# Laplace Transforms and Block Diagrams

**Laplace transform** is used for *transforming a function of time domain (t) into a function of frequency domain (s).*

**Why we use Laplace transforms?**

1- To transform ODE (Ordinary Differential Equation) to an algebraic equation.
2- Easy to analyze and design systems.
3- To deal with frequency responses.

**Laplace transforms table:**

| f(t) | F(s) |
|---|---|
| δ unit Impulse function | 1 |
| 1 | $\frac{1}{s}$ |
| $t$ | $\frac{1}{s^2}$ |
| $t^n$ | $\frac{n!}{s^{n+1}}$ |
| $e^{-\alpha t}$ | $\frac{1}{s+\alpha}$ |
| Sin $\omega t$ | $\frac{\omega}{s^2+\omega^2}$ |
| Cos $\omega t$ | $\frac{s}{s^2+\omega^2}$ |
| $te^{-\alpha t}$ | $\frac{1}{(s+\alpha)^2}$ |
| $\frac{dy}{dt}$ or $y'$ | $sY - y(0)$ |
| $\frac{d^2y}{dt^2}$ or $y''$ | $s^2Y - sy(0) - y'(0)$ |
| $\int_0^t f(\tau)\,d\tau$ | $\frac{F(s)}{s}$ |

## Some properties of Laplace transforms:

### 1- Linearity:

$$\mathcal{L}\{ \alpha_1 f_1(t) + \alpha_2 f_2(t) \} = \alpha_1 F_1 + \alpha_2 F_2$$

*Example:* $\mathcal{L}\{ 4t + 2e^{-6t} \}$

*Solution:* $\mathcal{L}\{ 4t + 2e^{-6t} \} = 4\mathcal{L}\{ \text{t} \} + 2\mathcal{L}\{ e^{-6t} \}$

$$= \frac{4}{s^2} + \frac{2}{s+6}$$

### 2- Time Delay:

$$\mathcal{L}\{ f(t-T)\} = e^{-Ts}F$$

*Example*: $\mathcal{L}\left\{ e^{-0.5(t-4)} \right\}$

*Solution*: $\mathcal{L}\left\{ e^{-0.5(t-4)} \right\} = e^{-4s} * \frac{1}{s+0.5} = \frac{e^{-4s}}{s+0.5}$

*Example*: $\mathcal{L}\{ \delta(t-4T) \}$

*Solution*: $\mathcal{L}\{ \delta(t-4T) \} = e^{-4Ts} * 1 = e^{-4Ts}$

### 3- Final Value Theorem:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Only if the poles of sF(s) are in left half plane (LHP)

*Example*: Find the final value for:

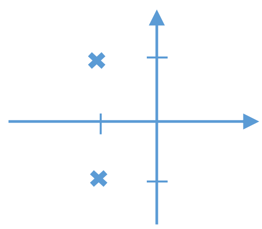
a- $F(s) = \frac{5}{s(s^2+s+2)}$

b- $F(s) = \frac{4}{s^2+4}$

*Solution*: a- $\lim_{t\to\infty} f(t) = \lim_{s\to 0} s * \frac{5}{s(s^2+s+2)}$

$= \lim_{s\to 0} \frac{5}{(s^2+s+2)}$ *the poles are :* $\frac{-1}{2} + \frac{\sqrt{7}}{2} i\ ,\ \frac{-1}{2} - \frac{\sqrt{7}}{2} i$

*all roots are in LHP:*



*Substitute s=0 in the equation* → $\lim_{s\to 0} \frac{5}{(s^2+s+2)} = \frac{5}{2}$

b- $\lim_{t\to\infty} f(t) = \lim_{s\to 0} s * \frac{4}{s^2+4}$

$= \lim_{s\to 0} \frac{4s}{s^2+4}$

*The roots are* $2i$ , $-2i$ . *not in LHP*

*So ,* $\lim_{t\to\infty} f(t) \neq \lim_{s\to 0} s * \frac{4}{s^2+4}$

**4- Initial value theorem:**

$$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$$

No need to check the poles locations.

