Mathematical Modeling

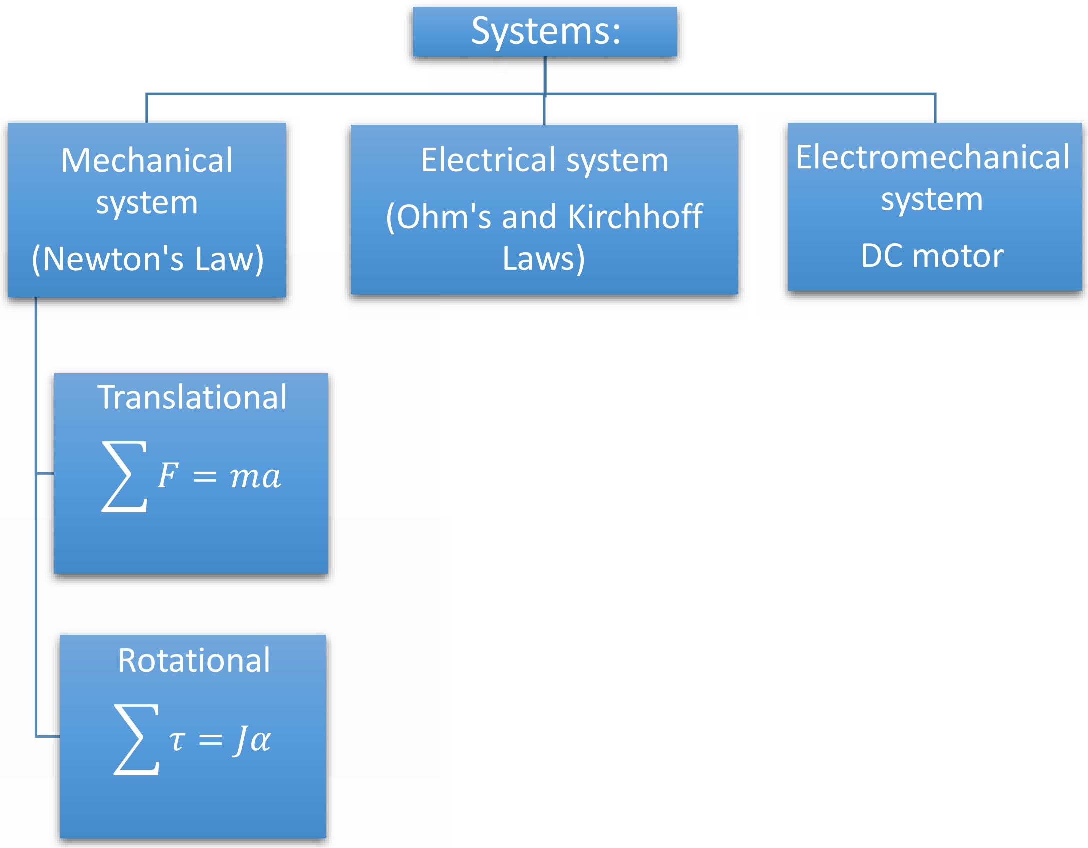
Mathematical modeling is simply *finding the transfer function between the input and the output of the physical system. In addition, it is an approximate representation of the physical system.*

Remark: Transfer function is only used for linear time-invariant system. It's also called LTI system.

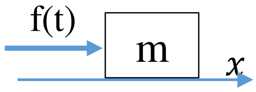
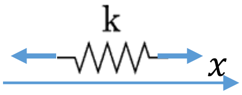
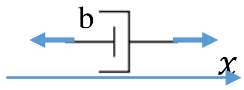
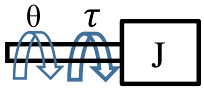
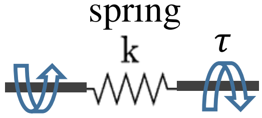
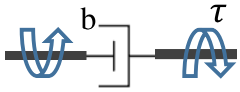
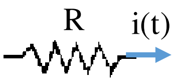
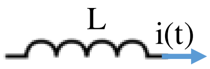
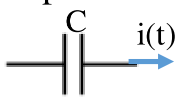
Linear system is *a system having a principle of superposition*. Real systems are nonlinear and difficult to deal with mathematically, so we will linearize them to be easy for designing and controlling.
Time-invariant system is *a system that does not change with time*. For example, you push a box with a mass (m), its mass will not change with time.
Time-varying system is *a system that change with time*. For example, a car with a mass (m) will lose its mass after long time by decreasing the fuel.

All modeling examples in this course are LTI system, and we will learn later how to deal with nonlinear system.

