 + x - 3 = 0$ has a root in the interval $[1, 2]$ and a root in the interval $[-2, -1]$.
(h) the equation $x^3 - 6x - 4 = 0$ has a root in the interval $[2, 3]$ and a root in the interval $[-1, 0]$. ($x = -2$ is a root)
(i) $\frac{667.38}{x}\left(1 - e^{-0.146843x}\right) - 40 = 0$ has a root in the interval $[12, 16]$

Exercise 2. Use the bisection method to solve the equation

$$\frac{667.38}{x}\left(1 - e^{-0.146843x}\right) - 40 = 0,$$

and continue until the approximate error $|\varepsilon_a|$ falls below a stopping criterion $\varepsilon_s = 0.5\%$.

Exercise 3. Show that $f(x) = x^3 + 4x^2 - 10 = 0$ has a root in $[1, 2]$, and use the Bisection method to determine an approximation to the root that is accurate to at least within 10^{-4}.

Exercise 4. Determine a formula which relates the number of iterations, n, required by the bisection method to converge to within an absolute error tolerance of ε, starting from the initial interval (a, b).

Exercise 5. Determine the number of iterations necessary to solve $f(x) = x^3 + 4x^2 - 10 = 0$ with accuracy 10^{-3} using $a = 1$ and $b = 2$.

Exercise 6. Approximate a real root of the equation

$$x^3 - 6x - 4 = 0,$$

by bisection method. Employ a stopping criterion ε_s that ensures that your result is correct to at least 2 significant figures.

Exercise 7. (HW) Use the bisection method to estimate the real root of $x^5 + x - 3 = 0$. Use a stopping criterion $\varepsilon_s = 0.05\%$.

Exercise 8. Use the false-position method to determine the root of the equation

$$\frac{667.38}{x}\left(1-e^{-0.146843x}\right)-40=0,$$

and continue until the approximate error $|\varepsilon_a|$ falls below a stopping criterion $\varepsilon_s = 0.5\%$.

Exercise 9. (HW) Use the false-position method to estimate the real root of $x^5+x-3=0$. Use a stopping criterion $\varepsilon_s = 0.05\%$.

Exercise 10. Use the Newton-Raphson method to estimate the root of $e^{-x}-x=0$, employing an initial guess of $x_0 = 0$.

Exercise 11. Use the Newton-Raphson method to estimate a root of $\cos(x)-x=0$. Use a stopping criterion $\varepsilon_s = 0.05\%$.

Exercise 12. (HW) Use the Newton-Raphson method to estimate the real root of $x^5+x-3=0$. Use a stopping criterion $\varepsilon_s = 0.05\%$.

Exercise 13. Rewrite each of the following equations in the form $x = g(x)$.

(a) $x^2-3x+2=0$
(b) $x^3-6x-4=0$
(c) $e^{2x}-x-5=0$

Exercise 14. Use simple fixed point iteration to approximate the root of $e^{-x}-x=0$.

Exercise 15. Use simple fixed point iteration to approximate a root of $\cos(x)-3x+1=0$.

Exercise 16. (HW) Approximate a root for $x^3-2x+5=0$ by fixed point method.

Exercise 17. Use the secant method to estimate the root of $f(x) = e^{-x}-x$. Start with initial estimates of $x_0 = 0$ and $x_1 = 1.0$.

Exercise 18. (HW) Use the secant method to estimate the real root of $x^5+x-3=0$. Use a stopping criterion $\varepsilon_s = 0.05\%$.

Exercise 19. Use fixed-point iteration to determine the roots of the following simultaneous nonlinear equations

$$\begin{cases} x^2+xy-10 &= 0 \\ y+3xy^2-57 &= 0 \end{cases}$$

Note that a correct pair of roots is $x = 2$ and $y = 3$. Initiate the computation with guesses of $x = 1.5$ and $y = 3.5$.

