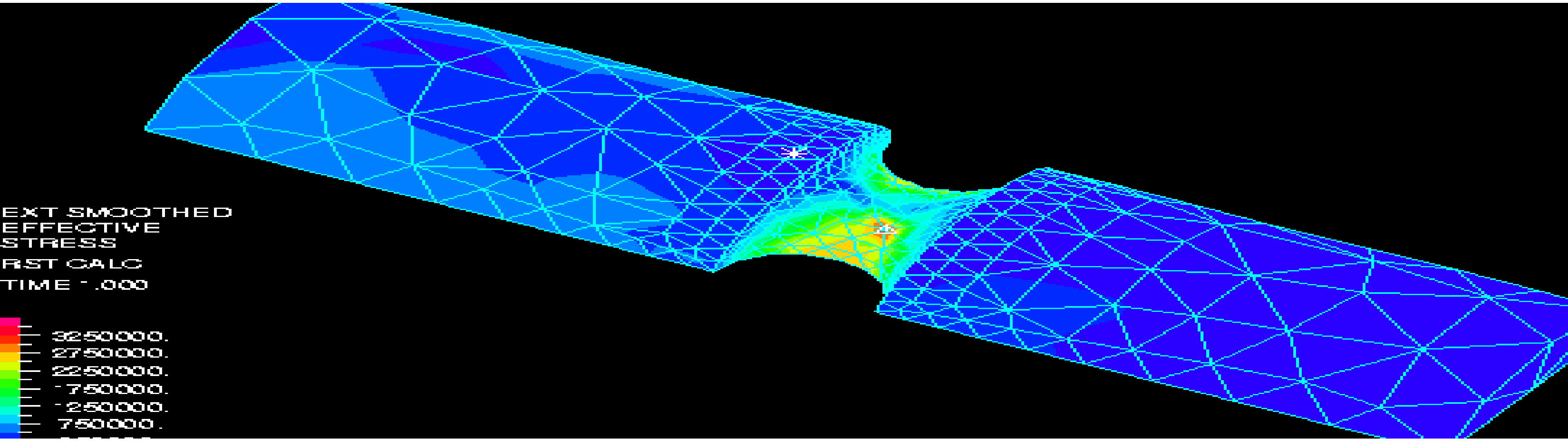
جامعة فهد بن سلطان
Fahad Bin Sultan University

COLLEGE OF ENGINEERING
DEPARTMENT OF CIVIL ENGINEERING



CIVE 517 : Finite Element Analysis

Shape functions in 1D

Prof. Khaldoon Bani-Hani

Fall, 2021

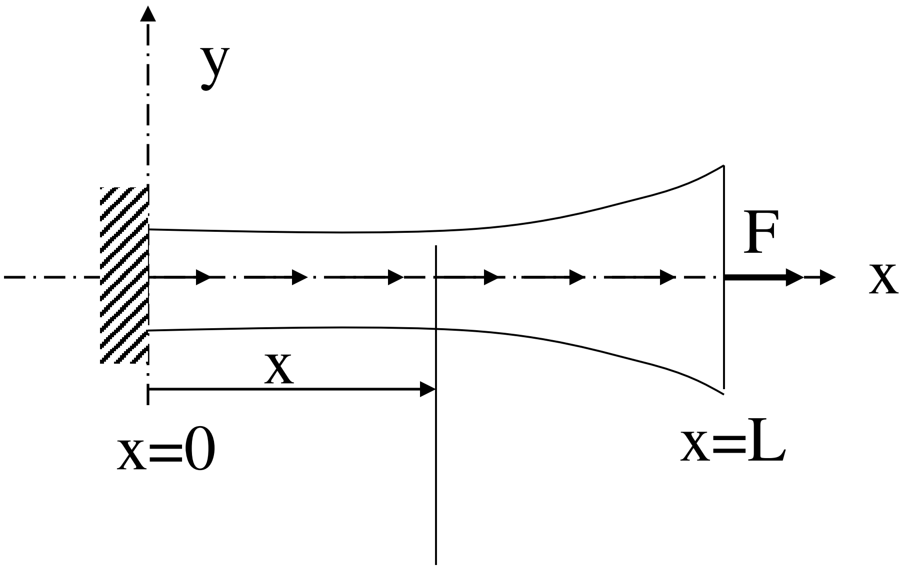
Reading assignment:

Lecture notes, Logan 2.2, 3.1

Summary:

- **Linear shape functions in 1D**
- **Quadratic and higher order shape functions**
- **Approximation of strains and stresses in an element**

Axially loaded elastic bar



A(x) = cross section at x
b(x) = body force distribution (force per unit length)
E(x) = Young's modulus

Potential energy of the axially loaded bar corresponding to the exact solution u(x)

$$\boxed{\Pi(u) = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx - \int_0^L bu\ dx - Fu(x = L)}$$

Finite element formulation, takes as its starting point, not the strong formulation, but the **Principle of Minimum Potential Energy**.

Task is to find the function 'w' that minimizes the potential energy of the system

$$\boxed{\Pi(\mathrm{w}) = \frac{1}{2}\int_0^L \mathrm{EA}\left(\frac{\mathrm{dw}}{\mathrm{dx}}\right)^2 \mathrm{dx} - \int_0^L \mathrm{bw}\ \mathrm{dx} - \mathrm{Fw}(\mathrm{x} = \mathrm{L})}$$

From the Principle of Minimum Potential Energy, that function 'w' is the exact solution.

Rayleigh-Ritz Principle

Step 1. Assume a solution

$$w(x) = a_0\varphi_o(x) + a_1\varphi_1(x) + a_2\varphi_2(x) + \ldots$$

Where $\varphi_o(x)$, $\varphi_1(x)$,… are "admissible" functions and a_o, a_1, etc are constants to be determined.

Step 2. Plug the approximate solution into the potential energy

$$\Pi(\mathrm{w}) = \frac{1}{2}\int_0^L \mathrm{EA}\left(\frac{\mathrm{dw}}{\mathrm{dx}}\right)^2 \mathrm{dx} - \int_0^L \mathrm{bw}\ \mathrm{dx} - \mathrm{Fw}(\mathrm{x} = \mathrm{L})$$

Step 3. Obtain the coefficients a_o, a_1, etc by setting

$$\frac{\partial \Pi(\mathrm{w})}{\partial a_i} = 0, \quad i = 0,1,2,\ldots$$

The approximate solution is

$$u(x) = a_0\varphi_o(x) + a_1\varphi_1(x) + a_2\varphi_2(x) + \ldots$$

Where the coefficients have been obtained from step 3

Need to find a systematic way of choosing the approximation functions.

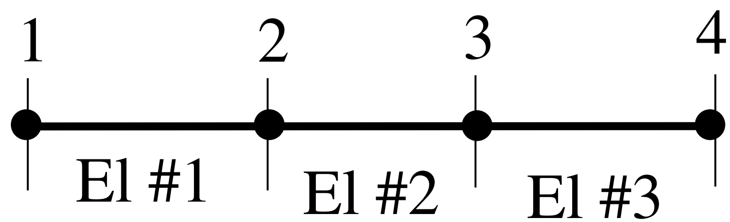
One idea: Choose polynomials!

$w(x) = a_0$ Is this good? (Is ‘1’ an “admissible” function?)

$w(x) = a_1 x$ Is this good? (Is ‘x’ an “admissible” function?)

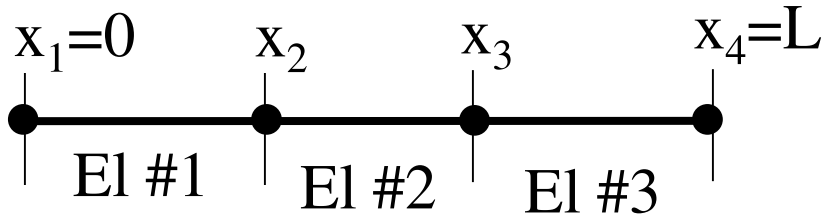
Finite element idea:

Step 1: Divide the truss into **finite elements** connected to each other through special points ("**nodes**")



Total potential energy=sum of potential energies of the elements

$$\Pi(w) = \frac{1}{2}\int_0^L EA\left(\frac{dw}{dx}\right)^2 dx - \int_0^L bw\ dx - Fw(x = L)$$



Total potential energy

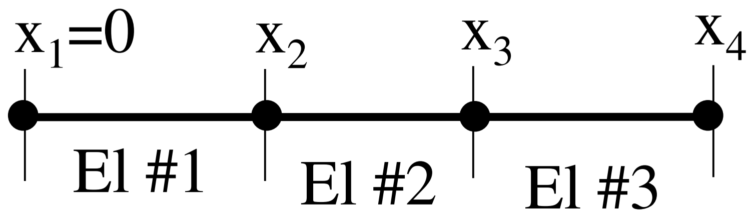
$$\Pi(w) = \frac{1}{2}\int_0^L EA\left(\frac{dw}{dx}\right)^2 dx - \int_0^L bw\ dx - Fw(x = L)$$

Potential energy of element 1:

$$\Pi_1(w) = \frac{1}{2}\int_{x_1}^{x_2} EA\left(\frac{dw}{dx}\right)^2 dx - \int_{x_1}^{x_2} bw\ dx$$

Potential energy of element 2:

$$\Pi_2(w) = \frac{1}{2}\int_{x_2}^{x_3} EA\left(\frac{dw}{dx}\right)^2 dx - \int_{x_2}^{x_3} bw\ dx$$



Potential energy of element 3:

$$\Pi_3(w) = \frac{1}{2}\int_{x_3}^{x_4} EA\left(\frac{dw}{dx}\right)^2 dx - \int_{x_3}^{x_4} bw\ dx - Fw(x = L)$$

Total potential energy=sum of potential energies of the elements

$$\Pi(w) = \Pi_1(w) + \Pi_2(w) + \Pi_3(w)$$

Step 2: Describe the behavior of each element

Recall that in the "**direct stiffness**" approach for a bar element, we derived the stiffness matrix of each element directly (See lecture on Trusses) using the following steps:

TASK 1: Approximate the displacement within each bar as a straight line

TASK 2: Approximate the strains and stresses and realize that a bar (with the approximation stated in Task 1) is exactly like a spring with k=EA/L

TASK 3: Use the principle of **force equilibrium** to generate the stiffness matrix

Now, we will show you a systematic way of deriving the stiffness matrix (sections 2.2 and 3.1 of Logan).

