Chapter 4
Motion in Two Dimensions

4.1 The Position, Velocity, and Acceleration Vectors

In one dimension, a single numerical value describes a particle's position, but in two dimensions, we indicate its position by its **position vector** $\vec{\mathbf{r}}$, drawn from the origin of some coordinate system to the location of the particle in the xy plane as in Figure 4.1.

We now define the **displacement vector** $\Delta\vec{\mathbf{r}}$ for a particle such as the one in Figure 4.1 as being the difference between its final position vector and its initial position vector:

$$\Delta\vec{\mathbf{r}} \equiv \vec{\mathbf{r}}_f - \vec{\mathbf{r}}_i \qquad (4.1)$$

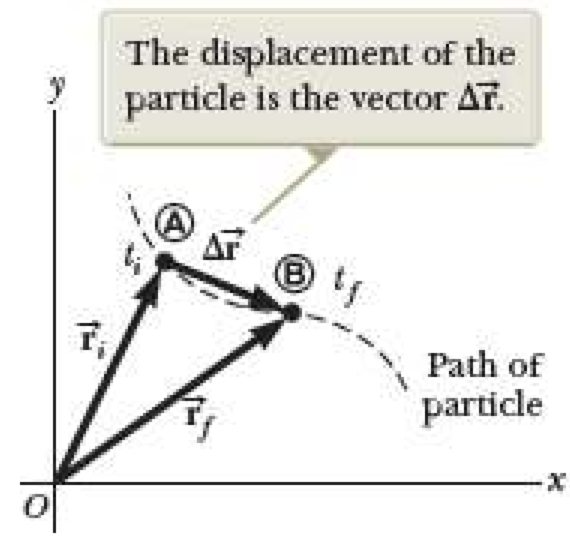


Figure 4.1

We define the **average velocity** $\vec{\mathbf{v}}_{avg}$ of a particle during the time interval Δt *as* the displacement of the particle divided by the time interval:

$$\vec{\mathbf{v}}_{avg} \equiv \frac{\Delta\vec{\mathbf{r}}}{\Delta t} \qquad (4.2)$$

The **instantaneous velocity** $\vec{\mathbf{v}}$ is defined as the limit of the average velocity $\Delta\vec{\mathbf{r}}/\Delta t$ as Δt approaches zero:

$$\vec{\mathbf{v}} \equiv \lim_{\Delta t \to 0} \frac{\Delta\vec{\mathbf{r}}}{\Delta t} = \frac{d\vec{\mathbf{r}}}{dt} \qquad (4.3)$$

The magnitude of the instantaneous velocity vector $v = |\vec{\mathbf{v}}|$ of a particle is called the *speed* of the particle, which is a scalar quantity.

The **average acceleration** $\vec{a}_{avg}$ of a particle is defined as the change in its instantaneous velocity vector $\Delta\vec{v}$ divided by the time interval Δt during which that change occurs:

$$\vec{a}_{avg} \equiv \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i} \qquad (4.4)$$

The **instantaneous acceleration** $\vec{a}$ is defined as the limiting value of the ratio $\Delta\vec{v}/\Delta t$ as Δt approaches zero:

$$\vec{a} \equiv \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} \qquad (4.5)$$

4.2 Two-Dimensional Motion with Constant Acceleration

motion in two dimensions can be modeled as two independent motions in each of the two perpendicular directions associated with the x and y axes. That is, any influence in the y direction does not affect the motion in the x direction and vice versa.
The position vector for a particle moving in the xy plane can be written

$$\vec{r} = x\hat{i} + y\hat{j} \qquad (4.6)$$

where x, y, and $\vec{r}$ change with time as the particle moves while the unit vectors $\hat{i}$ and $\hat{j}$ remain constant. If the position vector is known, the velocity of the particle can be obtained from Equations 4.3 and 4.6, which give

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} = v_x\hat{i} + v_y\hat{j} \qquad (4.7)$$

Substituting, from Equation 2.13, $v_{xf} = v_{xi} + a_x t$ and $v_{yf} = v_{yi} + a_y t$ into Equation 4.7 to determine the final velocity at any time t, we obtain

$$\vec{\mathbf{v}}_f = (v_{xi} + a_x t)\hat{\mathbf{i}} + (v_{yi} + a_y t)\hat{\mathbf{j}} = (v_{xi}\hat{\mathbf{i}} + v_{yi}\hat{\mathbf{j}}) + (a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}})t$$

$$\vec{\mathbf{v}}_f = \vec{\mathbf{v}}_i + \vec{\mathbf{a}}t \qquad (4.8)$$

Similarly, from Equation 2.16 we know that the x and y coordinates of a particle under constant acceleration are

$$x_f = x_i + v_{xi}t + \tfrac{1}{2}a_x t^2 \qquad y_f = y_i + v_{yi}t + \tfrac{1}{2}a_y t^2$$