Exercise 20. (HW) Use fixed-point iteration to determine the roots of the following simultaneous nonlinear equations

$$\begin{cases} y &= -x^2+x+0.75 \\ y+5xy &= x^2 \end{cases}$$

Worksheet N$^\circ$ 3

Topics:	Solving Systems of linear equations: Gauss and Gauss-Jorden elimination methods, Jaccobi and Gauss-Seidel iterative methods

Exercise 1. Use Gauss elimination to solve

$$\begin{cases} 3x_1 - 0.1x_2 - 0.2x_3 &= 7.85 \\ 0.1x_1 + 7x_2 - 0.3x_3 &= -19.3 \\ 0.3x_1 - 0.2x_2 + 10x_3 &= 71.4 \end{cases}$$

Carry six significant figures during the computation.

Exercise 2. Use Gauss-Jordan technique to solve

$$\begin{cases} 3x_1 - 0.1x_2 - 0.2x_3 &= 7.85 \\ 0.1x_1 + 7x_2 - 0.3x_3 &= -19.3 \\ 0.3x_1 - 0.2x_2 + 10x_3 &= 71.4 \end{cases}$$

Carry six significant figures during the computation.

Exercise 3. Use the Jaccobi method with initial guess $x_0 = y_0 = 0$ to approximate the solution of the system:

$$\begin{cases} 7x - y &= 6 \\ x - 5y &= -4 \end{cases}$$

Exercise 4. Use the Gauss-Seidel method with initial guess $x_0 = y_0 = 0$ to approximate the solution of the system:

$$\begin{cases} 7x - y &= 6 \\ x - 5y &= -4 \end{cases}$$

Exercise 5. Use the Gauss-Seidel method to solve

$$\begin{cases} 3x_1 - 0.1x_2 - 0.2x_3 &= 7.85 \\ 0.1x_1 + 7x_2 - 0.3x_3 &= -19.3 \\ 0.3x_1 - 0.2x_2 + 10x_3 &= 71.4 \end{cases}$$

Carry six significant figures during the computation. Recall that the true solution is $x_1 = 3$, $x_2 = -2.5$, and $x_3 = 7$.

Exercise 6. Apply the Jaccobi method with initial guess $x_0 = y_0 = 0$ to the system:

$$\begin{cases} x_1 - 5x_2 &= -4 \\ 7x_1 - x_2 &= 6 \end{cases}$$

Exercise 7. Which of the following systems has a strictly diagonally dominant coefficients matrix:

(a) $\begin{cases} 3x_1 - x_2 &= -4 \\ 2x_1 + 5x_2 &= 2 \end{cases}$

(b) $\begin{cases} x_1 - 5x_2 &= -4 \\ 7x_1 - x_2 &= 6 \end{cases}$

(c) $\begin{cases} 4x_1 + x_2 - x_3 &= 3 \\ 2x_1 + 7x_2 + x_3 &= 19 \\ x_1 - 3x_2 + 12x_3 &= 31 \end{cases}$

(d) $\begin{cases} 4x_1 + 2x_2 - x_3 &= -1 \\ x_1 + x_2 + 3x_3 &= -4 \\ 3x_1 - 5x_2 + x_3 &= 3 \end{cases}$

Exercise 8. Use the Jaccobi and Gauss-Seidel methods to approximate the solution of:

$$\begin{cases} 5x_1 - 2x_2 + 3x_3 &= -1 \\ -3x_1 + 9x_2 + x_3 &= 2 \\ 2x_1 - x_2 - 7x_3 &= 3 \end{cases}$$

Exercise 9. (HW) Use the the Jaccobi and Gauss-Seidel methods to approximate the solution of:

$$\begin{cases} 4x_1 + x_2 - x_3 &= 3 \\ 2x_1 + 7x_2 + x_3 &= 19 \\ x_1 - 3x_2 + 12x_3 &= 31 \end{cases}$$

Exercise 10. (HW) Use the Gauss-Seidel method to approximate the solution of each of the following systems:

(a) $\begin{cases} 3x_1 - x_2 &= -4 \\ 2x_1 + 5x_2 &= 2 \end{cases}$

(b) $\begin{cases} 4x_1 + 2x_2 - x_3 &= -1 \\ x_1 + x_2 + 3x_3 &= -4 \\ 3x_1 - 5x_2 + x_3 &= 3 \end{cases}$

Worksheet N$^\circ$ 4

Topics:	Curve fitting

Exercise 1. Given the data

x	1	6
$y = f(x) = \ln x$	0	1.791759

approximate $\ln 2$ by using linear interpolation.