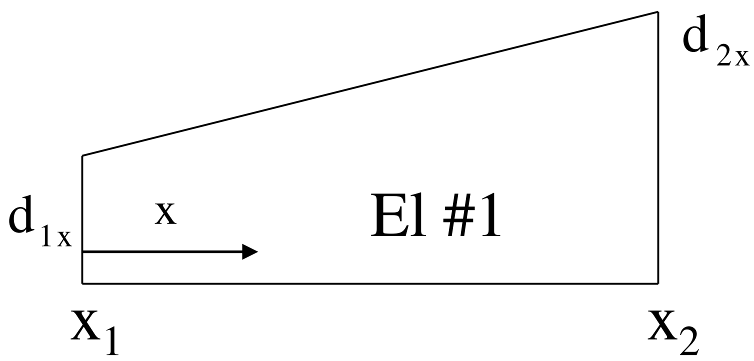
TASK 1: APPROXIMATE THE DISPLACEMENT WITHIN EACH ELEMENT
TASK 2: APPROXIMATE THE STRAIN and STRESS WITHIN EACH ELEMENT
TASK 3: DERIVE THE STIFFNESS MATRIX OF EACH ELEMENT (next class) USING THE PRINCIPLE OF MIN. POT ENERGY

Notice that the first two tasks are similar in the two methods. The only difference is that now we are going to use the principle of minimum potential energy, rather than force equilibrium, to derive the stiffness matrix.

TASK 1: APPROXIMATE THE DISPLACEMENT WITHIN EACH ELEMENT

Simplest assumption: displacement varying linearly inside each bar

$$w(x) = a_0 + a_1 x$$



How to obtain a_0 and a_1*?*

$$w(x_1) = a_0 + a_1 x_1 = d_{1x}$$
$$w(x_2) = a_0 + a_1 x_2 = d_{2x}$$

$$w(x_1) = a_0 + a_1 x_1 = d_{1x}$$
$$w(x_2) = a_0 + a_1 x_2 = d_{2x}$$

Solve simultaneously

$$a_0 = \frac{x_2}{x_2 - x_1} d_{1x} - \frac{x_1}{x_2 - x_1} d_{2x}$$
$$a_1 = -\frac{1}{x_2 - x_1} d_{1x} + \frac{1}{x_2 - x_1} d_{2x}$$

Hence

$$w(x) = a_0 + a_1 x = \underbrace{\frac{x_2 - x}{x_2 - x_1}}_{N_1(x)} d_{1x} + \underbrace{\frac{x - x_1}{x_2 - x_1}}_{N_2(x)} d_{2x} = N_1(x) d_{1x} + N_2(x) d_{2x}$$

"**Shape functions**" $N_1(x)$ and $N_2(x)$

In matrix notation, we write

$$\boxed{w(x) = \underline{N}\ \underline{d}} \qquad (1)$$

Vector of nodal shape functions

$$\underline{N} = \left[N_1(x) \quad N_2(x) \right] = \left[\frac{x_2 - x}{x_2 - x_1} \quad \frac{x - x_1}{x_2 - x_1} \right]$$

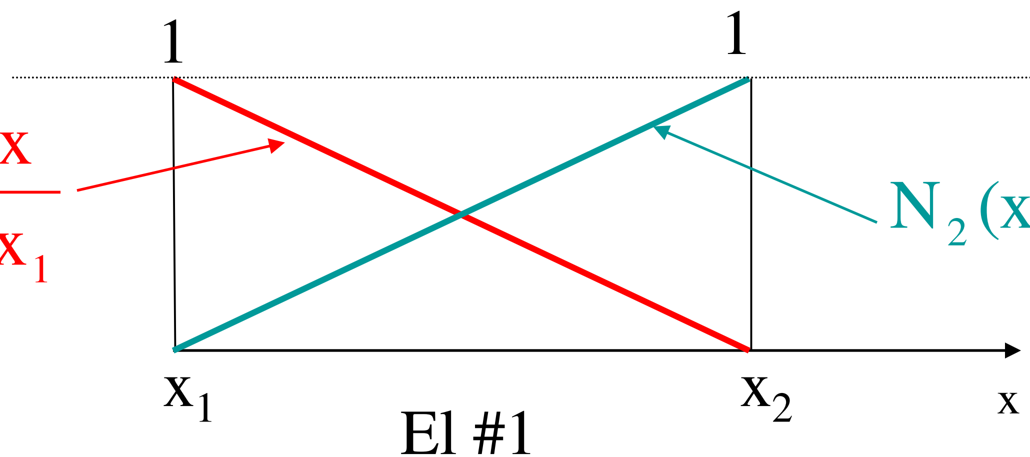
Vector of nodal displacements

$$\underline{d} = \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix}$$

NOTES: PROPERTIES OF THE SHAPE FUNCTIONS

1. <u>Kronecker delta property</u>: The shape function at any node has a value of 1 at that node and a value of zero at ALL other nodes.

$$N_1(x) = \frac{x_2 - x}{x_2 - x_1}$$



$$N_2(x) = \frac{x - x_1}{x_2 - x_1}$$

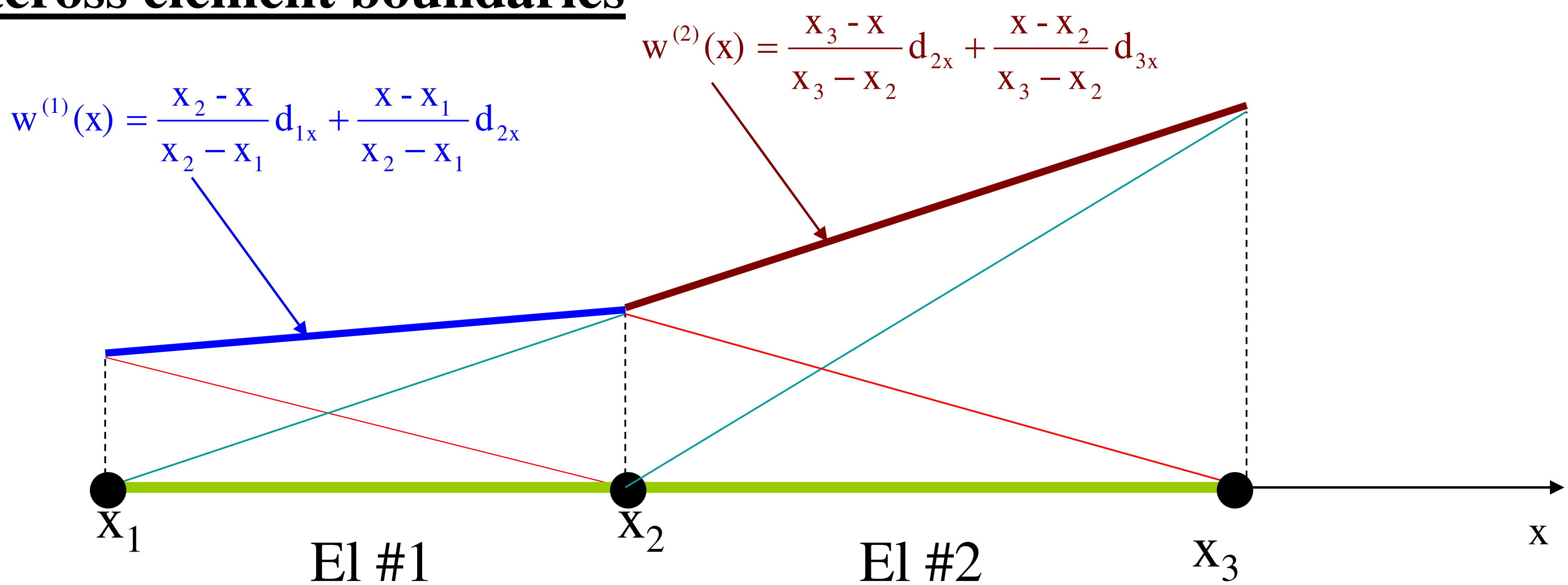
Check

$$N_1(x) = \frac{x_2 - x}{x_2 - x_1}$$

$$\Rightarrow N_1(x = x_1) = \frac{x_2 - x_1}{x_2 - x_1} = 1$$

$$\text{and } N_1(x = x_2) = \frac{x_2 - x_2}{x_2 - x_1} = 0$$

2. **<u>Compatibility</u>**: The displacement approximation is **<u>continuous across element boundaries</u>**



At $x=x_2$

$$w^{(1)}(x = x_2) = \frac{x_2 - x_2}{x_2 - x_1} d_{1x} + \frac{x_2 - x_1}{x_2 - x_1} d_{2x} = d_{2x}$$

$$w^{(2)}(x = x_2) = \frac{x_3 - x_2}{x_3 - x_2} d_{2x} + \frac{x_2 - x_2}{x_3 - x_2} d_{3x} = d_{2x}$$

Hence the displacement approximation is continuous across elements

3. Completeness

$$N_1(x) + N_2(x) = 1 \quad \text{for all } x$$
$$N_1(x)x_1 + N_2(x)x_2 = x \quad \text{for all } x$$

Use the expressions $N_1(x) = \dfrac{x_2 - x}{x_2 - x_1};$

$$N_2(x) = \frac{x - x_1}{x_2 - x_1}$$

And check

$$N_1(x) + N_2(x) = \frac{x_2 - x}{x_2 - x_1} + \frac{x - x_1}{x_2 - x_1} = 1$$

and $N_1(x)\, x_1 + N_2(x)\, x_2 = \dfrac{x_2 - x}{x_2 - x_1} x_1 + \dfrac{x - x_1}{x_2 - x_1} x_2 = x$

Rigid body mode

$$\boxed{N_1(x) + N_2(x) = 1 \qquad \text{for all } x}$$

What do we mean by "**rigid body modes**"?

