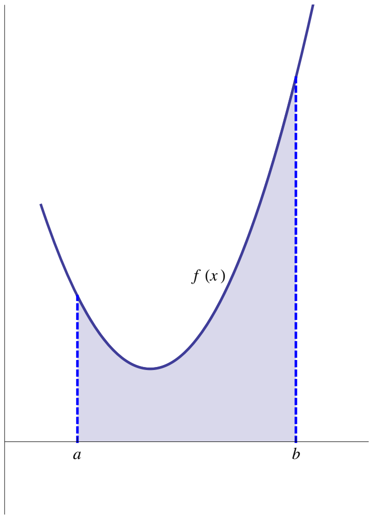
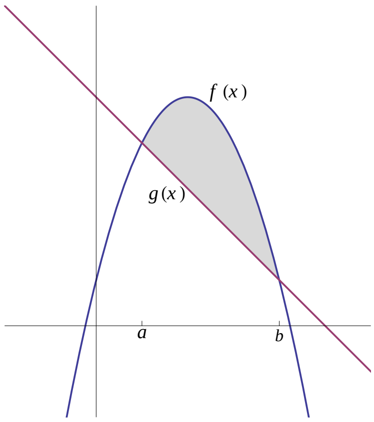
Chapter 5

APPLICATIONS OF INTEGRATION

5.1 Area



In the above figure the area under the graph of $f(x)$ on the interval $[a, b]$ is given by the definite integral $\int_a^b f(x)\ dx$

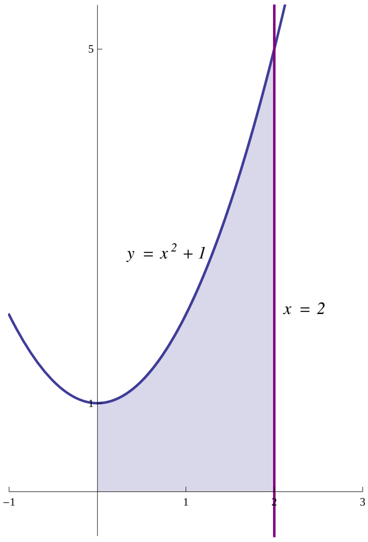


In the above figure the graphs of $f(x)$ and $g(x)$ intersect at the points $x = a$ and $x = b$.
The area bounded by the graphs of the curves of $f(x)$ and $g(x)$ equals

$$\int_a^b f(x)\ dx - \int_a^b g(x)\ dx = \int_a^b [f(x) - g(x)]\ dx$$

Examples :

1. Find the area of the region bounded by the graphs of $x = 0$, $y = 0$, $x = 2$ and $y = x^2 + 1$



$y = x^2 + 1$ is a parabola with vertex $(0, 1)$ and opens upwards.

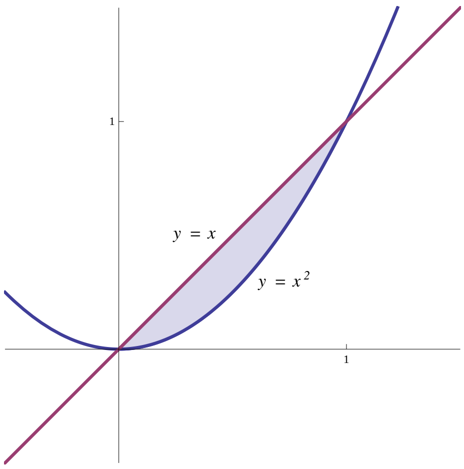
$x = 0$ is the y-axis and $y = 0$ is the x-axis.

$x = 2$ is a straight line parallel to the y-axis and passing through $(2, 0)$

$$\text{Area} = \int_0^2 (x^2 + 1)\, dx = \left[\frac{x^3}{3} + x\right]_0^2$$

$$\text{Area} = \left(\frac{2^3}{3} + 2\right) - \left(\frac{0^3}{3} + 0\right) = \frac{8}{3} + 2 = \frac{14}{3}$$

2. Find the area of the region bounded by the graphs of $y = x$ and $y = x^2$



$y = x^2$ is a parabola with vertex $(0,0)$ and opens upwards.

$y = x$ is a straight line passing through the origin with slope equals 1.

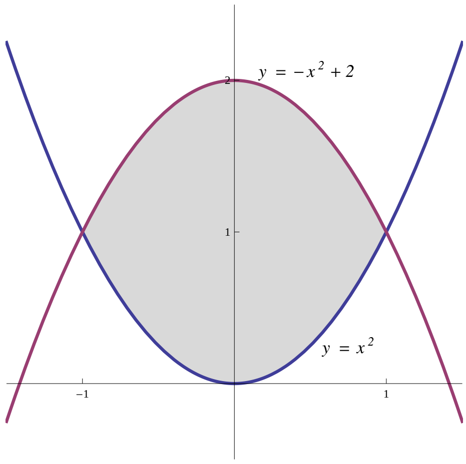
Points of intersection of $y = x^2$ and $y = x$:

$x^2 = x \Rightarrow x^2 - x = 0 \Rightarrow x(x-1) = 0 \Rightarrow x = 0 \, , \, x = 1$

$$\text{Area} = \int_0^1 (x - x^2)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$\text{Area} = \left(\frac{1^2}{2} - \frac{1^3}{3}\right) - \left(\frac{0^2}{2} - \frac{0^3}{3}\right) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

3. Find the area of the region bounded by the graphs of $y = x^2$ and $y = -x^2 + 2$



$y = -x^2 + 2$ is a parabola with vertex $(0,2)$ and opens downwards

$y = x^2$ is a parabola with vertex $(0,0)$ and opens upwards.