Types of system that will be covered in this course:



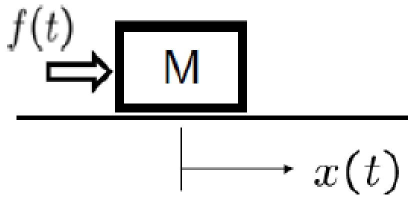
Systems Elements and their equations:

Translational Mechanical System			
Element	mass 	spring 	damper 
Diff-equation	$f(t) = m\frac{d^2x}{dt^2}$	$f(t) = kx$	$f(t) = b\frac{dx}{dt}$
By Laplace	$F(s) = Ms^2X$	$F(s) = KX$	$F(s) = BsX$
Rotational Mechanical System			
Element	inertia 	spring 	friction 
Diff-equation	$\tau(\theta) = J\frac{d^2\theta}{dt^2}$	$\tau(\theta) = k\theta$	$\tau(\theta) = b\frac{d\theta}{dt}$
By Laplace	$T(\theta) = Js^2\theta$	$T(\theta) = K\theta$	$T(\theta) = Bs\theta$
Electrical System			
Element	resistor 	inductor 	Capacitor 
Diff-equation	$v(t) = iR$	$v(t) = L\frac{di}{dt}$	$v(t) = i\int_0^t \frac{1}{C}dt$
By Laplace	$V(s) = IR$	$V(s) = LsI$	$V(s) = \frac{I}{sC}$

Remarks on systems elements:

1- For the mechanical systems, the forces of spring and damper always act in the opposite direction of the input force.
2- The number of equations in mechanical system depends on the number of masses.
3- If the system has more than one equation, we may use the block diagram instead of solving them.
4- All of the differential equations above assumed to be without initial conditions (ICs = zero).

Example: Modeling of translational mechanical system (mass only) with zero ICs:



Solution: The force f(t) is the input , and the distance x(t) is the output

Using Newton's Law: $\sum F = ma$

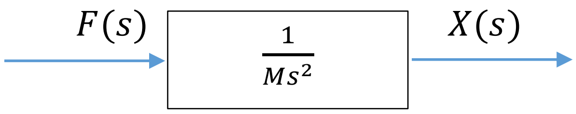
$$f(t) = m\frac{d^2x}{dt^2}$$

$$\mathcal{L} \rightarrow \quad F(s) = M(s^2X + sy(0) + y'(0))$$

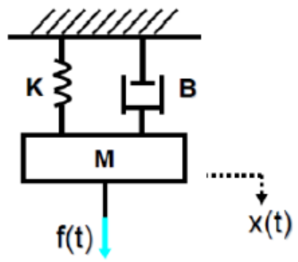
$$F(s) = Ms^2X$$

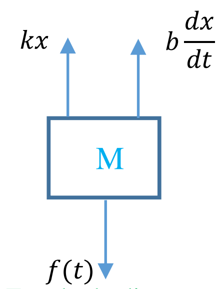
$$X = \frac{F(s)}{Ms^2}$$

Transfer function is $\frac{output}{input} = \frac{X(s)}{F(s)} = \frac{1}{Ms^2}$



Example: Modeling of translational mechanical system (mass-spring-damper) with zero ICs:





Free body diagram

Solution: f(t) is the input, and x(t) is the output.

Using Newton's Law: $\sum F = ma$

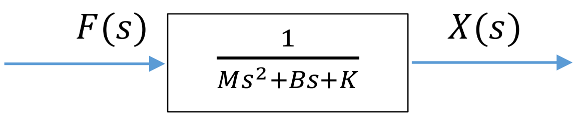
$$f(t) - b\frac{dx}{dt} - kx = m\frac{d^2x}{dt^2}$$

$$\mathcal{L} \rightarrow \quad F(s) - BsX - KX = Ms^2X$$

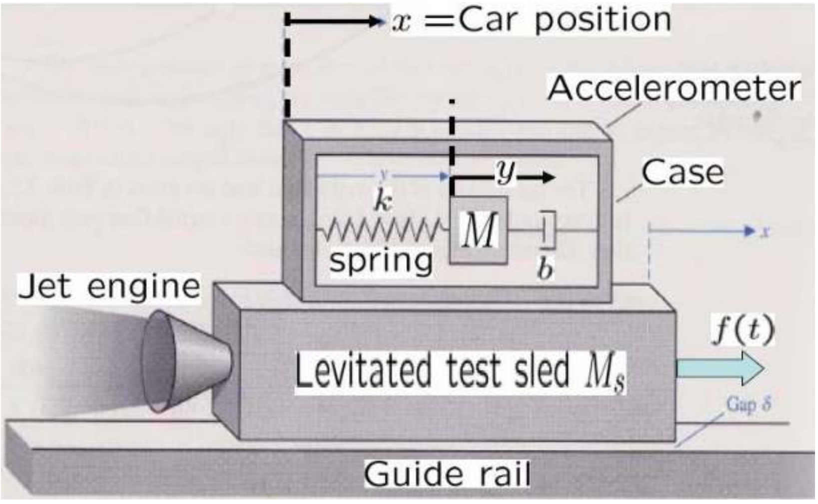
$$X(Ms^2 + Bs + K) = F(s)$$

$$X = \frac{F(s)}{Ms^2+Bs+K}$$

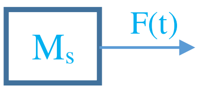
Transfer function is $\frac{output}{input} = \frac{X(s)}{F(s)} = \frac{1}{Ms^2+Bs+K}$



Example: Modeling of Accelerometer (two masses-spring-damper) system, assume the force f(t) is a unit input and zero ICs. Also we want to know how y(t) will move.