Assume that $d_{1x}=d_{2x}=1$, this means that the element should translate in the positive x direction by 1. Hence ***ANY point*** (x) on the bar should have unit displacement. Let us see whether the displacement approximation allows this.

$$w(x) = N_1(x)d_{1x} + N_2(x)d_{2x} = N_1(x) + N_2(x) = 1$$

YES!

Constant strain states

$$\boxed{N_1(x)x_1 + N_2(x)x_2 = x \quad \text{at all } x}$$

What do we mean by "**constant strain states**"?

Assume that $d_{1x}=x_1$ and $d_{2x}=x_2$. The strain at ***ANY point*** (x) within the bar is

$$\varepsilon(x) = \frac{d_{2x} - d_{1x}}{x_2 - x_1} = \frac{x_2 - x_1}{x_2 - x_1} = 1$$

Let us see whether the displacement approximation allows this.

$$w(x) = N_1(x)d_{1x} + N_2(x)d_{2x} = N_1(x)x_1 + N_2(x)x_2 = x$$

$$\text{Hence, } \varepsilon(x) = \frac{dw(x)}{dx} = 1$$

YES!

Completeness = Rigid body modes + Constant Strain states

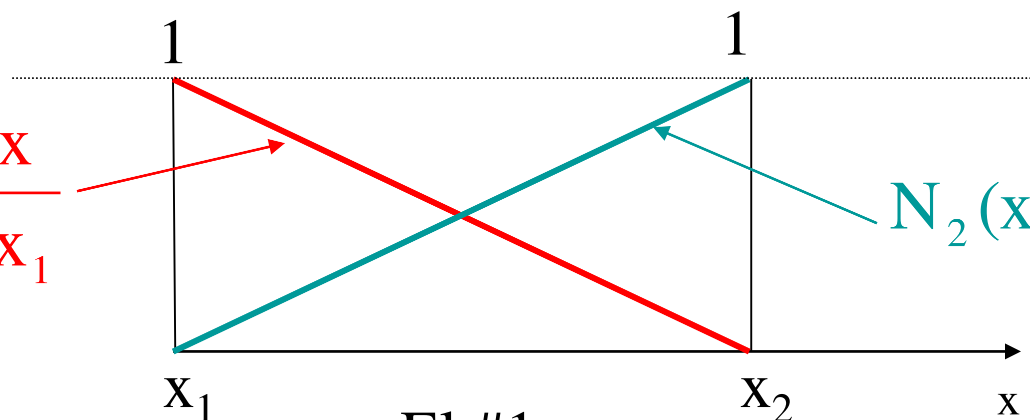
Compatibility + Completeness ⇒ Convergence

Ensure that the solution gets better as more elements are introduced and, in the limit, approaches the exact answer.

4. How to write the expressions for the shape functions easily (without having to derive them each time):

Start with the **Kronecker delta property** (the shape function at any node has value of 1 at that node and a value of zero at all other nodes)

$$N_1(x) = \frac{x_2 - x}{x_2 - x_1}$$



$$N_2(x) = \frac{x - x_1}{x_2 - x_1}$$

El #1

Node at which N_1 is 0

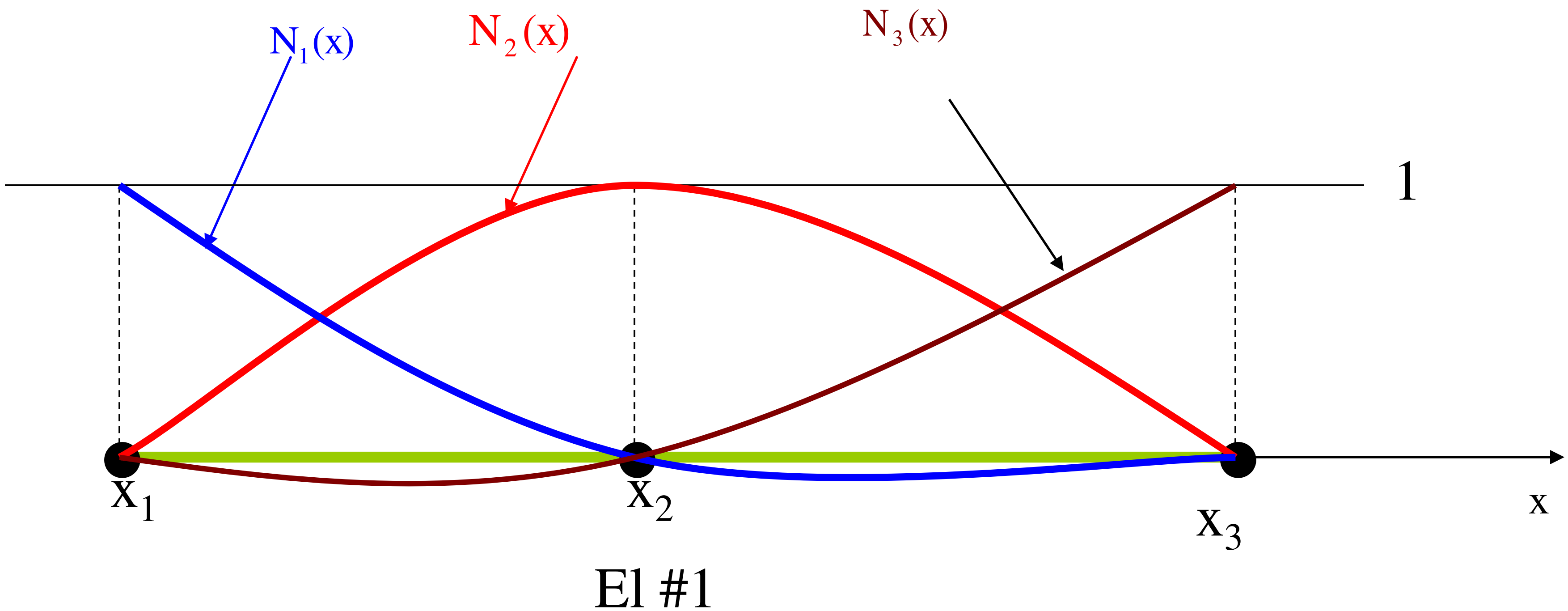
$$N_1(x) = \frac{(x_2 - x)}{(x_2 - x_1)}$$

Notice that the length of the element = x_2-x_1

$$N_2(x) = \frac{(x_1 - x)}{(x_1 - x_2)} = \frac{(x - x_1)}{(x_2 - x_1)}$$

The denominator is the numerator evaluated at the node itself

A slightly fancier assumption:
displacement varying **quadratically** inside each bar



$$N_1(x) = \frac{(x_2 - x)(x_3 - x)}{(x_2 - x_1)(x_3 - x_1)}$$

$$N_2(x) = \frac{(x_1 - x)(x_3 - x)}{(x_1 - x_2)(x_3 - x_2)}$$

$$N_3(x) = \frac{(x_1 - x)(x_2 - x)}{(x_1 - x_3)(x_2 - x_3)}$$

$$\boxed{w(x) = N_1(x)d_{1x} + N_2(x)d_{2x} + N_3(x)d_{3x}}$$

This is a **quadratic finite element** in 1D and it has three nodes and three associated shape functions per element.

TASK 2: APPROXIMATE THE STRAIN and STRESS WITHIN EACH ELEMENT

From equation (1), the displacement within each element

$$w(x) = \underline{N}\ \underline{d}$$

Recall that the **strain** in the bar $\varepsilon = \dfrac{dw}{dx}$

Hence

$$\boxed{\varepsilon = \left[\frac{d\underline{N}}{dx}\right]\underline{d} = \underline{B}\ \underline{d}} \qquad (2)$$

The matrix $\underline{B}$ is known as the "**strain-displacement matrix**"

$$\underline{B} = \left[\frac{d\underline{N}}{dx}\right]$$

For a linear finite element

$$\underline{N} = \left[N_1(x) \quad N_2(x) \right] = \left[\frac{x_2 - x}{x_2 - x_1} \quad \frac{x - x_1}{x_2 - x_1} \right]$$

Hence

$$\underline{B} = \left[\frac{-1}{x_2 - x_1} \quad \frac{1}{x_2 - x_1} \right] = \frac{1}{x_2 - x_1} \left[-1 \quad 1 \right]$$

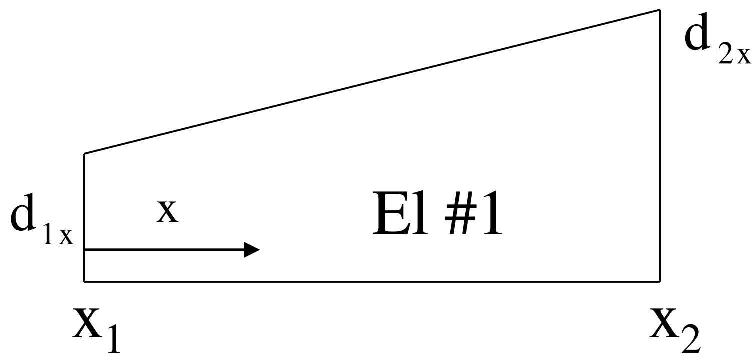
$$\varepsilon = \underline{B}\ \underline{d} = \left[\frac{-1}{x_2 - x_1} \quad \frac{1}{x_2 - x_1} \right] \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix}$$

$$= \frac{d_{2x} - d_{1x}}{x_2 - x_1}$$

Hence, strain is a **constant** within each element (only for a linear element)!