Exercise 2. Given the data

x	1	4
$y = f(x) = \ln x$	0	1.386294

approximate $\ln 2$ by using linear interpolation.

Exercise 3. Given the data

x	1	4	6
$y = f(x) = \ln x$	0	1.386294	1.791759

approximate $\ln 2$ by using quadratic interpolation.

Exercise 4. Given the data

x	1	2	3
$y = f(x)$	-1	1	2

construct the Newton's divided differences table and approximate $f(1.6)$ using the the second order Newton's interpolating polynomial.

Exercise 5. Consider the data

x	1	4	6	5
$y = f(x) = \ln x$	0	1.386294	1.791759	1.609438

construct the Newton's divided differences table and approximate $\ln 2$ using the the third order Newton's interpolating polynomial.

Exercise 6. Find $f(1.9)$ using Newton's interpolating polynomial for the following data.

x	1	3	4
$y = f(x)$	1	27	64

Exercise 7. Find $f(2.5)$ using Newton's interpolating polynomial for the following data.

x	1	2	3	4
$y = f(x)$	1	16	81	256

Exercise 8. (HW) Find $f(1.6)$ using Newton's interpolating polynomial for the following data.

x	1	1.4	1.8	2.2
$y = f(x)$	3.49	4.82	5.96	6.5

Exercise 9. Find $f(8)$ using Lagrange interpolating polynomial for the following data.

x	1	3	4
$y = f(x)$	1	27	64

Exercise 10. (HW) Find $f(10)$ using Lagrange interpolating polynomial for the following data.

x	1	7	15
$y = f(x)$	168	192	336

Exercise 11. Compute $f(3.5)$ from the following data

x	1	2	3	4
$y = f(x)$	1	2	9	28

using Lagrange interpolating polynomials of second and third order.

Exercise 12. Let $f(t)$ be integrable of $[-T, T]$. The **Fourier series** of f on $[-T, T]$ is

$$a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi t}{T} + b_n \sin \frac{n\pi t}{T} \right)$$

where the coefficients are given by:

$$a_0 = \frac{1}{2T} \int_{-T}^{T} f(t)dt, \quad \text{(the average value of } f \text{ on } [-T, T]).$$

$$a_n = \frac{1}{T} \int_{-T}^{T} f(t) \cos \frac{n\pi t}{T} dt,$$

$$b_n = \frac{1}{T} \int_{-T}^{T} f(t) \sin \frac{n\pi t}{T} dt.$$

Find the coefficients of the Fourier series of each of the following:

(a) $f(t) = t^2 - t$, $-2 \leq t \leq 2$.
(b) $f(t) = t$, $-\pi \leq t \leq \pi$.
(c) $f(t) = |t|$, $-\pi \leq t \leq \pi$.
(d) $f(t) = t(t^2 - 4)$, $-2 \leq t \leq 2$.