Points of intersection of $y = x^2$ and $y = -x^2 + 2$:

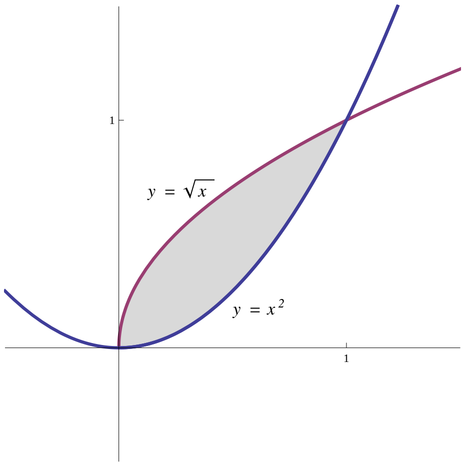
$x^2 = -x^2 + 2 \Rightarrow 2x^2 = 2 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

$$\text{Area} = \int_{-1}^1 \left[(-x^2 + 2) - x^2\right]\, dx = \int_{-1}^1 (2 - 2x^2)\, dx$$

$$\text{Area} = \left[2x - \frac{2x^3}{3}\right]_{-1}^1 = \left[\left(2 - \frac{2}{3}\right) - \left(-2 + \frac{2}{3}\right)\right]$$

$$\text{Area} = 2 - \frac{2}{3} + 2 - \frac{2}{3} = 4 - \frac{4}{3} = \frac{12 - 4}{3} = \frac{8}{3}$$

4. Find the area of the region bounded by the graphs of $y = x^2$ and $y = \sqrt{x}$



$y = x^2$ is a parabola with vertex $(0, 0)$ and opens upwards.

$y = \sqrt{x} \;\Rightarrow\; x = y^2$ is the upper half of the parabola with vertex $(0, 0)$ and opens to the right.

Points of intersection of $y = x^2$ and $y = \sqrt{x}$:

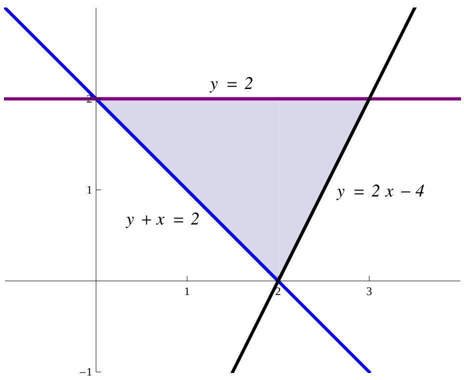
$x^2 = \sqrt{x} \;\Rightarrow\; x^4 = x \;\Rightarrow\; x^4 - x = 0 \;\Rightarrow\; x(x^3 - 1) = 0$

$\Rightarrow\; x = 0 \;, x^3 = 1 \;\Rightarrow\; x = 0 \;,\; x = 1$

$$\text{Area} = \int_0^1 \left(\sqrt{x} - x^2\right)\, dx = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1 = \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1$$

$$\text{Area} = \left(\frac{2}{3} - \frac{1}{3}\right) - (0 - 0) = \frac{1}{3}$$

5. Find the area of the region bounded by the graphs of $x + y = 2$, $y = 2$ and $y = 2x - 4$



$y = 2$, $y = 2x - 4$ and $y = -x + 2$ are three straight lines.

Point of intersection of $y = 2$ and $y = -x + 2$:

$-x + 2 = 2 \;\Rightarrow\; x = 0$

$y = 2$ and $y = -x + 2$ intersect at the point $(0, 2)$.

Point of intersection of $y = 2$ and $y = 2x - 4$:

$2x - 4 = 2 \;\Rightarrow\; x = 3$

$y = 2$ and $y = 2x - 4$ intersect at the point $(3, 2)$

Point of intersection of $y = -x + 2$ and $y = 2x - 4$:

$2x - 4 = -x + 2 \;\Rightarrow\; 3x = 6 \;\Rightarrow\; x = 2$

$y = -x + 2$ and $y = 2x - 4$ intersect at the point $(2, 0)$.

$$\text{Area} = \int_0^2 [2 - (-x + 2)] \; dx + \int_2^3 [2 - (2x - 4)] \; dx$$

$$\text{Area} = \int_0^2 x \; dx + \int_2^3 (6 - 2x) \; dx = \left[\frac{x^2}{2}\right]_0^2 + \left[6x - x^2\right]_2^3$$

$$\text{Area} = \left[\frac{2^2}{2} - \frac{0^2}{2}\right] + \left[(6 \times 3 - 3^2) - (6 \times 2 - 2^2)\right]$$

$$\text{Area} = (2 - 0) + [(18 - 9) - (12 - 4)] = 2 + (9 - 8) = 2 + 1 = 3$$

Another solution :

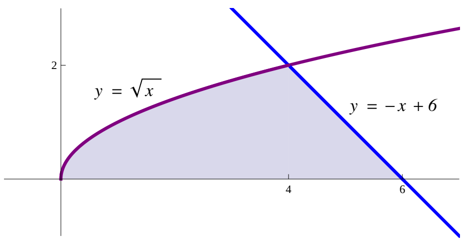
$y + x = 2 \;\Rightarrow\; x = -y + 2$ and $y = 2x - 4 \;\Rightarrow\; 2x = y + 4 \;\Rightarrow\; x = \frac{1}{2}y + 2$

$$\text{Area} = \int_0^2 \left[\left(\frac{1}{2}y + 2\right) - (-y + 2)\right] \; dy$$

$$\text{Area} = \int_0^2 \left(\frac{1}{2}y + y\right) \; dy = \int_0^2 \frac{3}{2}y \; dy$$

$$\text{Area} = \frac{3}{2}\left[\frac{y^2}{2}\right]_0^2 = \frac{3}{2}\left[\frac{2^2}{2} - \frac{0^2}{2}\right] = \frac{3}{2} \times 2 = 3$$

6. Find the area of the region bounded by the graphs of $y = 0$, $y = -x + 6$ and $y = \sqrt{x}$



$y = -x + 6$ is a straight line passing through $(0, 6)$ with slope equals -1.