Solution: The force $f(t)$ is the input, and distance y(t) is the output. First, we will draw the free body diagram for each mass:



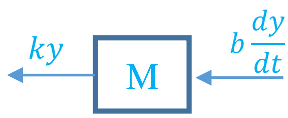
$$\sum F = ma$$

$$f(t) = M_s \frac{d^2x}{dt^2}$$

f(t) = 1:

$$\mathcal{L} \rightarrow \qquad \frac{1}{s} = M_s s^2 X$$

$$X = \frac{1}{M_s s^3} \qquad \textit{Equ 1}$$



$$\sum F = ma$$

$$-b\frac{dy}{dt} - ky = M\left(\frac{d^2y}{dt^2} + \frac{d^2x}{dt^2}\right)$$

$$M\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = -M\frac{d^2x}{dt^2}$$

$$Ms^2Y + BsY + KY = -Ms^2X$$

$$Y(Ms^2 + Bs + K) = -Ms^2X$$

$$Y = \frac{-Ms^2X}{Ms^2+Bs+K}$$

Taking M a common factor:

$$Y = \frac{-Ms^2X}{M(s^2+\frac{B}{M}s+\frac{K}{M})} = \frac{-s^2X}{s^2+\frac{B}{M}s+\frac{K}{M}} \qquad \textit{Equ 2}$$

Substituting Equ 1 in Equ 2 :

$$Y = \frac{-s^2\left(\frac{1}{M_s s^3}\right)}{s^2+\frac{B}{M}s+\frac{K}{M}} = \frac{-\frac{1}{M_s s}}{s^2+\frac{B}{M}s+\frac{K}{M}} = \frac{-1}{M_s s\left(s^2+\frac{B}{M}s+\frac{K}{M}\right)} \qquad \textit{Transfer Function}$$

If we want to know how y(t) moves, we must find y(t). That means solving the ODE.

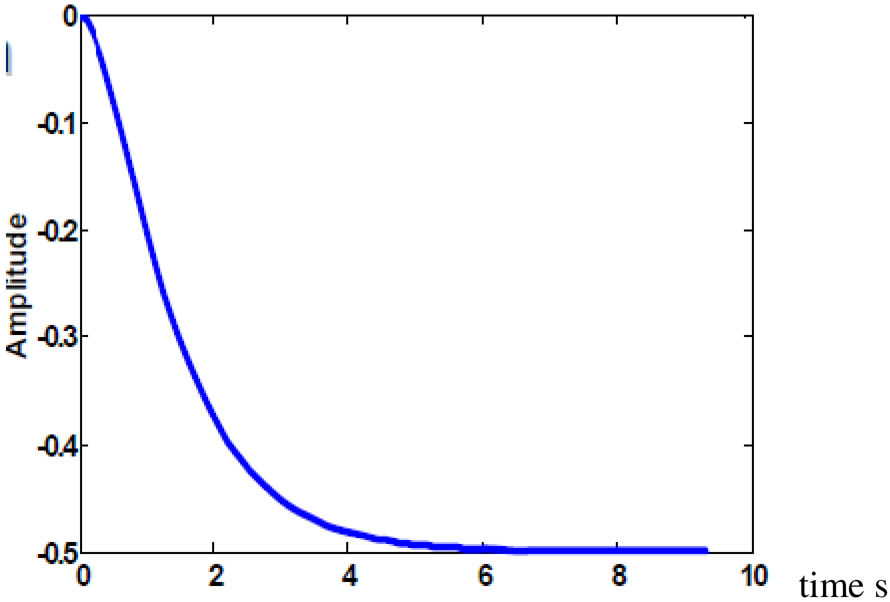
Suppose that: $\frac{B}{M} = 3, \quad \frac{K}{M} = 2, \quad M_s = 1:$

$$Y = \frac{-1}{s(s^2 + 3s + 2)} = \frac{-1}{s(s+1)(s+2)}$$

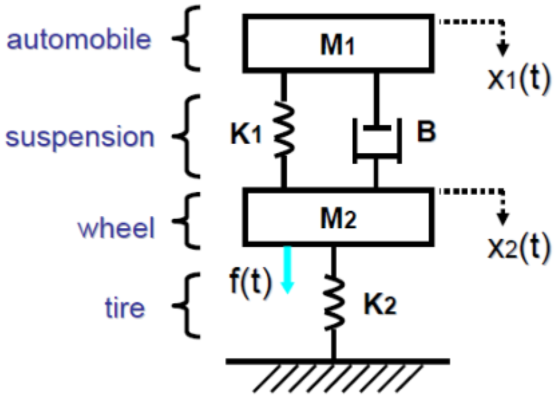
Using partial fraction we find:

$$Y = -\frac{1}{2s} + \frac{1}{s+1} - \frac{1}{2(s+2)}$$

$$\mathcal{L}^{-1} \rightarrow \qquad y(t) = -\frac{1}{2} + e^{-t} - \frac{1}{2}e^{-2t}$$