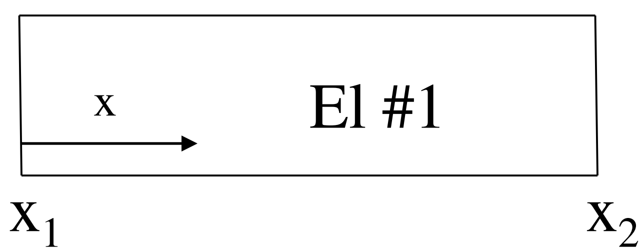
Displacement is linear

$$w(x) = a_0 + a_1 x$$



Strain is constant

$$\varepsilon = \frac{d_{2x} - d_{1x}}{x_2 - x_1}$$



Recall that the **stress** in the bar $\sigma = E\varepsilon = E\dfrac{du}{dx}$

Hence, inside the element, the approximate stress is

$$\boxed{\sigma = E\underline{B}\,\underline{d}} \qquad (3)$$

For a linear element the stress is also constant inside each element. This has the implication that the stress (and strain) is **discontinuous across element boundaries** in general.

Summary

Inside an element, the three most important approximations **in terms of the nodal displacements** ($\underline{d}$) are:

Displacement approximation in terms of shape functions

$$\boxed{u(x) = \underline{N}\ \underline{d}} \tag{1}$$

Strain approximation in terms of strain-displacement matrix

$$\boxed{\varepsilon(x) = \underline{B}\ \underline{d}} \tag{2}$$

Stress approximation in terms of strain-displacement matrix and Young's modulus

$$\boxed{\sigma = E\underline{B}\ \underline{d}} \tag{3}$$

Summary

For a **linear element**

Displacement approximation in terms of shape functions

$$u(x) = \begin{bmatrix} \dfrac{x_2 - x}{x_2 - x_1} & \dfrac{x - x_1}{x_2 - x_1} \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix}$$

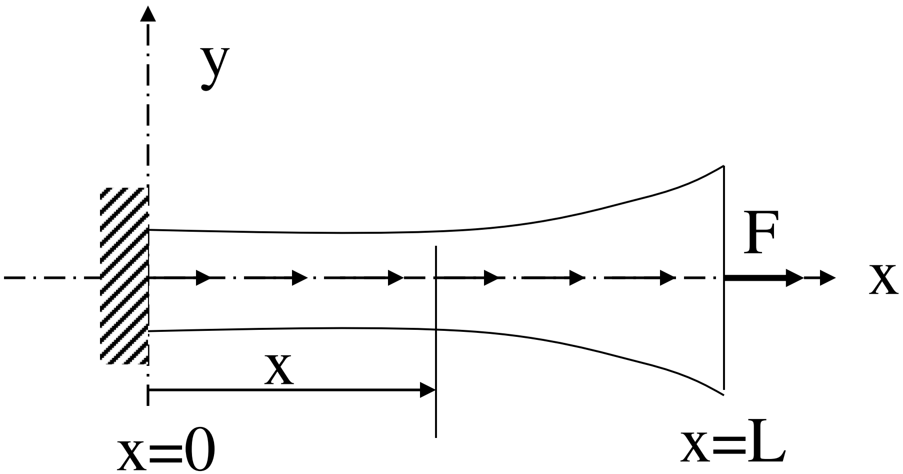
Strain approximation

$$\varepsilon = \frac{1}{x_2 - x_1} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix}$$

Stress approximation

$$\sigma = \frac{E}{x_2 - x_1} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix}$$

Axially loaded elastic bar



A(x) = cross section at x
b(x) = body force distribution (force per unit length)
E(x) = Young's modulus

Potential energy of the axially loaded bar corresponding to the exact solution u(x)

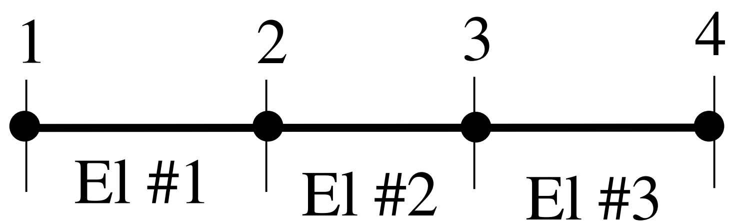
$$\Pi(u) = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx - \int_0^L bu\ dx - Fu(x = L)$$

Potential energy of the bar corresponding to an **admissible displacement** w(x)

$$\Pi(w) = \frac{1}{2}\int_0^L EA\left(\frac{dw}{dx}\right)^2 dx - \int_0^L bw\ dx - Fw(x = L)$$

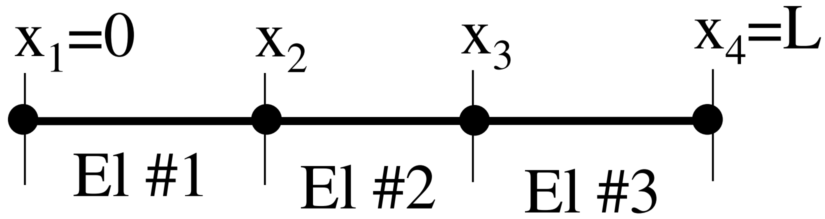
Finite element idea:

Step 1: Divide the truss into **finite elements** connected to each other through special points ("**nodes**")



Total potential energy=sum of potential energies of the elements

$$\Pi(\mathrm{w}) = \frac{1}{2}\int_0^L \mathrm{EA}\left(\frac{\mathrm{dw}}{\mathrm{dx}}\right)^2 \mathrm{dx} - \int_0^L \mathrm{bw}\ \mathrm{dx} - \mathrm{Fw}(\mathrm{x} = \mathrm{L})$$



Total potential energy

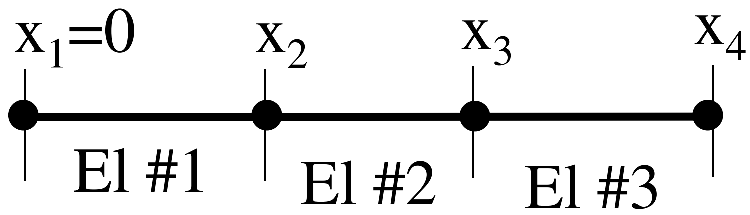
$$\Pi(w) = \frac{1}{2}\int_0^L EA\left(\frac{dw}{dx}\right)^2 dx - \int_0^L bw\ dx - Fw(x = L)$$

Potential energy of element 1:

$$\Pi_1(w) = \frac{1}{2}\int_{x_1}^{x_2} EA\left(\frac{dw}{dx}\right)^2 dx - \int_{x_1}^{x_2} bw\ dx$$

Potential energy of element 2:

$$\Pi_2(w) = \frac{1}{2}\int_{x_2}^{x_3} EA\left(\frac{dw}{dx}\right)^2 dx - \int_{x_2}^{x_3} bw\ dx$$



Potential energy of element 3:

$$\Pi_3(w) = \frac{1}{2}\int_{x_3}^{x_4} EA\left(\frac{dw}{dx}\right)^2 dx - \int_{x_3}^{x_4} bw\ dx - Fw(x = L)$$

Total potential energy=sum of potential energies of the elements

$$\Pi(w) = \Pi_1(w) + \Pi_2(w) + \Pi_3(w)$$

Step 2: Describe the behavior of each element

In the "**direct stiffness**" approach, we derived the stiffness matrix of each element directly (See lecture on Springs/Trusses).

Now, we will first approximate the **displacement** inside each element and then show you a systematic way of deriving the stiffness matrix (sections 2.2 and 3.1 of Logan).

TASK 1: APPROXIMATE THE DISPLACEMENT WITHIN EACH ELEMENT
TASK 2: APPROXIMATE THE STRAIN and STRESS WITHIN EACH ELEMENT

TASK 3: DERIVE THE STIFFNESS MATRIX OF EACH ELEMENT (this class) USING THE RAYLEIGH-RITZ PRINCIPLE

Summary

Inside an element, the three most important approximations **in terms of the nodal displacements** ($\underline{d}$) are:

Displacement approximation in terms of shape functions

$$\boxed{w(x) = \underline{N}\ \underline{d}} \qquad (1)$$

Strain approximation in terms of strain-displacement matrix

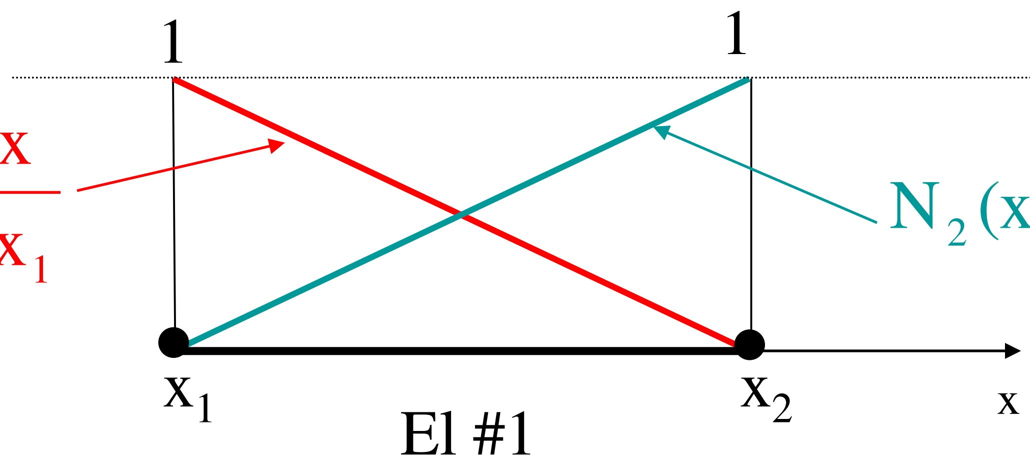
$$\boxed{\varepsilon(x) = \underline{B}\ \underline{d}} \qquad (2)$$