*Example*: Find the initial value for:

a- $F(s) = \frac{5}{s(s^2+s+2)}$

b- $F(s) = \frac{4}{s^2+4}$

*Solution*: a- $\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} s * \frac{5}{s(s^2+s+2)}$

$= \lim_{s\to\infty} \frac{5}{(s^2+s+2)} = 0$ *Because 5 is divided by a very large number.*

b- $\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} s * \frac{4}{s^2+4}$

$= \lim_{s\to\infty} \frac{4s}{s^2+4} = 0$

**5- Convolution:**

$$\mathcal{L}\{f_1(t) \times f_2(t)\} \neq F_1(s) \times F_2(s)$$

**6- Frequency shift theorem:**

$$\mathcal{L}\{f(t) \times e^{-\propto t}\} = F(s+\propto)$$

It means that you find the Laplace transform for the f(t) and change every (s) in the function with (s+α).

*Example*: Find the Laplace transform for:

a- $\mathcal{L}\{te^{-5t}\}$

b- $\mathcal{L}\{\sin 4t \cdot e^{4t}\}$

c- $\mathcal{L}\{3t^4 \cdot e^{-3it}\}$

d- $\mathcal{L}\{cos6t \cdot e^{8t}\}$

*Solution:* a- $\mathcal{L}\{te^{-5t}\} = \frac{1}{(s+5)^2}$ *first we find $\mathcal{L}\{t\} = \frac{1}{s^2}$ then we change (s) with (s+5). Do the same for each example.*

b- $\mathcal{L}\{\sin 4t \cdot e^{4t}\} = \frac{2}{(s-4)^2+4}$

c- $\mathcal{L}\{3t^4 \cdot e^{-3it}\} = 3\mathcal{L}\{3t^4 \cdot e^{-3it}\} = 3 \times \frac{4!}{(s+3i)^5} = \frac{72}{(s+3i)^5}$

d- $\mathcal{L}\{cos6t \cdot e^{8t}\} = \frac{(s-8)}{(s-8)^2+36}$

**Some exercises for finding $\mathcal{L}^{-1}$ :**

*Find:* a- $\mathcal{L}^{-1}\left\{\frac{5}{s}\right\}$ b- $\mathcal{L}^{-1}\left\{\frac{5}{s^2}\right\}$ c- $\mathcal{L}^{-1}\left\{\frac{5}{(s-3)^2}\right\}$

d- $\mathcal{L}^{-1}\left\{\frac{4}{s^2+4}\right\}$ e- $\mathcal{L}^{-1}\left\{\frac{4}{(s+1)^2+4}\right\}$ f- $\mathcal{L}^{-1}\left\{\frac{4}{s^2+4s+8}\right\}$

g- $\mathcal{L}^{-1}\left\{\frac{4s}{(s+1)^2+4}\right\}$

*Solution:* a- $\mathcal{L}^{-1}\left\{\frac{5}{s}\right\} = 5\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 5 \times 1 = 5$

b- $\mathcal{L}^{-1}\left\{\frac{5}{s^2}\right\} = 5\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = 5 \times t = 5t$

c- $\mathcal{L}^{-1}\left\{\frac{5}{(s-3)^2}\right\} = 5\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\} = 5 \times te^{3t} = 5te^{3t}$

d- $\mathcal{L}^{-1}\left\{\frac{4}{s^2+4}\right\} = 2\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = 2sin2t$

e- $\mathcal{L}^{-1}\left\{\frac{4}{(s+1)^2+4}\right\} = 2\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+4}\right\} = 2e^{-t}sin2t$

f- $\mathcal{L}^{-1}\left\{\frac{4}{s^2+4s+8}\right\} = \mathcal{L}^{-1}\left\{\frac{4}{(s^2+4s+4)+4}\right\} = \mathcal{L}^{-1}\left\{\frac{4}{(s+2)^2+4}\right\}$

$= 2\mathcal{L}^{-1}\left\{\frac{2}{(s+2)^2+4}\right\} = 2e^{-2t}sin2t$

g- $\mathcal{L}^{-1}\left\{\frac{4s}{(s+1)^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{4(s+1)-4}{(s+1)^2+4}\right\}$

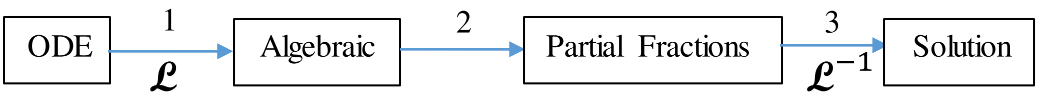
$= \mathcal{L}^{-1}\left\{\frac{4(s+1)}{(s+1)^2+4} - \frac{4}{(s+1)^2+4}\right\}$

$= 4\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+4}\right\} - 2\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+4}\right\}$

$= 4e^{-t}cos2t - 2e^{-t}sin2t = 2e^{-t}(2cos2t - sin2t)$

## Solving ODE by Laplace transforms:

**There are 3 steps to find the solution:**




NOTE FOR THE SECOND STEP:
To find the partial fractions, we have three cases:
1- Distinct roots.
2- Repeated roots.
3- Complex roots.