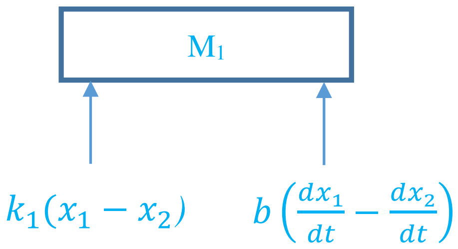


Example: Modeling of automobile suspension system (two masses-spring-damper) system.



Solution: f(t) is the input, and we choose $x_1(t)$ to be the output.

1- For the mass M_1 :



$$\sum F = ma$$

$$-\mathrm{k_1(x_1 - x_2) - b\left(\frac{dx_1}{dt} - \frac{dx_2}{dt}\right) = M_1 \frac{d^2x_1}{dt^2}}$$

$$\mathcal{L} \rightarrow \quad -K_1(X_1 - X_2) - Bs(X_1 - X_2) = M_1 s^2 X_1$$

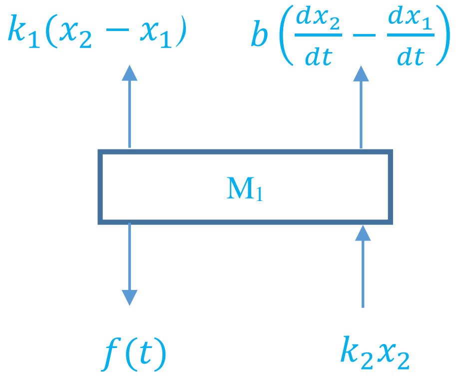
$$-K_1X_1 + K_1X_2 - BsX_1 + BsX_2 = M_1 s^2 X_1$$

$$X_1(M_1s^2 + Bs + K_1) = X_2(Bs + K_1)$$

$$X_1 = \frac{(Bs+K_1)}{M_1s^2+Bs+K_1} X_2 \qquad \textit{Equ 1}$$

Transfer function for M_1 : $\frac{X_1}{X_2} = \frac{Bs+K_1}{M_1s^2+Bs+K_1}$

2- For tha mass M_2 :



$$\sum F = ma$$

$$f(t) - k_2x_2 - b\left(\frac{dx_2}{dt} - \frac{dx_1}{dt}\right) - k_1(x_2 - x_1) = M_2\frac{d^2x_2}{dt^2}$$

$$\mathcal{L} \rightarrow \quad F(s) - K_2X_2 - Bs(X_2 - X_1) - K_1(X_2 - X_1) = M_2s^2X_2$$

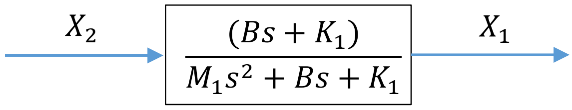
$$F(s) - K_2X_2 - BsX_2 + BsX_1 - K_1X_2 + k_1X_1 = M_2s^2X_2$$

$$X_2(M_2s^2 + Bs + K_1 + K_2) = F(s) + X_1(Bs + K_1)$$

$$X_2 = \frac{1}{M_2s^2+Bs+K_1+K_2}F(s) + \frac{Bs+K_1}{M_2s^2+Bs+K_1+K_2}X_1 \qquad \text{Equ 2}$$

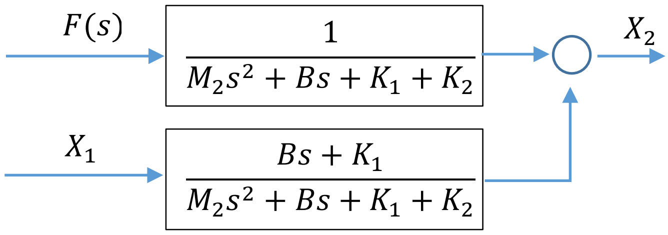
Looking at Equ 1 and 2 , we see that it is difficult to solve. So we will use the block diagram:

1- Block diagram for Equ 1: $X_1 = \frac{(Bs+K_1)}{M_1s^2+Bs+K_1}X_2$

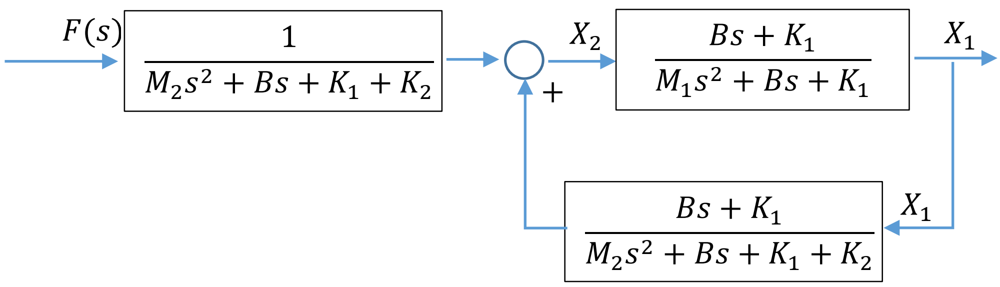


2- Block Diagram for Equ 2:

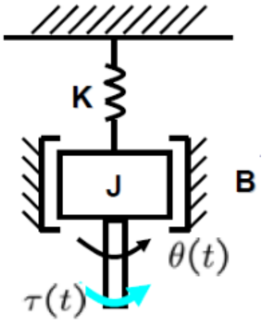
$$X_2 = \frac{1}{M_2s^2+Bs+K_1+K_2}F(s) + \frac{Bs+K_1}{M_2s^2+Bs+K_1+K_2}X_1$$



∴ By adding the two diagrams, the block diagram for the system is:



Example: Rotational mechanical system: Torsional Pendulum (inertia-friction-rotational spring):



Solution: The input is a torque $\tau(t)$, and the output is $\theta(t)$.

$$\sum \tau = J\frac{d^2\theta}{dt^2}$$

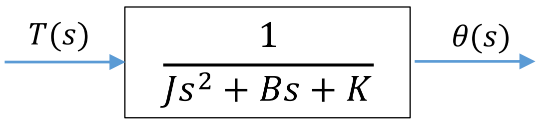
$$\tau(t) - b\frac{d\theta}{dt} - k\theta = J\frac{d^2\theta}{dt^2}$$

$\mathcal{L} \rightarrow$ $$T(s) - Bs\theta - K\theta = Js^2\theta$$

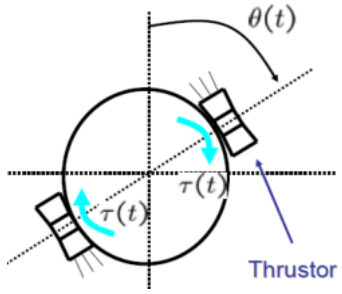
$$\theta(Js^2 + Bs + K) = T(s)$$

$$\theta = \frac{T(s)}{Js^2+Bs+K}$$

Transfer function: $\frac{\theta(s)}{T(s)} = \frac{1}{Js^2+Bs+K}$



Example: Rotational mechanical system: Rigid satellite (inertia only):



Solution: The input is a torque $\tau(t)$, and the output is $\theta(t)$.

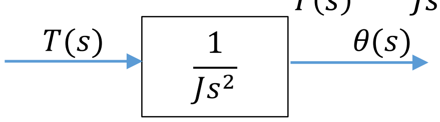
$$\sum \tau = J\frac{d^2\theta}{dt^2}$$

$$\tau(t) = J\frac{d^2\theta}{dt^2}$$

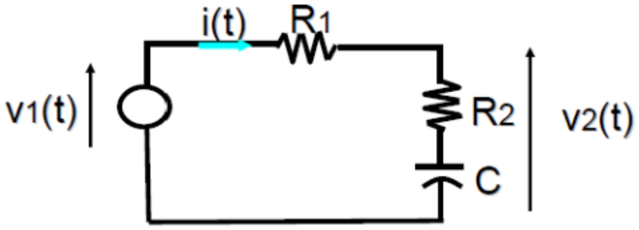
$\mathcal{L} \rightarrow$ $$T(s) = Js^2\theta$$

$$\theta = \frac{T(s)}{Js^2}$$

Transfer function: $\frac{\theta(s)}{T(s)} = \frac{1}{Js^2}$



Example: Electrical system (RC circuit):



Solution: v1(t) is the input, and v2(t) is the output:

Using Kirchhoff voltage law: $\sum v \ in\ the\ loop = 0$

$$v1(t) = v_{R1} + v_{R2} + v_c$$

$$v1(t) = iR_1 + iR_2 + i\int_0^t \frac{1}{C} dt$$

$$\mathcal{L} \rightarrow \quad V1(s) = IR_1 + IR_2 + \frac{I}{sC} \quad \textit{The input}$$

$$v2(t) = iR_2 + i\int_0^t \frac{1}{C} dt$$

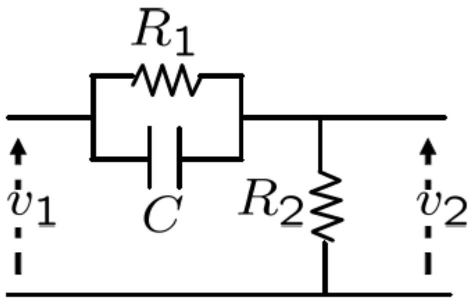
$$\mathcal{L} \rightarrow \quad V2(s) = IR_2 + \frac{I}{sC} \quad \textit{The output}$$