$y = \sqrt{x} \;\Rightarrow\; x = y^2$ is the upper half of the parabola with vertex $(0, 0)$ and opens to the right.

Points of intersection of $x = y^2$ and $x = -y + 6$:

$y^2 = -y + 6 \;\Rightarrow\; y^2 + y - 6 = 0 \;\Rightarrow\; (y - 2)(y + 3) = 0 \;\Rightarrow\; y = 2\;,\; y = -3$

(Note that $y = -3$ is not in the desired region).

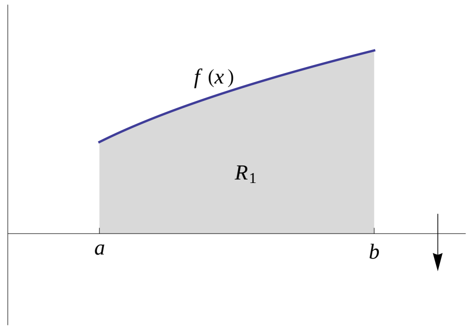
$$\text{Area} = \int_0^2 \left[(-y + 6) - y^2\right]\, dy = \left[6y - \frac{y^2}{2} - \frac{y^3}{3}\right]_0^2$$

$$\text{Area} = \left(12 - \frac{4}{2} - \frac{8}{3}\right) - (0 - 0 - 0) = 12 - 2 - \frac{8}{3} = 10 - \frac{8}{3} = \frac{30 - 8}{3} = \frac{22}{3}$$

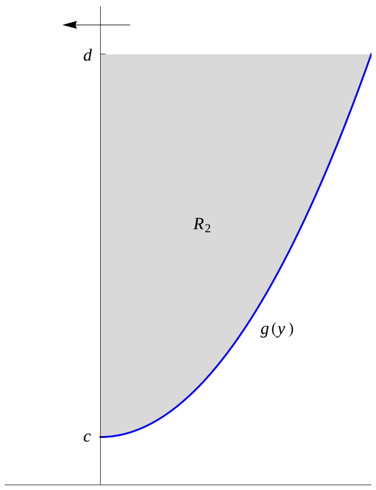
5.2 Volume of a solid of revolution (using disk or washer method)

5.2.1 Disk Method

Recall that the volume of a right circular cylinder equals $\pi r^2 h$ where r is the radius of the base (which is a circle) and h is the height of the cylinder .



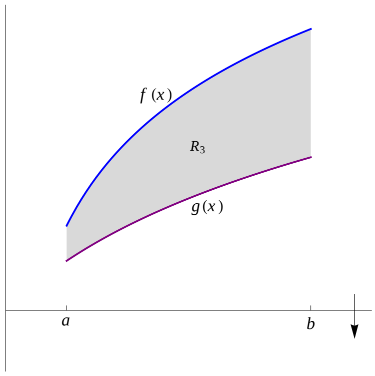
In the above figure R_1 is the region bounded by the graphs of the curves of $f(x)$, $x = a$, $x = b$ and the x-axis.
Using disk method , the volume of the solid of revolution generated by revolving the region R_1 around the x-axis is $V = \pi \int_a^b [f(x)]^2 \ dx$



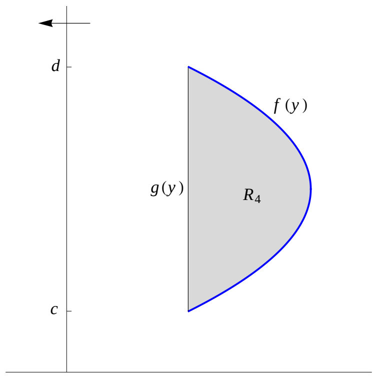
In the above figure R_2 is the region bounded by the graphs of the curves of $g(y)$, $y = d$ and the y-axis.
Using disk method , the volume of the solid of revolution generated by revolving the region R_2 around the y-axis is $V = \pi \int_c^d [g(y)]^2 \ dy$

5.2.2 Washer Method

$Volume\ of\ a\ washer\ = \pi \left[(outer\ radius)^2 - (inner\ radius)^2\right]\ (thickness)$



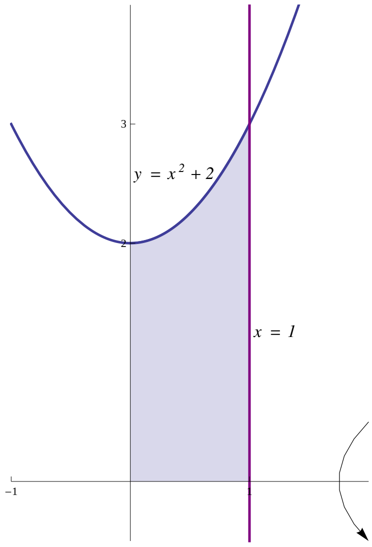
In the above figure R_3 is the region bounded by the graphs of the curves of $f(x)$, $g(x)$, $x = a$ and $x = b$.
Using washer method , the volume of the solid of revolution generated by revolving the region R_3 around the x-axis is $V = \pi \int_a^b \left[(f(x))^2 - (g(x))^2\right]\ dx$