**First Case: Distinct roots:**

*Example*: Solve the ODE:

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 5 \quad , \quad \text{ICs are: } y(0) = -1 \ , y'(0) = 2$$

*Solution:*

$$y'' + 3y' + 2y = 5$$

$$\mathcal{L} \rightarrow \quad \{s^2Y - sy(0) - y'(0)\} + 3\{sY - y(0)\} + 2Y = \frac{5}{s}$$

$$\{s^2Y + s - 2\} + 3\{sY + 1\} + 2Y = \frac{5}{s} \quad \textit{Substitute ICs}$$

$$s^2Y + 3sY + 2Y + s + 1 = \frac{5}{s}$$

$$Y(s^2 + 3s + 2) = \frac{5}{s} - s - 1$$

$$Y(s^2 + 3s + 2) = \frac{5}{s} - \frac{s^2}{s} - \frac{s}{s} = \frac{5 - s^2 - s}{s}$$

$$Y = \frac{5 - s^2 - s}{s(s^2 + 3s + 2)} = \frac{5 - s^2 - s}{s(s+1)(s+2)} \quad \textit{Roots are 0 , -1 , -2}$$

$$\frac{5 - s^2 - s}{s(s^2 + 3s + 2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

*Multiply both sides by* $s(s^2 + 3s + 2)$:

$$5 - s^2 - s = A(s^2 + 3s + 2) + Bs(s+2) + Cs(s+1)$$

*Now choose values of s that makes other unknowns = 0 :*

*When s = 0:*

$$5 = A(2) \rightarrow A = \frac{5}{2}$$

*When s = -1:*

$$5 = B(-1)(1) \rightarrow B = -5$$

*When s = -2:*

$$3 = -2C(-1) \rightarrow C = \frac{3}{2}$$

*Partial fractions are:* $\frac{5}{2s} - \frac{5}{s+1} + \frac{3}{2(s+2)}$

$$\mathcal{L}^{-1} \rightarrow \quad \mathcal{L}^{-1}\left\{\frac{5}{2s} - \frac{5}{s+1} + \frac{3}{2(s+2)}\right\}$$

$\frac{5}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 5\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$

$\frac{5}{2} - 5e^{-t} + \frac{3}{2}e^{-2t}$

$\therefore\ y = \frac{5}{2} - 5e^{-t} + \frac{3}{2}e^{-2t}$ *This is the solution for ODE*

**Second Case: Repeated roots:**

*Example*: Solve the ODE:

$$\frac{d^3y}{dt^3} + 5\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 4y = 2\delta \ ,\quad \text{with Zero ICs}$$

$$y(0) = y'(0) = y''(0) = 0$$

*Solution:*

$y''' + 5y'' + 8y' + 4y = 2\delta$

$\mathcal{L} \rightarrow$ $s^3Y + 5s^2Y + 8sY + 4Y = 2$ *Because no initial conditions.*

$Y(s^3 + 5s^2 + 8s + 4) = 2$

$Y = \frac{2}{s^3+5s^2+8s+4}$

$Y = \frac{2}{(s+1)(s+2)^2}$ *See the Appendix in last page to know how to Factor the third-degree equation.*

$\frac{2}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{(s+2)^2} + \frac{C}{s+2}$

*Multiply by:* $(s+1)(s+2)^2$ :

$2 = A(s+2)^2 + B(s+1) + C(s+1)(s+2)$

*When s = -1:*

$2 = A(1)^2 \quad \rightarrow \quad A = 2$

*When s = -2:*

$2 = b(-1) \quad \rightarrow \quad B = -2$

*When s = 1:*

$2 = 2(9) - 2(2) + C(2)(3) \quad \rightarrow \quad C = -2$

*Partial fractions are:* $\frac{2}{s+1} - \frac{2}{(s+2)^2} - \frac{2}{s+2}$

$\mathcal{L}^{-1} \rightarrow$ $\mathcal{L}^{-1}\left\{\frac{2}{s+1} - \frac{2}{(s+2)^2} - \frac{2}{s+2}\right\}$