Transfer function:
$$\frac{output}{input} = \frac{V2}{V1} = \frac{IR_2 + \frac{I}{sC}}{IR_1 + IR_2 + \frac{I}{sC}} = \frac{I(R_2 + \frac{1}{sC})}{I(R_1 + R_2 + \frac{1}{sC})}$$

$$= \frac{R_2 + \frac{1}{sC}}{R_1 + R_2 + \frac{1}{sC}} = \frac{\frac{R_2 sC + 1}{sC}}{\frac{(R_1 + R_2)sC + 1}{sC}}$$

$$= \frac{R_2 sC + 1}{(R_1 + R_2)sC + 1}$$

Example: Electrical system(RC circuit):
If the input v1=10, find the steady state value of output v2 .



Solution: The input is v1 = 10, and the output is v2 at steady state.
At steady state, the capacitor will be an open circuit, so no current will pass through it.

$v1 = iR_1 + iR_2$

$\mathcal{L} \rightarrow \quad V1 = IR_1 + IR_2$ *The input.*

$v2 = iR_2$

$\mathcal{L} \rightarrow \quad V2 = IR_2$ *The output.*

Transfer function: $\frac{V2}{V1} = \frac{IR_2}{IR_1+IR_2} = \frac{IR_2}{I(R_1+R_2)} = \frac{R_2}{R_1+R_2}$

But $v1 = 10$, $\mathcal{L} \rightarrow V1 = \frac{10}{s}$

$$\frac{V2}{\frac{10}{s}} = \frac{R_2}{R_1+R_2}$$

$$V2 = \frac{10R_2}{s(R_1+R_2)}$$

Steady state means t $\rightarrow \infty$ *. To find the steady state value of output* $V2$ *we will use the final value theorem:*

$$\lim_{t\to\infty} v2(t) = \lim_{s\to\infty} sV2 = \lim_{s\to\infty} \frac{10R_2}{(R_1+R_2)}$$

$$= \frac{10R_2}{(R_1+R_2)}$$

Modeling of DC motor (Electromechanical system):

DC Motor *is an actuator converting electrical energy into rotational mechanical energy.*

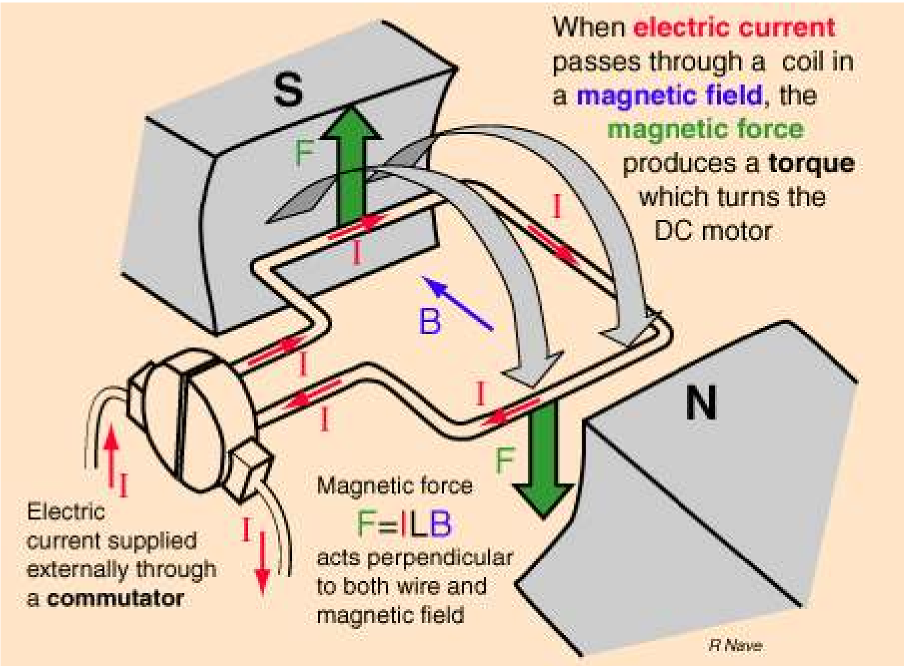
Advantages:

1- High torque.
2- Speed controllability.
3- Portability.