In the above figure R_4 is the region bounded by the graphs of the curves of $f(y)$ and $g(y)$, where $f(y)$ and $g(y)$ intersect at the points $y = c$ and $y = d$.
Using washer method , the volume of the solid of revolution generated by revolving the region R_4 around the y-axis is $V = \pi \int_c^d \left[(f(y))^2 - (g(y))^2\right]\ dy$

Examples : Use Disk or washer method to calculate the volume of the solid of revolution generated by revolving the region bounded by the graphs of :

1. $y = x^2 + 2$, $y = 0$, $x = 0$, $x = 1$, around the x-axis



$y = x^2 + 2$ is a parabola with vertex $(0, 2)$ and opens upwards.

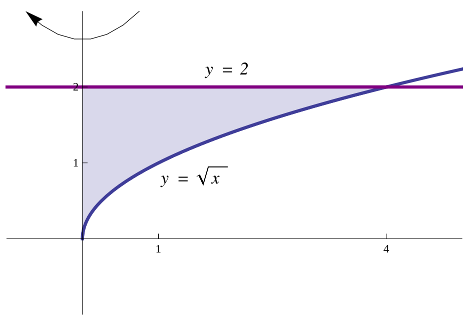
$x = 1$ is a straight line parallel to the y-axis and pasing through $(1, 0)$

Using Disk method :

$$\text{Volume} = \pi \int_0^1 (x^2 + 2)^2 \, dx = \pi \int_0^1 (x^4 + 4x^2 + 4) \, dx$$

$$= \pi \left[\frac{x^5}{5} + \frac{4x^3}{3} + 4x \right]_0^1 = \pi \left[\left(\frac{1}{5} + \frac{4}{3} + 4 \right) - (0 + 0 + 0) \right] = \frac{83\pi}{15}$$

2. $y = \sqrt{x}$, $y = 2$ and $x = 0$, around the y-axis



$y = \sqrt{x}$ is the upper half of the parabola $x = y^2$ with vertex $(0, 0)$ and opens to the right

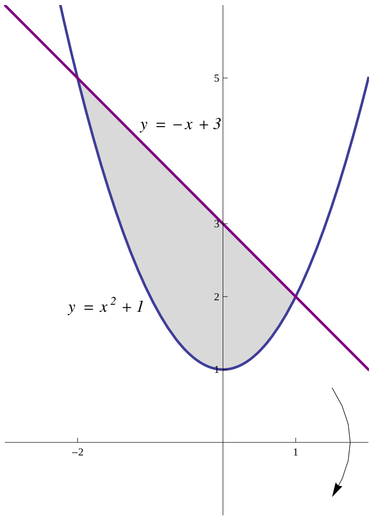
$y = 2$ is a straight line parallel to the x-axis and passing through $(0, 2)$

Using Disk method :

$$\text{Volume} = \pi \int_0^2 (y^2)^2 \; dy = \pi \int_0^2 y^4 \; dy$$

$$= \pi \left[\frac{y^5}{5} \right]_0^2 = \pi \left[\frac{2^5}{5} - 0 \right] = \frac{32\pi}{5}$$

3. $y = x^2 + 1$ and $y = -x + 3$, around the x-axis



$y = x^2 + 1$ is a parabola with vertex $(0, 1)$ and opens upwards.

$y = -x + 3$ is a straight line with slope -1 and passing through $(0, 3)$.

Points of intersection of $y = x^2 + 1$ and $y = -x + 3$:

$x^2 + 1 = -x + 3 \;\Rightarrow\; x^2 + x - 2 = 0 \;\Rightarrow\; (x+2)(x-1) = 0 \;\Rightarrow\; x = -2\,,\; x = 1$

Using Washer method :

$$\text{volume} = \pi \int_{-2}^{1} \left[(-x+3)^2 - (x^2+1)^2 \right] \; dx$$

$$\text{Volume} = \pi \int_{-2}^{1} \left[(x^2 - 6x + 9) - (x^4 + 2x^2 + 1) \right] \; dx$$

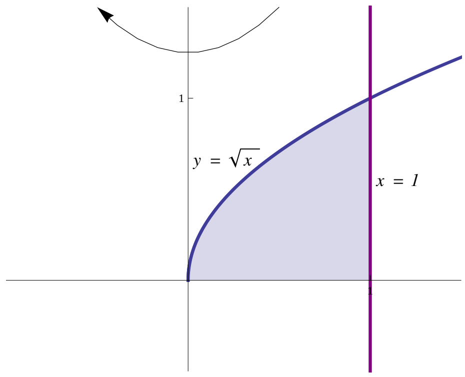
$$\text{Volume} = \pi \int_{-2}^{1} \left(-x^4 - x^2 - 6x + 8 \right) \; dx = \pi \left[-\frac{x^5}{5} - \frac{x^3}{3} - 3x^2 + 8x \right]_{-2}^{1}$$

$$= \pi \left[\left(-\frac{1}{5} - \frac{1}{3} - 3 + 8 \right) - \left(\frac{32}{5} + \frac{8}{3} - 12 - 16 \right) \right]$$

$$= \pi \left(-\frac{1}{5} - \frac{1}{3} + 5 - \frac{32}{5} - \frac{8}{3} + 28 \right)$$

$$= \pi\left(33 - 3 - \frac{33}{5}\right) = \pi\left(30 - \frac{33}{5}\right) = \frac{150-33}{5}\pi = \frac{117\pi}{5}$$

4. $y = \sqrt{x}$, $y = 0$ and $x = 1$, around the y-axis



$y = \sqrt{x}$ is the upper half of the parabola $x = y^2$ with vertex $(0, 0)$ and opens to the right

$x = 1$ is a straight line parallel to the y-axis and passing through $(1, 0)$

Note that $y = \sqrt{x}$ intersects $x = 1$ at the point $(1, 1)$.