$2\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$

$2e^{-t} - 2te^{-2t} - 2e^{-2t}$

$\therefore\ y = 2(e^{-t} - te^{-2t} - e^{-2t})$ *This is the solution for ODE*

**Third case: Complex roots:**

*Example*: Solve the ODE:

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 3 \quad , \quad \text{with Zero ICs}$$

$$y(0) = y'(0) = 0$$

*Solution:*

$$y'' + 2y' + 5y = 3$$

$$\mathcal{L} \rightarrow \qquad s^2Y + 2sY + 5Y = \frac{3}{s}$$

$$Y(s^2 + 2s + 5) = \frac{3}{s}$$

$$Y = \frac{3}{s(s^2+2s+5)}$$

*Roots are: 0 , -1+2i , -1-2i*

$$\frac{3}{s(s^2+2s+5)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+5}$$

*Multiply both sides by:* $s(s^2 + 2s + 5)$

$$3 = A(s^2 + 2s + 5) + (Bs + c)s$$

*When s = 0 :*

$$3 = A(5) \quad \rightarrow \quad A = \frac{3}{5}$$

*When s = 1 :*

$$3 = \frac{3}{5}(8) + B + C \quad \rightarrow \textit{equation } 1$$

*When s = -1 :*

$$3 = \frac{3}{5}(4) + B - C \quad \rightarrow \textit{equation } 2$$

*Solving the equation 1 and 2 :*

$$\rightarrow \quad B = -\frac{3}{5}$$

$$\rightarrow \quad C = -\frac{6}{5}$$

*Partial fractions are:* $\frac{3}{5s} - \frac{\frac{3}{5}s+\frac{6}{5}}{(s^2+2s+5)}$

$$\frac{3}{5s} - \frac{\frac{3}{5}s+\frac{6}{5}}{s^2+2s+1+4} = \frac{3}{5s} - \frac{\frac{3}{5}s+\frac{6}{5}}{(s+1)^2+4} = \frac{3}{5s} - \frac{\frac{3}{5}(s+1)+\frac{6}{5}-\frac{3}{5}}{(s+1)^2+4}$$

$$\frac{3}{5s} - \frac{3}{5}\left(\frac{(s+1)}{(s+1)^2+4}\right) - \frac{3}{10}\left(\frac{2}{(s+1)^2+4}\right)$$

$$\mathcal{L}^{-1} \rightarrow \mathcal{L}^{-1}\left\{\frac{3}{5s} - \frac{3}{5}\left(\frac{(s+1)}{(s+1)^2+4}\right) - \frac{3}{10}\left(\frac{2}{(s+1)^2+4}\right)\right\}$$

$$= \frac{3}{5} - \frac{3}{5}e^{-t}cos2t - \frac{3}{10}e^{-t}sin2t$$

$$\therefore y = \frac{3}{5} - \frac{3}{5}e^{-t}\left(cos2t + \frac{1}{2}sin2t\right)$$

# Block Diagrams:

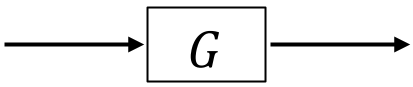
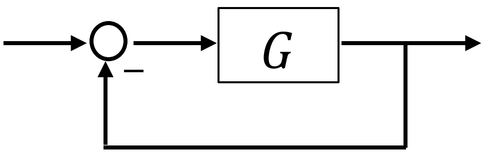
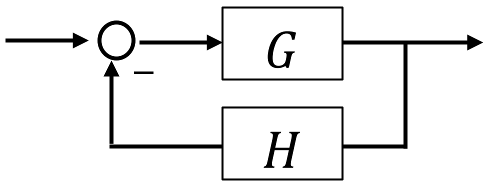
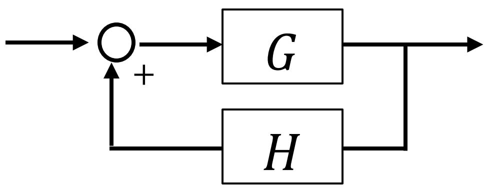
Graphical representation of transfer functions.

$$\text{Transfer function} = \frac{output\ of\ the\ system}{input\ of\ the\ system}$$

For example, if you have a mass and you push it by a force $f(t)$ and you want to know the distance $x(t)$, the force will be the input of the system and the distance will be the output:


We will learn how to find the transfer function in mathematical modeling.