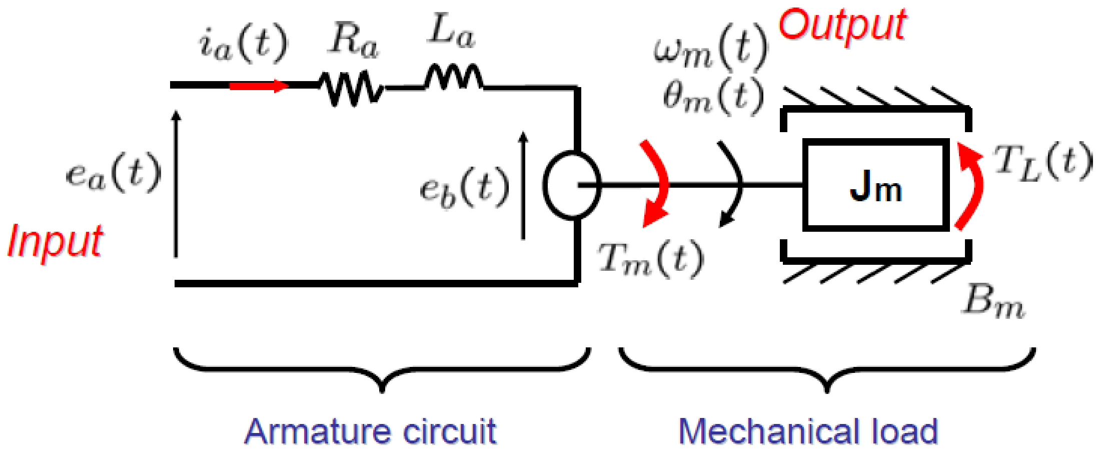
Applications:

1- Robot
2- Tape drivers
3- Printers
4- Machine tool industries
5- Radar tracking system

How does it work?



Example: Modeling of Electromechanical system: DC motor:



Where: e_a: applied voltage e_b: back EMF voltage
T_m:torque of mechanical load.
T_L :disturbance torque.
ω : angular velocity. θ: angular position.

And the connection between two systems are:

$T_m = K_i i$

$e_b = K_b \omega$

$\omega = \theta'$ $\mathcal{L} \rightarrow \ \Omega = \theta s \ \rightarrow \ \theta = \frac{\Omega}{s}$

Solution: 1- For the electrical system:

Input is the voltage $(e_a - e_b)$, and output is current i.

$$e_a = iR + L\frac{di}{dt} + e_b$$

$$e_a - e_b = iR + L\frac{di}{dt}$$

$\mathcal{L} \rightarrow$ $$E_a - E_b = IR + LsI$$

$$I(R + Ls) = E_a - E_b \quad \rightarrow \quad I = \frac{1}{R+Ls}(E_a - E_b) \quad \textit{Equ 1}$$

Transfer function: $\frac{I(s)}{E_a - E_b} = \frac{1}{R+Ls}$

2- For the mechanical system:

Input is torque $(\tau_M - \tau_L)$, output is angular velocity ω.

$\sum \tau = J\omega'$ *ω' is the same of the acceleration θ''*

$$\tau_M - \tau_L - b\omega = J\omega'$$

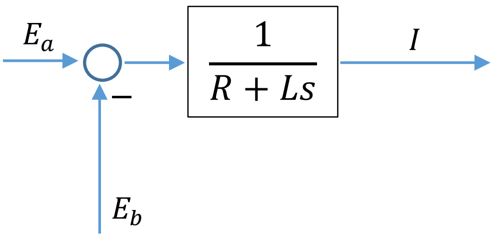
$\mathcal{L} \rightarrow$ $$T_M - T_L - B\Omega = Js\Omega$$

$$\Omega(Js + B) = T_M - T_L \quad \rightarrow \quad \Omega = \frac{1}{Js+B}(T_M - T_L) \quad \textit{Equ 2}$$

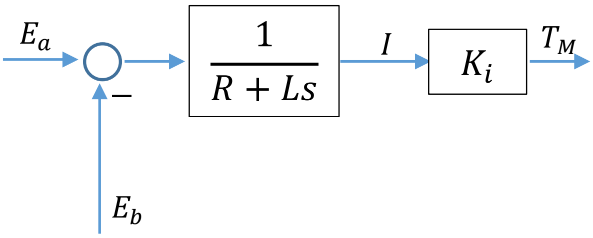
Transfer function: $\frac{\Omega(s)}{(T_M - T_L)} = \frac{1}{Js+B}$

After finding the transfer function for both systems, now we will make the block diagram for the DC motor.

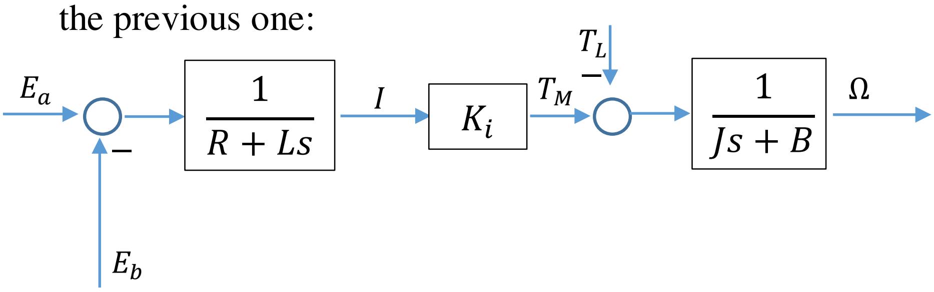
1- Block diagram for electrical system:



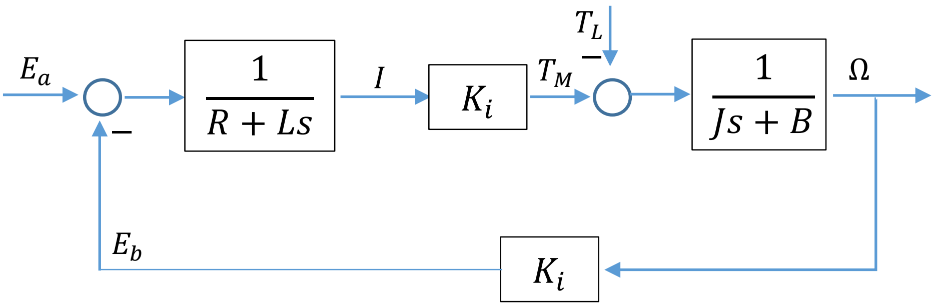
2- From the connecting equation: $T_m = K_i i$, if the current I is multiplied by K_i will give us the torque T_M:



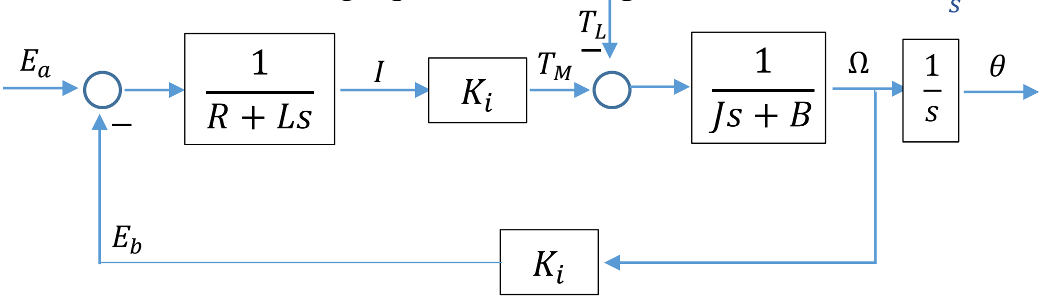
3- Block diagram of mechanical system can be connected now with the previous one:



4- From the connecting equation: $e_b = K_b\omega$:



5- From the connecting equation after Laplace transform: $\theta = \frac{\Omega}{s}$:



Linearization and Modeling of Nonlinear System:

To linearize a nonlinear system we use the first two terms of Taylor series:

Linearization around a point $x = \alpha$:
$y(x) \approx y(\alpha) + y'(\alpha)(x - \alpha)$

Example: Linearize these functions:

1- $y(x) = x^3$ around $x = 2$

2- $y(x) = sin2x$ around $x = 60$

Solution:

1- $y(2) = 2^3 = 8$

$y'(x) = 3x^2 \quad \rightarrow \quad y'(2) = 3(2)^2 = 12$

$\therefore y(x) \approx 8 + 12(x - 2)$

$\approx 12x - 16$

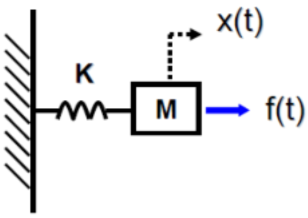
2- $y(60) = sin120 = \frac{\sqrt{3}}{2}$

$y'(x) = 2cos2x \quad \rightarrow \quad y'(60) = 2cos120 = -1$

$\therefore y(x) \approx \frac{\sqrt{3}}{2} - 1(x - 60)$

$\approx -x + 60.87$

Example: Linearization of nonlinear spring:



$M\frac{d^2x}{dt^2} = f(t) - \frac{1}{3}kx^3$, Find the transfer function.

Solution: we must linearize x^3 around $x = \alpha$.

By Taylor series: $x^3 \approx \alpha^3 + 3\alpha^2(x - \alpha)$

The linearized function: $M\frac{d^2x}{dt^2} \approx \underbrace{f(t) - \frac{1}{3}k\alpha^3}_{\hat{f}(t)} - k\alpha^2\underbrace{(x - \alpha)}_{\hat{x}(t)}$

$$M\frac{d^2\hat{x}}{dt^2} \approx \hat{f}(t) - k\alpha^2\hat{x}(t)$$

$\mathcal{L} \rightarrow \quad Ms^2\hat{X} = \hat{F}(s) - K\alpha^2\hat{X}(s) \quad \rightarrow \quad \hat{X}(Ms^2 + K\alpha^2) = \hat{F}(s)$

Transfer function : $\frac{\hat{X}(s)}{\hat{F}(s)} = \frac{1}{Ms^2 + K\alpha^2}$