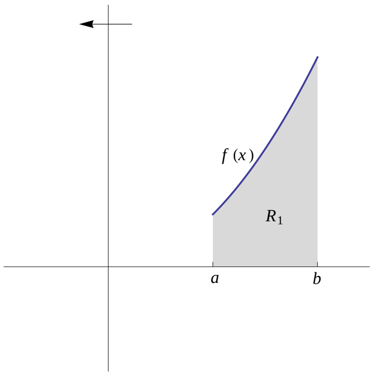
Using Washer method :

$$\text{Volume} = \pi \int_0^1 \left[(1)^2 - (y^2)^2\right]\ dy = \pi \int_0^1 (1 - y^4)\ dy$$

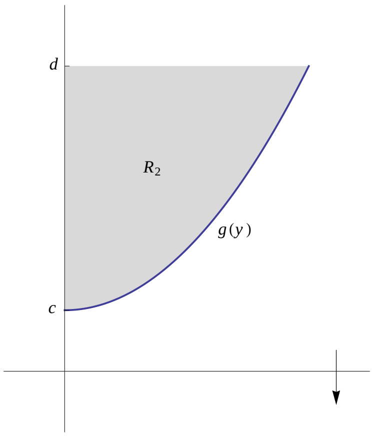
$$= \pi\left[y - \frac{y^5}{5}\right]_0^1 = \pi\left[\left(1 - \frac{1}{5}\right) - (0 - 0)\right] = \pi\left(1 - \frac{1}{5}\right) = \frac{4\pi}{5}$$

5.3 Volume of a solid of revolution (using cylindrical shells method)

Volume of a shell = 2π (average radius) (altitude) (thickness)



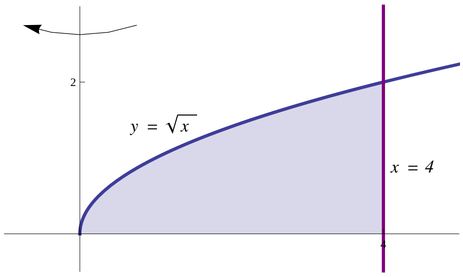
In the above figure R_1 is the region bounded by the graphs of the curves of $f(x)$, $x = a$, $x = b$ and the x-axis.
Using cylindrical shells method , the volume of the solid of revolution generated by revolving the region R_1 around the y-axis is $V = 2\pi \int_a^b x\ f(x)\ dx$



In the above figure R_2 is the region bounded by the graphs of the curves of $g(y)$, $y = d$ and the y-axis.
Using cylindrical shells method , the volume of the solid of revolution generated by revolving the region R_2 around the x-axis is $V = 2\pi \int_c^d y\ g(y)\ dy$

Examples : Use cylindrical shells method to calculate the volume of the solid of revolution generated by revolving the region bounded by the graphs of :

1. $y = \sqrt{x}$, $y = 0$ and $x = 4$, around the y-axis.



$y = 0$ is the x-axis

$y = \sqrt{x}$ is the upper half of the parabola $x = y^2$ with vertex $(0, 0)$ and opens to the right.

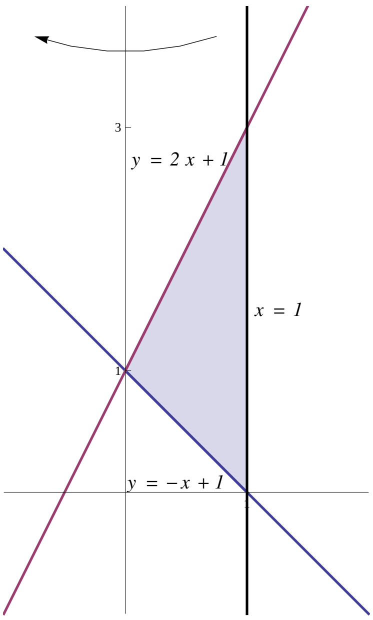
$x = 4$ is a straight line parallel to the y-axis and passing through $(4, 0)$.

Using Cylindrical shells method

$$\text{Volume} = 2\pi \int_0^4 x\sqrt{x}\ dx = 2\pi \int_0^4 x^{\frac{3}{2}}\ dx$$

$$\text{Volume} = 2\pi \left[\frac{2}{5}x^{\frac{5}{2}}\right]_0^4 = 2\pi\ \frac{2}{5}\ (4)^{\frac{5}{2}} = 2\pi\ \frac{2}{5}\ (32) = \frac{128\pi}{5}$$

2. $x + y = 1$, $x = 1$ and $y = 2x + 1$, around the y-axis .



$y = -x + 1$ is a straight line with slope -1 and passing through $(0, 1)$.

$y = 2x + 1$ is a straight line with slope 2 and passing through $(0, 1)$.

$x = 1$ is a straight line parallel to the y-axis and passing through $(1, 0)$.

Point of intersection of $x = 1$ and $y = -x + 1$ is $(1, 0)$.

Point of intersection of $x = 1$ and $y = 2x + 1$ is $(1, 3)$.