Stress approximation in terms of strain-displacement matrix and Young's modulus

$$\boxed{\sigma = E\underline{B}\ \underline{d}} \qquad (3)$$

The shape functions for a **1D linear element**

$$N_1(x) = \frac{x_2 - x}{x_2 - x_1}$$



$$N_2(x) = \frac{x - x_1}{x_2 - x_1}$$

Within the element, the displacement approximation is

$$w(x) = \frac{x_2 - x}{x_2 - x_1} d_{1x} + \frac{x - x_1}{x_2 - x_1} d_{2x}$$

For a **linear element**

Displacement approximation in terms of shape functions

$$\boxed{w(x) = \begin{bmatrix} \dfrac{x_2 - x}{x_2 - x_1} & \dfrac{x - x_1}{x_2 - x_1} \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix}}$$

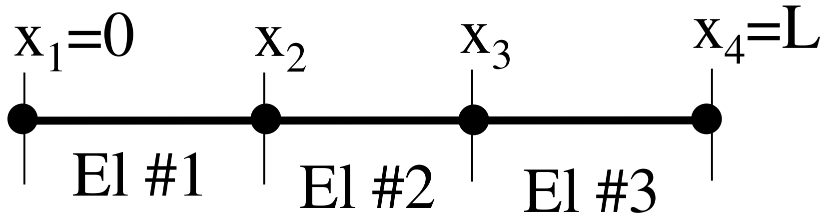
Strain approximation

$$\boxed{\varepsilon = \frac{dw}{dx} = \frac{1}{x_2 - x_1} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix}}$$

Stress approximation

$$\boxed{\sigma = E\varepsilon = \frac{E}{x_2 - x_1} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix}}$$

Why is the approximation “admissible”?



For the entire bar, the displacement approximation is

$$w(x) = w^{(1)}(x) + w^{(2)}(x) + w^{(3)}(x)$$

Where $w^{(i)}(x)$ is the displacement approximation within element (i). Let use set d_{1x}=0. Then, can you seen that the above approximation does satisfy the two conditions of being an admissible function on the entire bar, i.e.,

$$(1)\ w(x = 0) = 0$$

$$(2)\ \frac{dw}{dx} \ \text{ exists}$$

TASK 3: DERIVE THE STIFFNESS MATRIX OF EACH ELEMENT USING THE RAYLEIGH-RITZ PRINCIPLE

Potential energy of element 1:

$$\Pi_1(\mathrm{w}) = \frac{1}{2}\int_{x_1}^{x_2} \sigma \;\; \varepsilon \;\; A\mathrm{dx} - \int_{x_1}^{x_2} \mathrm{bw} \;\; \mathrm{dx}$$

Lets plug in the **approximation**

$\boxed{\mathrm{w(x)} = \underline{\mathrm{N}}\ \underline{\mathrm{d}}}$ $\boxed{\varepsilon\mathrm{(x)} = \underline{\mathrm{B}}\ \underline{\mathrm{d}}}$ $\boxed{\sigma = \mathrm{E}\underline{\mathrm{B}}\ \underline{\mathrm{d}}}$

$$\Pi_1(\underline{\mathrm{d}}) = \frac{1}{2}\underline{\mathrm{d}}^T\left(\int_{x_1}^{x_2} \underline{\mathrm{B}}^{\mathrm{T}}\mathrm{E}\underline{\mathrm{B}} \;\; A\mathrm{dx}\right)\underline{\mathrm{d}} - \underline{\mathrm{d}}^T\left(\int_{x_1}^{x_2} \underline{\mathrm{N}}^T \;\; \mathrm{b} \;\; \mathrm{dx}\right)$$

Lets see what the matrix

$$\int_{x_1}^{x_2} \underline{\mathrm{B}}^{\mathrm{T}} \mathrm{E} \underline{\mathrm{B}} \ A \mathrm{dx}$$

is for a 1D linear element

Recall that

$$\boxed{\underline{\mathrm{B}} = \frac{1}{\mathrm{x}_2 - \mathrm{x}_1} \begin{bmatrix} -1 & 1 \end{bmatrix}}$$

Hence

$$\underline{\mathrm{B}}^T \ \mathrm{E} \ \underline{\mathrm{B}} = \frac{1}{\mathrm{x}_2 - \mathrm{x}_1} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \mathrm{E} \ \frac{1}{\mathrm{x}_2 - \mathrm{x}_1} \begin{bmatrix} -1 & 1 \end{bmatrix}$$

$$= \frac{\mathrm{E}}{(\mathrm{x}_2 - \mathrm{x}_1)^2} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} = \frac{\mathrm{E}}{(\mathrm{x}_2 - \mathrm{x}_1)^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\int_{x_1}^{x_2} \underline{B}^T E \underline{B} \; A dx = \frac{1}{(x_2 - x_1)^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \int_{x_1}^{x_2} AEdx = \left(\int_{x_1}^{x_2} AEdx \right) \frac{1}{(x_2 - x_1)^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Now, if we **assume** E and A are constant

$$\int_{x_1}^{x_2} \underline{B}^T E \underline{B} \; A dx = \left(\int_{x_1}^{x_2} AEdx \right) \frac{1}{(x_2 - x_1)^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{AE(x_2 - x_1)}{(x_2 - x_1)^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{AE}{(x_2 - x_1)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Remembering that $(x_2 - x_1)$ is the length of the element, this is the **stiffness matrix** we had derived directly before using the **direct stiffness** approach!!

Then why is it necessary to go through this complicated procedure??
1. Easy to handle **nonuniform** E and A
2. Easy to handle **distributed loads**
For nonuniform E and A, i.e. E(x) and A(x), the **stiffness matrix** of the linear element will **NOT** be

$$\frac{EA}{(x_2 - x_1)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

But it will **ALWAYS be**

$$\boxed{\underline{k} = \int_{x_1}^{x_2} \underline{B}^T E \underline{B} \; A dx}$$

Now lets go back to

$$\Pi_1(\underline{d}) = \frac{1}{2}\underline{d}^T \left(\underbrace{\int_{x_1}^{x_2} \underline{B}^T E \underline{B} \; A dx}_{\underline{k}} \right) \underline{d} - \underline{d}^T \left(\underbrace{\int_{x_1}^{x_2} \underline{N}^T \; b \; dx}_{\underline{f_b}} \right)$$

$$= \frac{1}{2}\underline{d}^T \underline{k}\,\underline{d} - \underline{d}^T \underline{f_b}$$

Element **stiffness matrix**

$$\boxed{\underline{k} = \int_{x_1}^{x_2} \underline{B}^T E \underline{B} \; A dx}$$

Element **nodal load vector due to distributed body force**

$$\boxed{\underline{f_b} = \int_{x_1}^{x_2} \underline{N}^T \; b \; dx}$$

Apply Rayleigh-Ritz principle for the 1D linear element

$$\left.\begin{array}{l}\dfrac{\partial \Pi_1(\underline{\mathrm{d}})}{\partial \mathrm{d}_{1x}} = 0 \\[2ex] \dfrac{\partial \Pi_1(\underline{\mathrm{d}})}{\partial \mathrm{d}_{2x}} = 0\end{array}\right\} \Rightarrow \frac{\partial \Pi_1(\underline{\mathrm{d}})}{\partial \underline{\mathrm{d}}} = 0$$

Recall from linear algebra (Lecture notes on Linear Algebra)

$$\Pi_1(\underline{\mathrm{d}}) = \frac{1}{2}\underline{\mathrm{d}}^T \underline{k}\,\underline{\mathrm{d}} - \underline{\mathrm{d}}^T \underline{f_b}$$

$$\Rightarrow \frac{\partial \Pi_1(\underline{\mathrm{d}})}{\partial \underline{\mathrm{d}}} = \underline{k}\,\underline{\mathrm{d}} - \underline{f_b}$$

Hence

$$\frac{\partial \Pi_1(\underline{\mathrm{d}})}{\partial \underline{\mathrm{d}}} = 0$$

$$\boxed{\Rightarrow \underline{k}\,\underline{\mathrm{d}} = \underline{f_b}}$$

Exactly the same equation that we had before, except that the stiffness matrix and nodal force vectors are more general

Recap of the properties of the element stiffness matrix

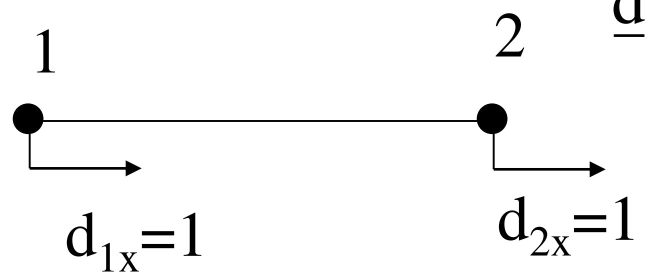
$$\boxed{\underline{k} = \int_{x_1}^{x_2} \underline{\mathrm{B}}^{\mathrm{T}} \mathrm{E} \underline{\mathrm{B}} \; A \mathrm{dx}}$$

1. The stiffness matrix is **singular** and is therefore non-invertible
2. The stiffness matrix is **symmetric**
3. **Sum of any row** (or **column**) of the stiffness matrix is zero! k_{11}

   Why?