(e) $f(t) = \begin{cases} -1 & \text{if } -\frac{T}{2} < t < \frac{-T}{4} \\ 1 & \text{if } -\frac{T}{4} < t < \frac{T}{4} \\ -1 & \text{if } \frac{T}{4} < t < \frac{T}{2} \end{cases}$

(f) $f(x) = \begin{cases} -x & \text{if } -2 < t < 0 \\ \frac{1}{2} & \text{if } 0 < x < 2 \end{cases}$

(g) $f(x) = \begin{cases} 0 & \text{if } -1 < t < -\frac{1}{2} \\ 1 & \text{if } -\frac{1}{2} < x < \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} < x < 1 \end{cases}$

(h) $f(x) = \begin{cases} 2 & \text{if } -2 < t < -1 \\ 1 & \text{if } -1 < x < 1 \\ -1 & \text{if } 1 < x < 2 \end{cases}$

Worksheet N$^\circ$ 5

Topics:	Numerical Integration

Exercise 1. Evaluate $I = \displaystyle\int_0^1 e^{-x^2}\, dx$ using

(a) one-segment Trapezoidal rule
(b) two-segments Trapezoidal rule
(c) five-segments Trapezoidal rule

Exercise 2. (HW) Use the Trapezoidal rule with $n = 5$ subintervals to approximate $\displaystyle\int_1^2 \frac{1}{x}\, dx$

Exercise 3. (HW) Use the Trapezoidal rule with $n = 10$ subintervals to approximate $\displaystyle\int_0^1 e^{x^2}\, dx$

Exercise 4. (HW) Use the Trapezoidal rule with $n = 10$ subintervals to approximate $\displaystyle\int_0^1 \sqrt{1 + x^3}\, dx$

Exercise 5. Use the Trapezoidal rule with $n = 6$ subintervals to approximate $\displaystyle\int_0^\pi x \sin x\, dx$

Exercise 6. Use the Trapezoidal rule with $n = 10$ subintervals to approximate $\displaystyle\int_1^6 2 + \sin(2\sqrt{x})\, dx$

Exercise 7. Some values of a function $y = f(x)$ are tabulated as follows:

x	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
$y = f(x)$	1.23	1.58	2.03	4.32	6.25	8.38	10.23	12.45

Evaluate $\displaystyle\int_{0.6}^2 f(x)\, dx$ using Trapezoidal rule.

Exercise 8. Calculate the area bounded by the curve $y = f(x)$ and the x-axis from $x = 7.47$ to $x = 7.52$, where $f(x)$ is tabulated as follows:

x	7.47	7.48	7.49	7.50	7.51	7.52
$y = f(x)$	1.93	1.95	1.98	2.01	2.03	2.06

Exercise 9. A rocket is launched from the ground. Its acceleration measured every 5 seconds is tabulated below.

t	0	5	10	15	10	25	30	35	40
$a(t)$	40.0	45.25	48.50	51.25	54.35	59.48	61.5	64.3	68.7

Find velocity of the rocket at $t = 40$ seconds using Trapezoidal rule.

Exercise 10. Use Simpson's $\frac{1}{3}$ rule with $n = 2$ subintervals to estimate $\displaystyle\int_2^3 \frac{x}{x^5+2}\,dx$.

Exercise 11. Use Simpson's $\frac{1}{3}$ rule with $n = 6$ subintervals to estimate $\displaystyle\int_1^4 \sqrt{1+x^3}\,dx$.

Exercise 12. (HW) Use Simpson's $\frac{1}{3}$ rule with $n = 8$ subintervals to estimate $\displaystyle\int_0^2 e^{-x^2}\,dx$.

Exercise 13. (HW) Use Simpson's $\frac{1}{3}$ rule with $n = 10$ subintervals to estimate $\displaystyle\int_0^1 \sqrt{1+x^3}\,dx$.

Exercise 14. The table below shows the temperature $f(t)$ as a function of time t.

x	1	2	3	4	5	6	7
$y = f(x)$	81	75	80	83	78	70	60

Use Simpson's $\frac{1}{3}$ rule to estimate $\displaystyle\int_1^7 f(t)\,dt$.

Exercise 15. Use Simpson's $\frac{3}{8}$ rule with $n = 3$ and $n = 6$ subintervals respectively to evaluate $\displaystyle\int_0^1 \sqrt{1+x^4}\,dx$.