## Types of block diagrams:

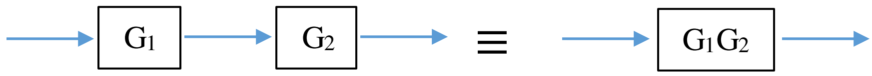
| | |
|---|---|
|   | Open-Loop system (OL) |
|   | Unity Feedback system. Closed-Loop (CL) |
|   | Negative Feedback system. Closed-Loop (CL) |
|   | Positive Feedback system. Closed- Loop (CL) |

Where G and H are transfer functions.

## Block diagrams reduction rules:

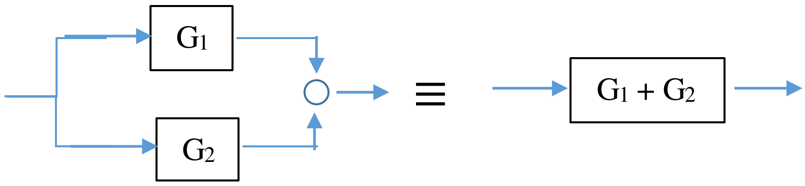
We can reduce a large block diagram to a simple open-loop (OL) system by using these rules:

1- Between every two blocks (**in series**) is multiplication (×)

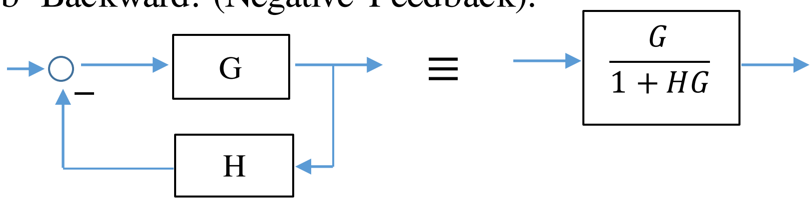



2- If there are two blocks **in parallel**, they have four cases:

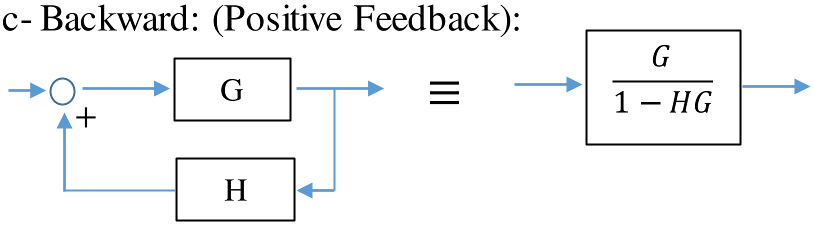
a- Forward:



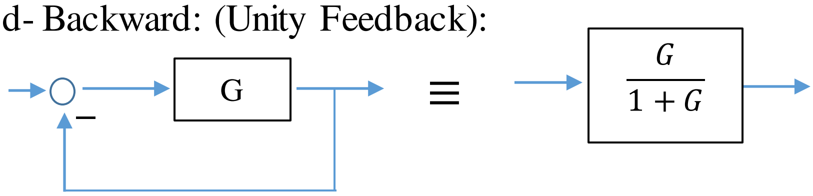

b- Backward: (Negative Feedback):




c- Backward: (Positive Feedback):

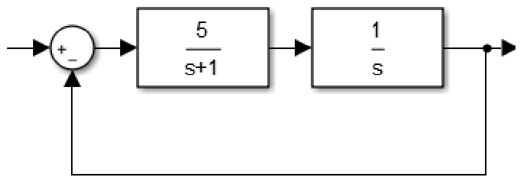



d- Backward: (Unity Feedback):

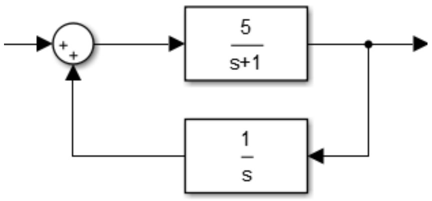


*Example*: Reduce these block diagrams to a single block (OL transfer function) :