Sum of any row (or column) of the stiffness matrix is zero

Consider a **rigid body motion** of the element



$$\underline{d} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Element strain $\varepsilon = 0 = \underline{B}\ \underline{d}$

$$\Rightarrow \underline{k}\ \underline{d} = \left(\int_{x_1}^{x_2} \underline{B}^T E \underline{B}\ A dx \right) \underline{d}$$

$$= \int_{x_1}^{x_2} \underline{B}^T E \left(\underline{B} \underline{d} \right)\ A dx$$

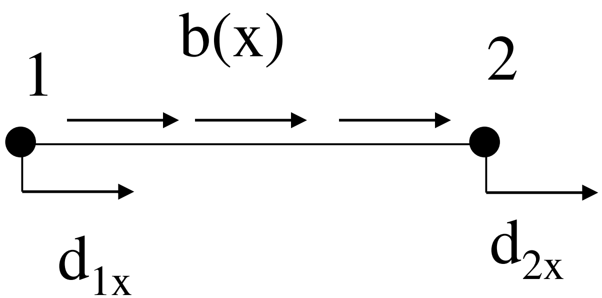
$$= \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\underline{k}\ \underline{d} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow k_{11} + k_{12} = 0 \ \text{ and } \quad k_{21} + k_{22} = 0$$

The nodal load vector

$$\boxed{\underline{f_b} = \int_{x_1}^{x_2} \underline{\mathrm{N}}^T \ \mathrm{b\ dx}}$$

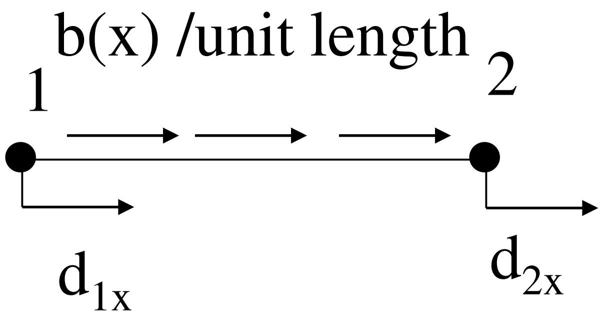


$$\underline{f_b} = \int_{x_1}^{x_2} \underline{\mathrm{N}} \ \mathrm{b\ dx} = \int_{x_1}^{x_2} \begin{Bmatrix} N_1(x) \\ N_2(x) \end{Bmatrix} \mathrm{b\ dx}$$

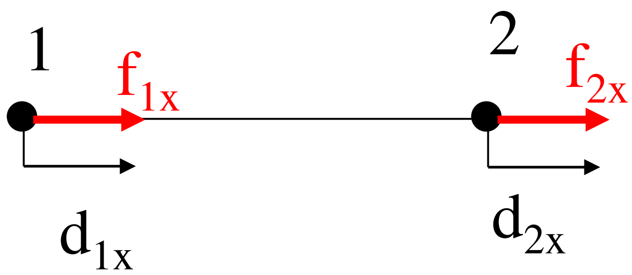
$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{Bmatrix} \int_{x_1}^{x_2} N_1(x) \ \mathrm{b\ dx} \\ \int_{x_1}^{x_2} N_2(x) \ \mathrm{b\ dx} \end{Bmatrix}$$

$$\boxed{\begin{aligned} f_{1x} &= \int_{x_1}^{x_2} N_1(x) \ \mathrm{b\ dx} \\ f_{2x} &= \int_{x_1}^{x_2} N_2(x) \ \mathrm{b\ dx} \end{aligned}}$$

"**Consistent**" nodal loads



Replaced by



A distributed load is represented by two nodal loads in a consistent manner

e.g., if b=1

$$f_{1x} = \int_{x_1}^{x_2} N_1(x)\, \mathrm{b}\, \mathrm{dx} = \int_{x_1}^{x_2} N_1(x)\, \mathrm{dx} = \frac{\mathrm{x}_2 - x_1}{2}$$

$$f_{2x} = \int_{x_1}^{x_2} N_2(x)\, \mathrm{b}\, \mathrm{dx} = \int_{x_1}^{x_2} N_2(x)\, \mathrm{dx} = \frac{\mathrm{x}_2 - x_1}{2}$$

Divide the **total force** into two equal halves and lump them at the nodes

What happens if b(x)=x?

Summary: For each element

Displacement approximation in terms of shape functions

$$\boxed{w(x) = \underline{N}\ \underline{d}}$$

Strain approximation in terms of strain-displacement matrix

$$\boxed{\varepsilon(x) = \underline{B}\ \underline{d}}$$

Stress approximation

$$\boxed{\sigma = \mathrm{E}\underline{B}\ \underline{d}}$$

Element stiffness matrix

$$\boxed{\underline{k} = \int_{x_1}^{x_2} \underline{B}^{\mathrm{T}}\mathrm{E}\underline{B}\ \ A\,dx}$$

Element nodal load vector

$$\boxed{\underline{f_b} = \int_{x_1}^{x_2} \underline{N}^{T}\ \ b\ dx}$$

What happens for element #3?

$$\Pi_3(w) = \frac{1}{2}\int_{x_3}^{x_4} EA\left(\frac{dw}{dx}\right)^2 dx - \int_{x_3}^{x_4} bw \; dx - Fw(x = L)$$

For element 3

$$\boxed{w(x) = \begin{bmatrix} \dfrac{x_4 - x}{x_4 - x_3} & \dfrac{x - x_3}{x_4 - x_3} \end{bmatrix} \begin{Bmatrix} d_{3x} \\ d_{4x} \end{Bmatrix}}$$

$$\Rightarrow w(x = L) = d_{4x}$$

The discretized form of the potential energy

$$\Pi_3(\underline{d}) = \frac{1}{2}\underline{d}^T\left(\int_{x_3}^{x_4} \underline{B}^T E \underline{B} \;\; A dx\right)\underline{d} - \underline{d}^T\left(\int_{x_3}^{x_4} \underline{N}^T \;\; b \;\; dx\right) - Fd_{4x}$$

What happens for element #3?

Now apply Rayleigh-Ritz principle

$$\frac{\partial \Pi_3(\underline{\mathrm{d}})}{\partial \underline{\mathrm{d}}} = 0$$

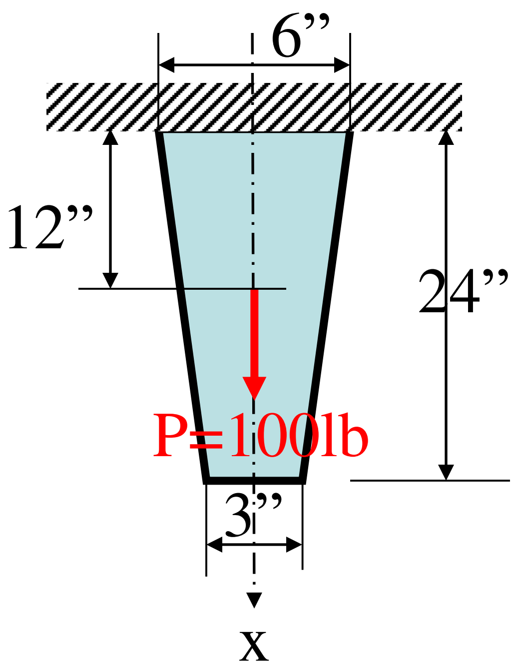
$$\boxed{\Rightarrow \underline{k}\,\underline{\mathrm{d}} = \underline{f_b} + \begin{Bmatrix} 0 \\ \mathrm{F} \end{Bmatrix}}$$

Hence there is an extra load term on the right hand side due to the concentrated force F applied to the right end of the bar.

NOTE that whenever you have a concentrated load at ANY node, that load should be applied as an extra right hand side term.

Step3:Assembly exactly as you had done before, assemble the global stiffness matrix and global load vector and solve the resulting set of equations by properly taking into account the displacement boundary conditions

Problem:



$E=30\text{x}10^6$ psi
$\rho=0.2836$ lb/in^3
Thickness of plate, t=1”

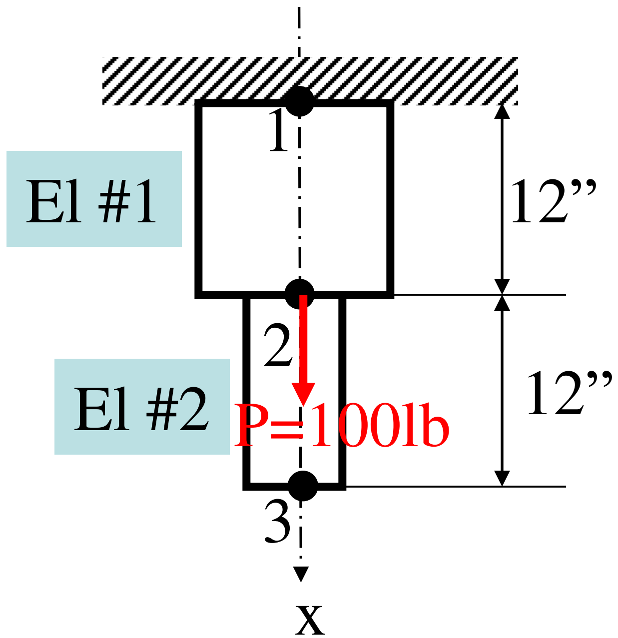
Model the plate as 2 finite elements and

(1) Write the expression for element stiffness matrix and body force vectors
(2) Assemble the global stiffness matrix and load vector
(3) Solve for the unknown displacements
(4) Evaluate the stress in each element
(5) Evaluate the reaction in each support

Solution (1)