Point of intersection of $y = -x + 1$ and $y = 2x + 1$:

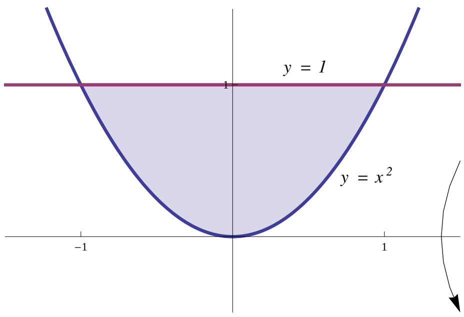
$2x + 1 = -x + 1 \;\Rightarrow\; 3x = 0 \;\Rightarrow\; x = 0.$

Using Cylindrical shells method

$$\text{Volume} = 2\pi \int_0^1 x[(2x+1) - (-x+1)]\ dx = 2\pi \int_0^1 x(3x)\ dx = 2\pi \int_0^1 3x^2\ dx$$

$$\text{Volume} = 2\pi \left[x^3\right]_0^1 = 2\pi[1 - 0] = 2\pi$$

3. $y = x^2$ and $y = 1$, around the x-axis.



$y = x^2$ is a parabola with vertex $(0, 0)$ and opens upwards.

$y = 1$ is a straight line parallel to the x-axis and passing through $(0, 1)$.

Since the bounded region is symmetric with respect to the y-axis, consider the right half of the parabola $y = x^2$ which is $x = \sqrt{y}$.

Using Cylindrical shells method

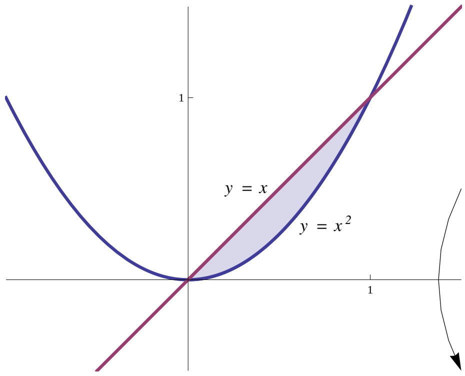
$$\text{Volume} = 2\left(2\pi \int_0^1 y\sqrt{y}\ dy\right) = 4\pi \int_0^1 y^{\frac{3}{2}}\ dy$$

$$\text{Volume} = 4\pi \left[\frac{2}{5} y^{\frac{5}{2}}\right]_0^1 = 4\pi\left(\frac{2}{5} - 0\right) = \frac{8\pi}{5}$$

4. $y = x^2$ and $y = x$, around the x-axis.

$y = x^2$ is a parabola with vertex $(0, 0)$ and opens upwards.

$y = x$ is a straigh line passing through the origin with solpe 1.



Consider $x = \sqrt{y}$ which is the right half of the parabola $y = x^2$.

Points of intersection of $x = \sqrt{y}$ and $x = y$:

$y = \sqrt{y} \;\Rightarrow\; y^2 = y \;\Rightarrow\; y^2 - y = 0 \;\Rightarrow\; y(y-1) = 0 \;\Rightarrow\; y = 0\ ,\ y = 1$

Using Cylindrical shells method

$$\text{Volume} = 2\pi \int_0^1 y\left(\sqrt{y} - y\right)\, dy = 2\pi \int_0^1 \left(y^{\frac{3}{2}} - y^2\right)\, dy$$

$$\text{Volume} = 2\pi \left[\frac{2}{5} y^{\frac{5}{2}} - \frac{y^3}{3}\right]_0^1 = 2\pi \left[\left(\frac{2}{5} - \frac{1}{3}\right) - (0 - 0)\right]$$

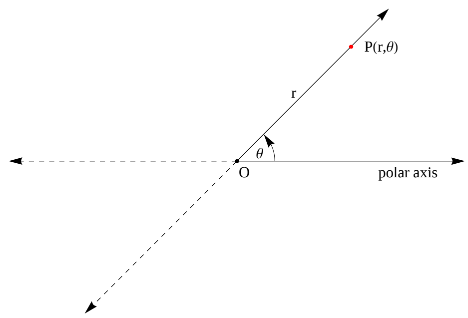
$$\text{Volume} = 2\pi \left(\frac{2}{5} - \frac{1}{3}\right) = 2\pi \left(\frac{6-5}{15}\right) = \frac{2\pi}{15}$$

5.4 Polar Coordinates and Applications

5.4.1 Polar coordinates system :
In the recatangular coordinates system the ordered pair (a, b) represents a point, where "a" is the x-coordinat and "b" is the y-coordinate.

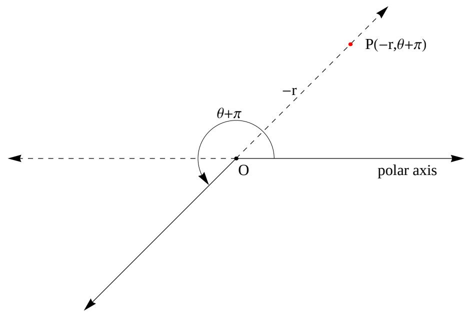
The polar coordinates system can be used also to represents points in the plane. The **pole** in the polar coordinates system is the origin in the rectangular coordinates system , and the **polar axis** is the directed half-line (the non-negative part of the x-axis).

If P is any point in the plane different from the origin, then its polar coordinates consists of two components r and θ , where r is the distance between P and the pole O , and θ is the measure of the angle determined by the polar axis and OP.