Exercise 16. Some values of a function $y = f(x)$ are tabulated as follows:

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7
$y = f(x)$	1.543	1.669	1.811	1.971	2.151	2.352	2.577	2.828

Approximate $\displaystyle\int_1^{1.7} f(x)\,dx$ by

(a) Trapezoidal rule

(b) combining Simpson's $\frac{1}{3}$ and Simpson's $\frac{3}{8}$ rules.

Exercise 17. (HW) Evaluate $I = \displaystyle\int_0^{1.2} (x^3 - x + 1)\,dx$ by

(a) exact integration

(b) Trapezoidal rule with $n = 12$ subintervals.

(c) Simpson's $\frac{1}{3}$ rule with $n = 12$ subintervals.

(d) Simpson's $\frac{3}{8}$ rule with $n = 12$ subintervals.

Exercise 18. Approximate

$$\int_0^1 e^{-x^2} dx,$$

using

(a) n-segment Trapezoidal rule with $n = 1$, $n = 2$, $n = 4$ and $n = 8$ respectively.

(b) Richardson's extrapolation using 1, 2, 4, 8 segments.

(c) Romberg integration.

Exercise 19. Let

$$I = \int_{-0.5}^{0} x \ln(x+1)\, dx.$$

Use Romberg integration to approximate I. Use the 1, 2, 4 and 8-segment Trapezoidal rule.

Exercise 20. (HW) Construct the Romberg integration table for the integral $\displaystyle\int_{-1}^{1} \sin(1 + e^{3x})\, dx$. Use the 1, 2, 4 and 8-segment Trapezoidal rule.

Exercise 21. Construct the Romberg integration table for the integral $\displaystyle\int_1^2 \frac{1}{x^2}\, dx$. Use the 1, 2, 4 and 8-segment Trapezoidal rule.

Exercise 22. Employ two- through six-point Gauss-Legendre formulas to obtain estimations for

$$I = \int_{-3}^{3} \frac{1}{1+x^2}\, dx.$$

Worksheet N$^\circ$ 6

Topics:	Solving differential equations numerically

Exercise 1. Rewrite each of the following equations in the form $y' = f(x, y)$.

(a) $y' + y = x$
(b) $2y' + y = t$
(c) $xy' - 3y = x^2$
(d) $y' - y^2 = x$
(e) $y' + 2y = x^3e^{-2x}$
(f) $y' + 2xy = 1$
(g) $(1 + y^2)(y' - 2y) = x$
(h) $y' + 3x^2y = 1 + y^2$
(i) $y' + 2y^2 - xy = x^2$

Exercise 2. Consider the initial value problem:

$$y' + y = x, \quad y(0) = 1.$$

Use Euler's method to approximate $y(0.2)$, $y(0.4)$, $y(0.6)$, $y(0.8)$, $y(1.0)$.

Exercise 3. Consider the initial value problem:

$$y' = xy, \quad y(0) = 1.$$

Use Euler's method with step size $h = 0.1$ to approximate $y(0.5)$.

Exercise 4. (HW) Consider the initial value problem:

$$y' + y = 1, \quad y(0) = 0.$$

Use Euler's method with step size $h = 0.1$ to approximate $y(0.4)$.

Exercise 5. Consider the initial value problem:

$$y' - y + x = 0, \quad y(0) = \frac{1}{2}.$$

Approximate $y(0.3)$ using Euler's method with step size:

(a) 0.1
(b) 0.05

Exercise 6. Consider the initial value problem:

$$y' = xy, \quad y(0) = 1.$$

Use improved Euler's method (Heun's method) with step size $h = 0.1$ to approximate $y(0.5)$.

Exercise 7. Consider the initial value problem:

$$y' - y + x = 0, \quad y(0) = \frac{1}{2}.$$

Approximate $y(0.2)$ using Heun's method with step size $h = 0.1$.

Exercise 8. Consider the initial value problem:

$$y' - y + x = 0, \quad y(0) = \frac{1}{2}.$$

Approximate $y(0.2)$ using the fourth order Runge-Kutta method with step size $h = 0.1$