Node-element connectivity chart

Finite element model



Element #	Node 1	Node 2
1	1	2
2	2	3

Stiffness matrix of El #1

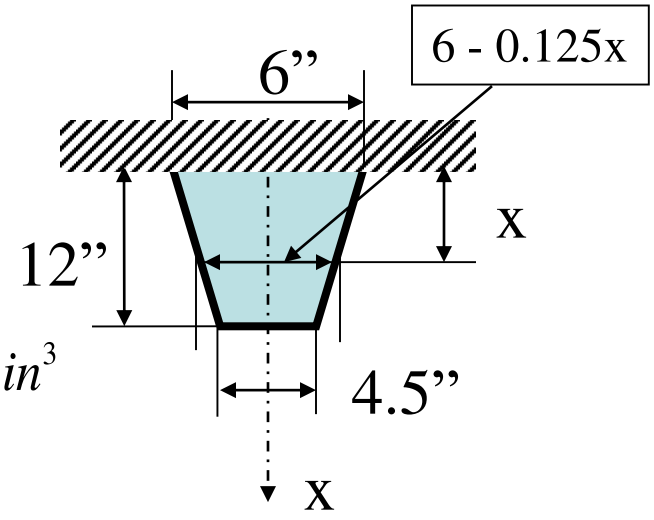
$$\underline{k}^{(1)} = \int_0^{12} \underline{B}^T E \underline{B} \; A dx = \frac{E}{(12)^2}\left(\int_0^{12} A(x)dx\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\int_0^{12} A(x)dx = \int_0^{12} t(6-0.125x)dx = t\int_0^{12}(6-0.125x)dx = 63\, in^3$$

$$\Rightarrow \underline{k}^{(1)} = \frac{E}{(12)^2}(63)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 13.125\times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Stiffness matrix of El #2

$$\underline{k}^{(2)} = \int_{12}^{24} \underline{B}^T E \underline{B}\ A dx = \frac{E}{(12)^2}\left(\int_{12}^{24} A(x)dx\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$



$$\int_{12}^{24} A(x)dx = \int_{12}^{24} t(6-0.125x)dx = t\int_{12}^{24}(6-0.125x)dx = 45\ in^3$$

$$\Rightarrow \underline{k}^{(2)} = \frac{E}{(12)^2}(45)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 9.375\times 10^6\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Now compute the element load vector due to distributed body force (weight)

$$\boxed{\underline{f_b} = \int_{x_1}^{x_2} \underline{N}^T\ b\ dx}$$

For element #1

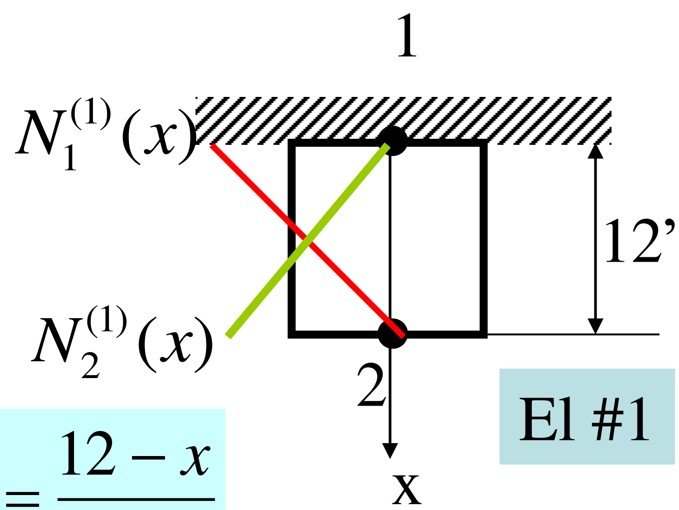
$$\underline{f_b}^{(1)} = \int_0^{12} \underline{\mathrm{N}}^T \ \mathrm{b\,dx} \ = \int_0^{12} \underline{\mathrm{N}}^T \left(\rho \mathrm{A}\right) \mathrm{dx}$$

$$= \rho \int_0^{12} \underline{\mathrm{N}}^T \ A \ \mathrm{dx}$$

$$= \rho \int_0^{12} \begin{Bmatrix} N_1^{(1)}(x) \\ N_2^{(1)}(x) \end{Bmatrix} \underbrace{t(6-0.125x)}_{A(x)} \mathrm{dx}$$

$$= 0.2836 \begin{Bmatrix} 33 \\ 30 \end{Bmatrix} \ lb$$

$$= \begin{Bmatrix} 9.3588 \\ 8.508 \end{Bmatrix} lb$$



$$N_1^{(1)}(x) = \frac{12-x}{12}$$

$$N_2^{(1)}(x) = \frac{x}{12}$$

Superscript in parenthesis indicates element number

For element #2

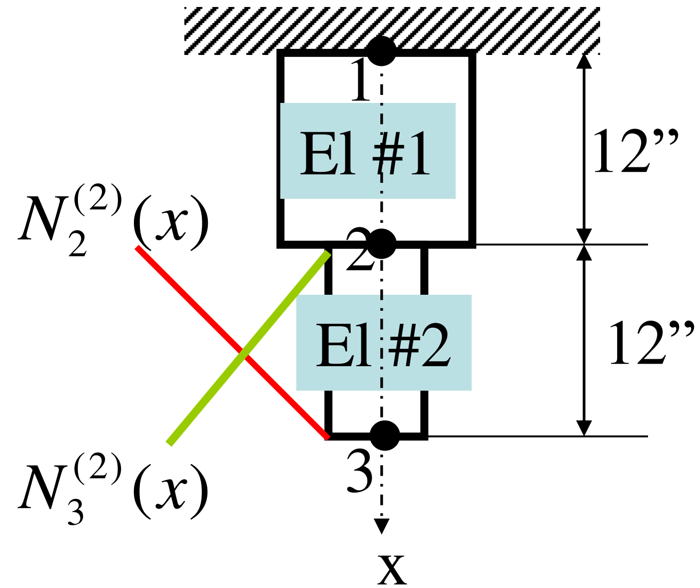
$$\underline{f_b}^{(2)} = \int_{12}^{24} \underline{\mathrm{N}}^T \ \mathrm{b}\, \mathrm{dx} \ = \int_{12}^{24} \underline{\mathrm{N}}^T \left(\rho \mathrm{A}\right) \mathrm{dx}$$

$$= \rho \int_{12}^{24} \underline{\mathrm{N}}^T \ A\, \mathrm{dx}$$

$$= \rho \int_{12}^{24} \begin{Bmatrix} N_2^{(2)}(x) \\ N_3^{(2)}(x) \end{Bmatrix} \underbrace{t(6-0.125x)}_{A(x)} \,\mathrm{dx}$$

$$= 0.2836 \begin{Bmatrix} 24 \\ 21 \end{Bmatrix} \ lb$$

$$= \begin{Bmatrix} 6.8064 \\ 5.9556 \end{Bmatrix} lb$$



$$N_2^{(2)}(x) = \frac{24-x}{12}$$

$$N_3^{(2)}(x) = \frac{x-12}{12}$$

Solution (2) Assemble the system equations

$$\underline{K} = 10^6 \times \begin{bmatrix} 13.125 & -13.125 & 0 \\ -13.125 & 22.5 & -9.375 \\ 0 & -9.375 & 9.375 \end{bmatrix}$$

$$\underline{f} = \underline{f_b} + \underline{f}_{concentrated\ load}$$

$$\underline{f_b} = \begin{Bmatrix} 9.3588 \\ 8.508 + 6.8064 \\ 5.9556 \end{Bmatrix} lb$$

$$\underline{f}_{concentrated\ load} = \begin{Bmatrix} 0 \\ 100 \\ 0 \end{Bmatrix} lb$$

$$\Rightarrow \underline{f} = \begin{Bmatrix} 9.3588 \\ 115.3144 \\ 5.9556 \end{Bmatrix} lb$$

Solution (3)

Hence we need to solve

$$10^6 \times \begin{bmatrix} 13.125 & -13.125 & 0 \\ -13.125 & 22.5 & -9.375 \\ 0 & -9.375 & 9.375 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \end{Bmatrix} = \begin{Bmatrix} 9.3588 + R_1 \\ 115.3144 \\ 5.9556 \end{Bmatrix}$$

R_1 is the reaction at node 1.
Notice that since the boundary condition at x=0 (d_{1x}=0) has not been taken into account, the system matrix is not invertible.
Incorporating the boundary condition d_{1x}=0 we need to solve the following set of equations

$$10^6 \times \begin{bmatrix} 22.5 & -9.375 \\ -9.375 & 9.375 \end{bmatrix} \begin{Bmatrix} d_{2x} \\ d_{3x} \end{Bmatrix} = \begin{Bmatrix} 115.3144 \\ 5.9556 \end{Bmatrix}$$

Solve to obtain

$$\begin{Bmatrix} d_{2x} \\ d_{3x} \end{Bmatrix} = \begin{Bmatrix} 0.92396 \times 10^{-5} \\ 0.98749 \times 10^{-5} \end{Bmatrix} in$$

Solution (4) Stress in elements

Notice that since we are using linear elements, the stress within each element is constant.

In element #1

$$\sigma^{(1)} = E \underline{B}^{(1)} \underline{d}^{(1)}$$

$$= \frac{E}{x_2 - x_1} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} \mathrm{d}_{1x} \\ \mathrm{d}_{2x} \end{Bmatrix}$$

$$= \frac{30 \times 10^6}{12} \mathrm{d}_{2x} \quad \because \mathrm{d}_{1x} = 0$$

$$= 23.099 \ psi$$

In element #2

$$\sigma^{(2)} = E\underline{B}^{(2)}\underline{d}^{(2)}$$

$$= \frac{E}{x_3 - x_2}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{Bmatrix}\mathrm{d}_{2x} \\ \mathrm{d}_{3x}\end{Bmatrix}$$

$$= \frac{30\times 10^6}{12}\left(\mathrm{d}_{3x} - \mathrm{d}_{2x}\right)$$

$$= 1.5882\ psi$$

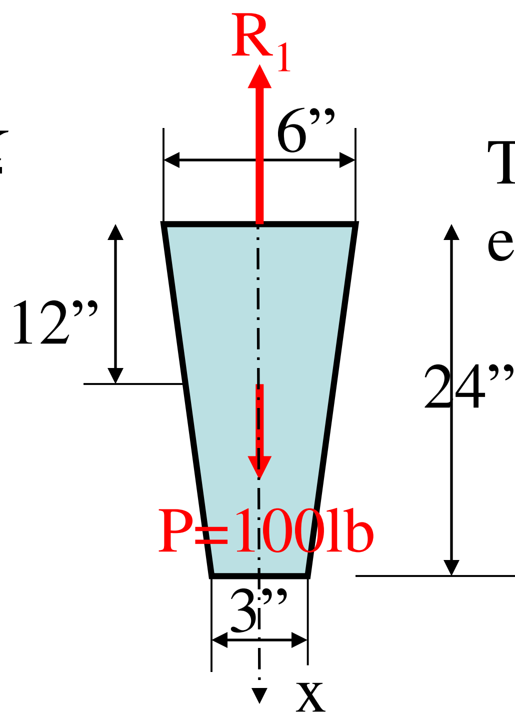
Solution (5) Reaction at support

Go back to the ***first line*** of the global equilibrium equations…

$$10^6 \times \begin{bmatrix} 13.125 & -13.125 & 0 \\ -13.125 & 22.5 & -9.375 \\ 0 & -9.375 & 9.375 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \end{Bmatrix} = \begin{Bmatrix} 9.3588 + R_1 \\ 115.3144 \\ 5.9556 \end{Bmatrix}$$

$\Rightarrow R_1 = -130.6288\, lb$ (The –ve sign indicates that the force is in the –ve x-direction)

Check



The reaction at the wall from force equilibrium in the x-direction

$$R_1 = P + \int_{x=0}^{24} \rho A(x)\, dx$$

$$= 100 + \rho t \int_{x=0}^{24} (6 - 0.125x)\, dx$$

$$= 130.6288\, lb$$

Problem: Can you solve for the displacement and stresses analytically?