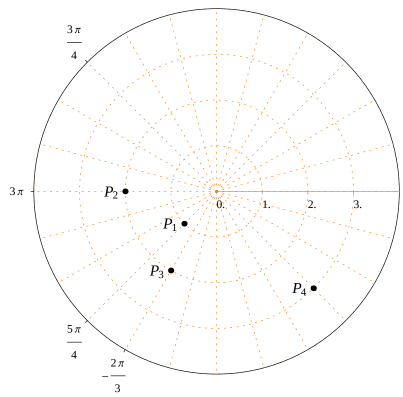


The meaning of polar coordinates (r, θ) can be extended to the case in which r is negative by considering the points (r, θ) and $(-r, \theta)$ lying on the same line through O and at a same distance $|r|$ from O but in opposite directions.

Remark : In this case the representation of a point using polar coordinates is not unique, for instance if $P(r, \theta)$ then other possible represenations are
$(-r, \pi + \theta)$, $(-r, \theta - \pi)$ $(r, \theta - 2\pi)$ and $(r, \theta \pm 2n\pi)$ where $n \in \mathbb{N}$.



Example 1: Plot the points whose polar coordinates are given :
$P_1\left(1, \dfrac{5\pi}{4}\right)$, $P_2(2, 3\pi)$, $P_3\left(2, -\dfrac{2\pi}{3}\right)$ and $P_4\left(-3, \dfrac{3\pi}{4}\right)$.
Solution :



Example 2: Write other polar reprsentations of the point $\left(1, \dfrac{\pi}{4}\right)$.
Solution :

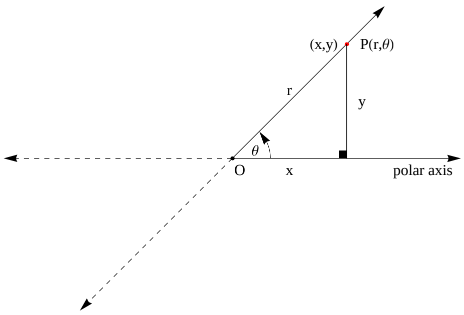
$$\left(-1, \frac{\pi}{4} + \pi\right) = \left(-1, \frac{5\pi}{4}\right).$$

$$\left(-1, \frac{\pi}{4} - \pi\right) = \left(-1, -\frac{3\pi}{4}\right)$$

$$\left(1, \frac{\pi}{4} - 2\pi\right) = \left(1, -\frac{7\pi}{4}\right)$$

$$\left(1, \frac{\pi}{4} + 3\pi\right) = \left(1, \frac{13\pi}{4}\right)$$

5.4.2 Relationship with Cartesian coordinates :



From the above figure , the relationship between the polar and cartesian coordinates is given by the formulas :

$\cos\theta = \frac{x}{r} \implies x = r\cos\theta$

$\sin\theta = \frac{y}{r} \implies y = r\sin\theta$

$r^2 = x^2 + y^2 \implies r = \sqrt{x^2+y^2}$

$\tan\theta = \frac{y}{x} \implies \theta = \tan^{-1}\left(\frac{y}{x}\right)$ where $x \neq 0$.

Examples :

1. Convert the point $\left(2, \frac{\pi}{3}\right)$ from polar to Cartesian coordinates.
2. Convert the point $(1,1)$ from Cartesian to polar coordinates.

Solution :

1. The point $\left(2, \frac{\pi}{3}\right)$ is written in polar coordinates where $r = 2$ and $\theta = \frac{\pi}{3}$

   $x = r\cos\theta = 2\cos\left(\frac{\pi}{3}\right) = 2 \times \frac{1}{2} = 1.$

   $y = r\sin\theta = 2\sin\left(\frac{\pi}{3}\right) = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}.$

   The Cartesian coordinates of the point $\left(2, \frac{\pi}{3}\right)$ is $(1, \sqrt{3})$.

2. The point $(1,1)$ is written in Cartesian coordinates where $x = 1$ and $y = 1$

   $r = \sqrt{x^2+y^2} = \sqrt{(1)^2+(1)^2} = \sqrt{1+1} = \sqrt{2}$

   $\tan\theta = \frac{y}{x} = \frac{1}{1} = 1 \implies \theta = \tan^{-1}(1) = \frac{\pi}{4}$

   The polar coordinates of the point $(1,1)$ is $\left(\sqrt{2}, \frac{\pi}{4}\right)$

5.4.3 Polar curves:
A polar curve is an equation of r and θ of the form $r = r(\theta)$ or $r = f(\theta)$ where $\theta_1 \leq \theta \leq \theta_2$.

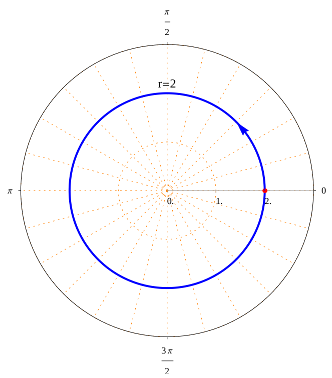
This section focuses on the circles centered at the origin and of radius $a > 0$.
The polar curve $r = a$ where $a > 0$ represents a circle with center $(0, 0)$ and its radius equals a.