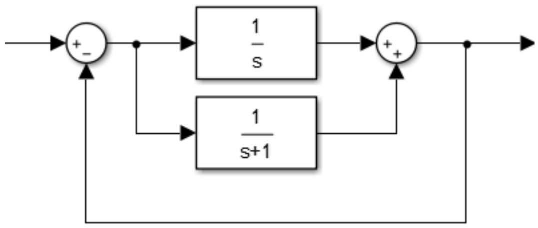
a-




b-

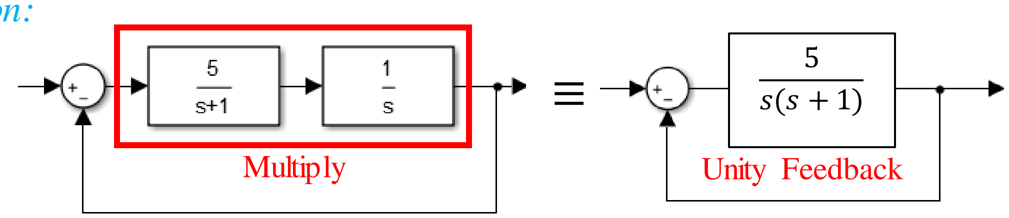



c-

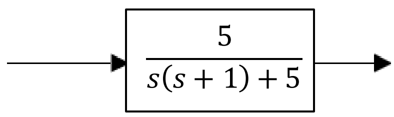



*Solution:*

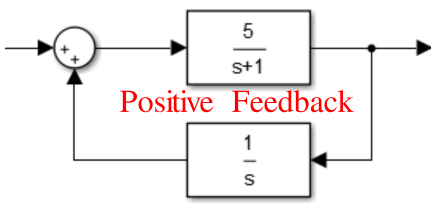
a-




$$\frac{\frac{5}{s(s+1)}}{1+\frac{5}{s(s+1)}} = \frac{\frac{5}{s(s+1)}}{\frac{s(s+1)+5}{s(s+1)}} = \frac{5}{s(s+1)+5} \equiv$$

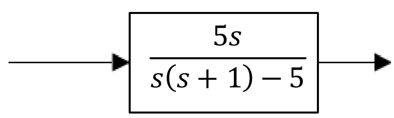



b-

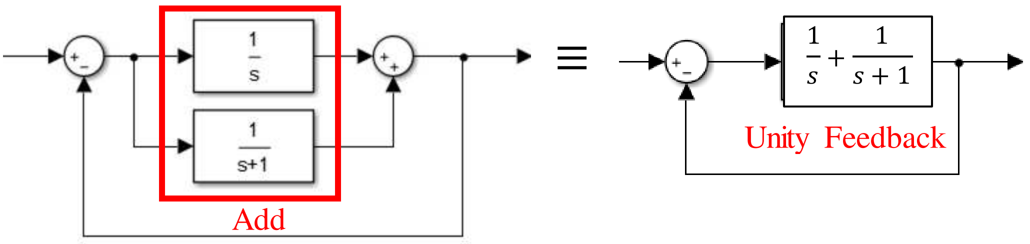



$$\frac{\frac{5}{s+1}}{1-\left(\frac{1}{s}\right)\left(\frac{5}{s+1}\right)} = \frac{\frac{5}{s+1}}{1-\left(\frac{5}{s(s+1)}\right)} = \frac{\frac{5}{s+1}}{\frac{s(s+1)-5}{s(s+1)}} = \frac{5}{\frac{s(s+1)-5}{s}} = \frac{5s}{s(s+1)-5}$$

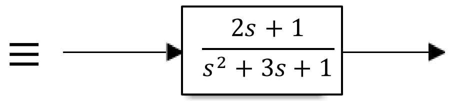
$$\equiv$$



c-




$$\frac{\frac{1}{s}+\frac{1}{s+1}}{1+\left(\frac{1}{s}+\frac{1}{s+1}\right)}=\frac{\frac{2s+1}{s(s+1)}}{1+\frac{2s+1}{s(s+1)}}=\frac{\frac{2s+1}{s(s+1)}}{\frac{s(s+1)+2s+1}{s(s+1)}}=\frac{2s+1}{s^2+3s+1}$$




*Example:* If the transfer function of an open-loop system is: $\frac{2s+1}{s^2+3s+1}$ , find the unity feedback block diagram.

*Solution:* The rule of the unity feedback is: $\frac{G}{1+G}$, we need to find the Value of G.

$$\frac{2s+1}{s^2+3s+1}=\frac{G}{1+G} \rightarrow G(s^2+3s+1)=(1+G)(2s+1)$$

$$G(s^2+3s+1)=2Gs+G+2s+1$$
$$G(s^2+3s+1)-2Gs-G=2s+1$$
$$G(s^2+s)=2s+1$$

$$\therefore\ G=\frac{2s+1}{(s^2+s)}=\frac{2s+1}{s(s+1)}$$