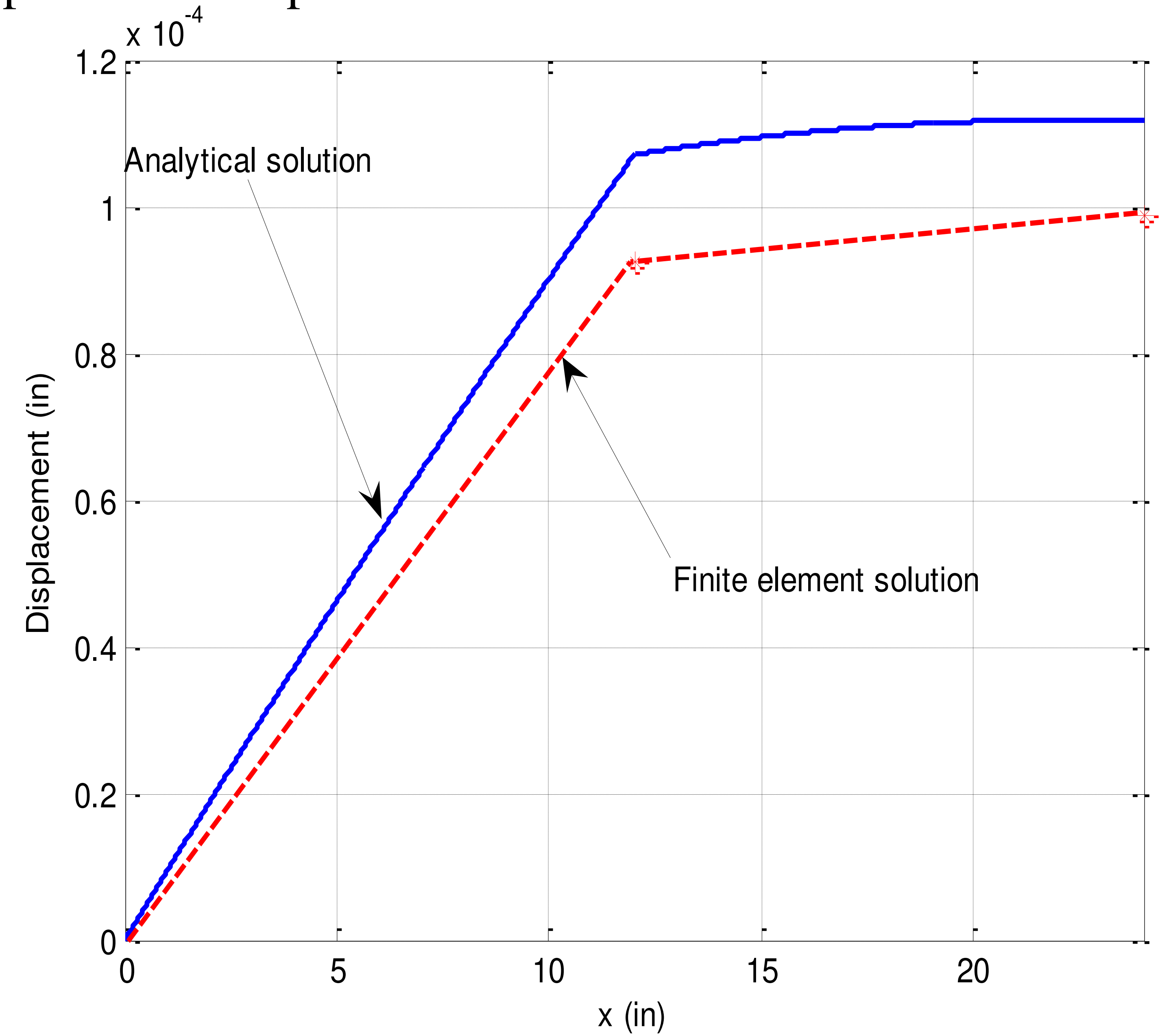
Check out

$$u_{anal} = \begin{cases} -4.727\times10^{-9}x^2 + 9.487\times10^{-7}x & for\ 0 \le x < 12 \\ -4.727\times10^{-9}x^2 + 2.0797\times10^{-7}x + 8.89\times10^{-6} & for\ 12 \le x \le 24 \end{cases}$$

Stress

$$\sigma(x)_{anal} = E\frac{du_{anal}}{dx} = 30\times10^{6}\frac{du_{anal}}{dx}$$

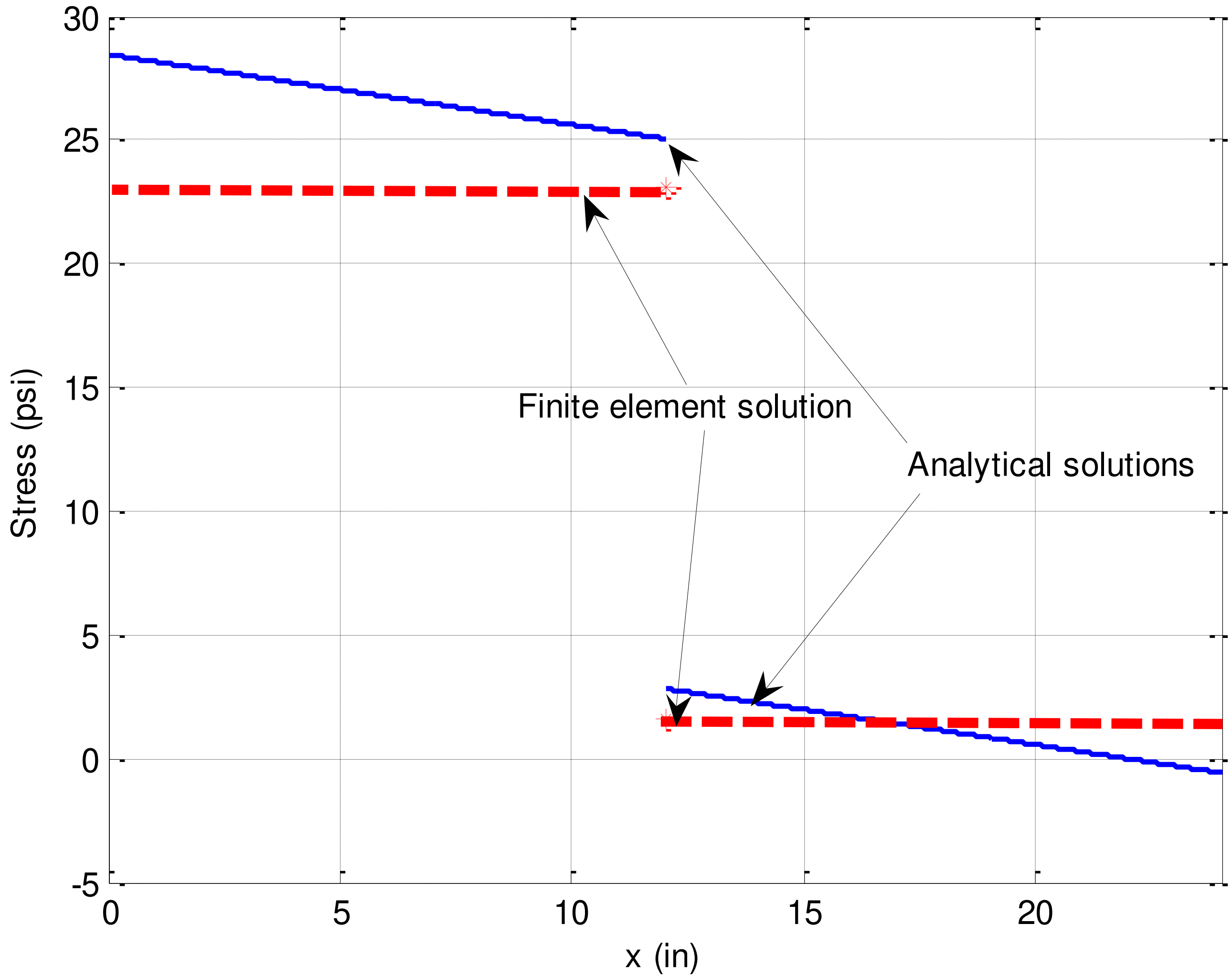
Comparison of displacement solutions



Notice:

1. Slope discontinuity at x=12 (why?)
2. The finite element solution does not produce the exact solution even at the nodes
3. We may improve the solution by
   (1) Increasing the number of elements
   (2) Using higher order elements (e.g., quadratic instead of linear)

Comparison of stress solutions



The analytical as well as the finite element stresses are discontinuous across the elements