Substituting these expressions into Equation 4.6 (and labeling the final position vector $\vec{\mathbf{r}}_f$) gives

$$\vec{\mathbf{r}}_f = (x_i + v_{xi}t + \tfrac{1}{2}a_x t^2)\hat{\mathbf{i}} + (y_i + v_{yi}t + \tfrac{1}{2}a_y t^2)\hat{\mathbf{j}}$$

$$= (x_i\hat{\mathbf{i}} + y_i\hat{\mathbf{j}}) + (v_{xi}\hat{\mathbf{i}} + v_{yi}\hat{\mathbf{j}})t + \tfrac{1}{2}(a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}})t^2$$

$$\vec{\mathbf{r}}_f = \vec{\mathbf{r}}_i + \vec{\mathbf{v}}_i t + \tfrac{1}{2}\vec{\mathbf{a}}t^2 \qquad (4.9)$$

Example 4.1 Motion in a Plane

A particle moves in the xy plane, starting from the origin at $t = 0$ with an initial velocity having an x component of 20 m/s and a y component of −15 m/s. The particle experiences an acceleration in the x direction, given by $a_x = 4.0\ m/s^2$.

(A) Determine the total velocity vector at any time.

Solution

To begin the mathematical analysis, we set $v_{xi} = 20$ m/s, $v_{yi} = -15$ m/s, $a_x = 4.0\ m/s^2$, and $a_y = 0$.

Use Equation 4.8 for the velocity vector: $\vec{\mathbf{v}}_f = \vec{\mathbf{v}}_i + \vec{\mathbf{a}}t = (v_{xi} + a_x t)\hat{\mathbf{i}} + (v_{yi} + a_y t)\hat{\mathbf{j}}$

Substitute numerical values with the velocity in meters per second and the time in seconds:

$$\vec{\mathbf{v}}_f = [20 + (4.0)t]\hat{\mathbf{i}} + [-15 + (0)t]\hat{\mathbf{j}}$$

$$(1) \quad \vec{\mathbf{v}}_f = [(20 + 4.0t)\hat{\mathbf{i}} - 15\hat{\mathbf{j}}]$$

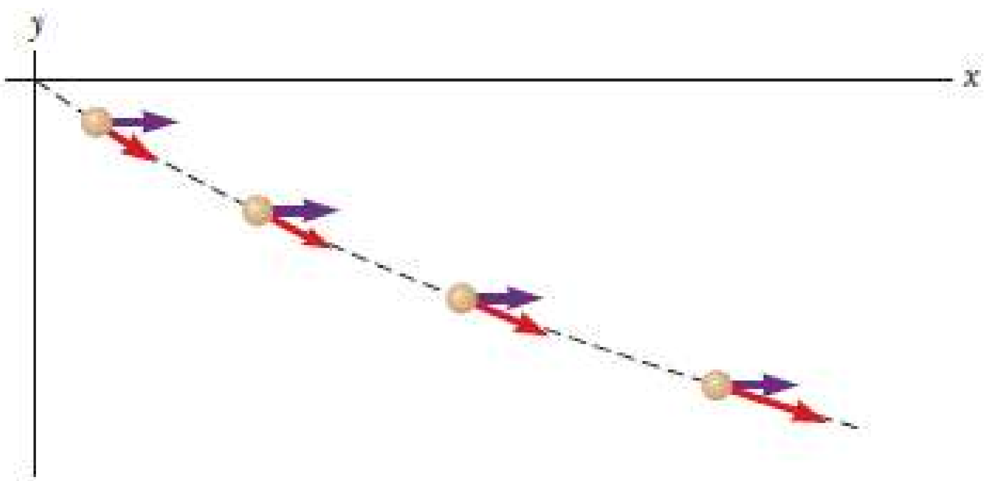


Figure 4.6 (Example 4.1) Motion diagram for the particle.

(B) Calculate the velocity and speed of the particle at $t = 5.0\ s$ and the angle the velocity vector makes with the x axis.