Examples : Sketch the following polar curves :

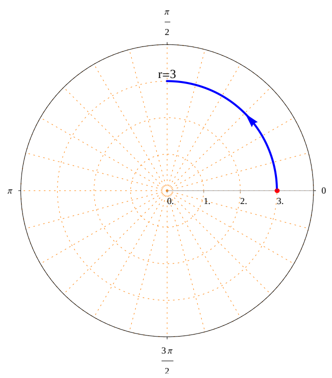
1. $r = 2$ where $0 \leq \theta \leq 2\pi$.
2. $r = 3$ where $0 \leq \theta \leq \dfrac{\pi}{2}$

Solution :

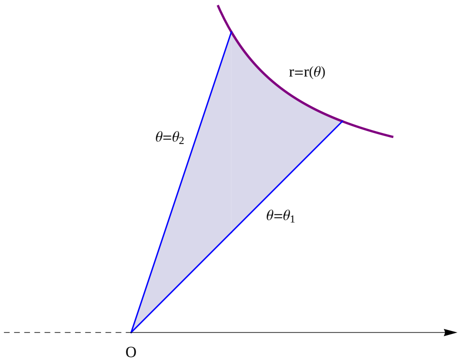
1. $r = 2$ where $0 \leq \theta \leq 2\pi$ represents a whole circle centered at $(0, 0)$ and its radius is 2.



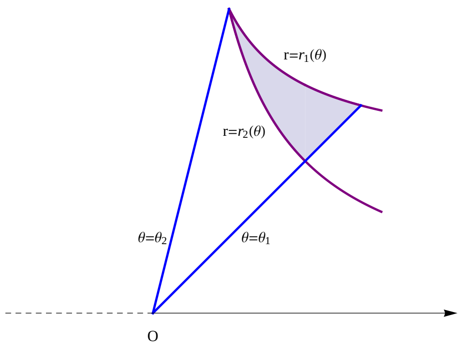
2. $r = 3$ where $0 \leq \theta \leq \dfrac{\pi}{2}$ represents the first quarter of a circle centered at $(0, 0)$ and its radius is 3.



5.4.4 Area with polar coordinates :

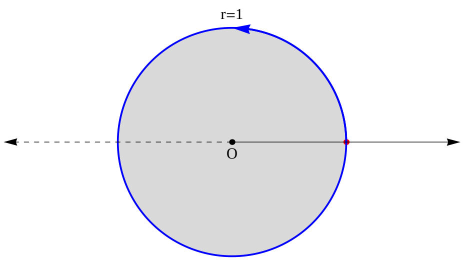


The area of the region bounded by the graph of $r = r(\theta)$, and the two lines $\theta = \theta_1$, $\theta = \theta_2$ is given by the formula
$$\text{Area} = \frac{1}{2}\int_{\theta_1}^{\theta_2} [r(\theta)]^2 \; d\theta$$



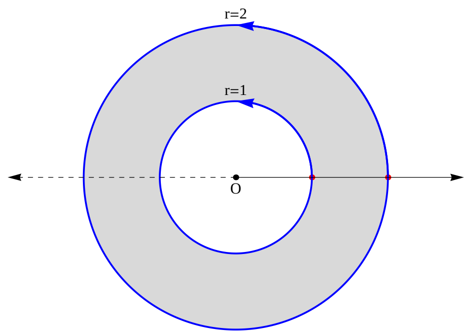
The area of the region bounded by the graphs of $r_1 = r_1(\theta)$, $r_2 = r_2(\theta)$ and the two lines $\theta = \theta_1$, $\theta = \theta_2$ is given by the formula
$$\text{Area} = \frac{1}{2}\int_{\theta_1}^{\theta_2} \left([r_1(\theta)]^2 - [r_2(\theta)]^2\right) \; d\theta$$

Example 1 : Find the area of the region inside the polar curve $r = 1$.
Solution : $r = 1$ is a whole circle centered at $(0, 0)$ and its radius is 1.



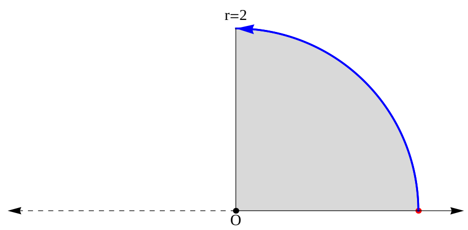
$\text{Area} = \frac{1}{2}\int_0^{2\pi} (1)^2 \; d\theta = \frac{1}{2}\int_0^{2\pi} 1 \; d\theta$
$= \frac{1}{2}\left[\theta\right]_0^{2\pi} = \frac{1}{2}\left[2\pi - 0\right] = \frac{1}{2} \times 2\pi = \pi$

Example 2 : Find the area of the region inside the polar curve $r = 2$ and outside the polar curve $r = 1$.
Solution : $r = 1$ is a whole circle centered at $(0, 0)$ and its radius is 1.
$r = 2$ is a whole circle centered at $(0, 0)$ and its radius is 2.



$\text{Area} = \frac{1}{2}\int_0^{2\pi} \left[(2)^2 - (1)^2\right] \; d\theta = \frac{1}{2}\int_0^{2\pi} (4-1) \; d\theta = \frac{1}{2}\int_0^{2\pi} 3 \; d\theta$
$= \frac{1}{2}\left[3\theta\right]_0^{2\pi} = \frac{1}{2}\left[3 \times 2\pi - 0\right] = \frac{1}{2} \times 6\pi = 3\pi$

Example 3 : Find the area of the region inside the polar curve $r = 2$ and at the first quadrant.
Solution : $r = 2$ is a circle centered at $(0, 0)$ and its radius is 2.
The region in the first quadrant means that it is bounded by the two lines $\theta = 0$ and $\theta = \frac{\pi}{2}$



$\text{Area} = \frac{1}{2}\int_0^{\frac{\pi}{2}} (2)^2 \; d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} 4 \; d\theta$
$= \frac{1}{2}\left[4\theta\right]_0^{\frac{\pi}{2}} = \frac{1}{2}\left[4 \times \frac{\pi}{2} - 0\right] = \frac{1}{2} \times 2\pi = \pi$