Solution

Evaluate the result from Equation (1) at $t = 5.0\ s$:

$$\vec{v}_f = [(20 + 4.0(5.0))\hat{i} - 15\hat{j}] = (40\hat{i} - 15\hat{j})\ \text{m/s}$$

Determine the angle θ that $\vec{v}_f$ makes with the x axis at t = 5.0 s:

$$\theta = \tan^{-1}\left(\frac{v_{yf}}{v_{xf}}\right) = \tan^{-1}\left(\frac{-15\ \text{m/s}}{40\ \text{m/s}}\right) = -21°$$

Evaluate the speed of the particle as the magnitude of $\vec{v}_f$:

$$v_f = |\vec{v}_f| = \sqrt{v_{xf}^2 + v_{yf}^2} = \sqrt{(40)^2 + (-15)^2}\ \text{m/s} = 43\ \text{m/s}$$

(C) Determine the x and y coordinates of the particle at any time t and its position vector at this time.

Solution

Use the components of Equation 4.9 with $x_i = y_i = 0$ at $t = 0$ and with x and y in meters and t in seconds:

$$x_f = v_{xi}t + \tfrac{1}{2}a_x t^2 = 20t + 2.0t^2$$

$$y_f = v_{yi}t = -15t$$

Express the position vector of the particle at any time t: $\vec{r}_f = x_f\hat{i} + y_f\hat{j} = (20t + 2.0t^2)\hat{i} - 15t\hat{j}$

4.3 Projectile Motion

Projectile motion of an object is simple to analyze if we make two assumptions: (1) the free-fall acceleration is constant over the range of motion and is directed downward, and (2) the effect of air resistance is negligible. With these assumptions, we find that the path of a projectile, which we call its trajectory, is always a parabola as shown in Figure 4.7.

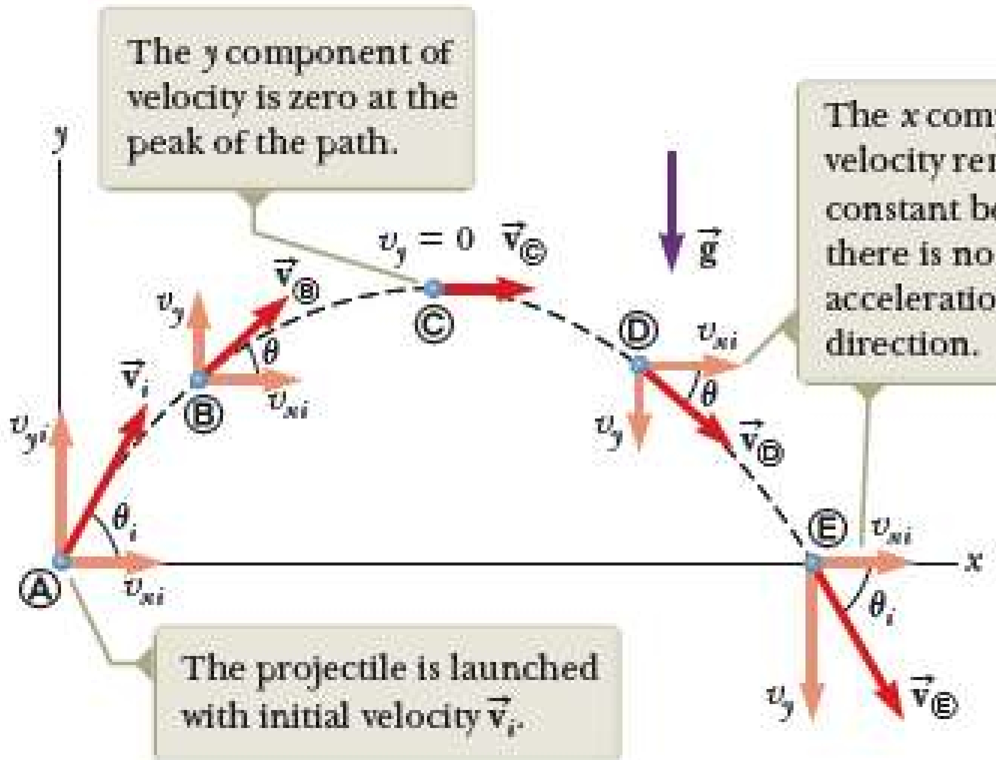


Figure 4.7

Therefore, when solving projectile motion problems, use two analysis models: **(1)** the particle under constant velocity in the horizontal direction (Eq. 2.7):

$$x_f = x_i + v_{xi}t$$

and **(2)** the particle under constant acceleration in the vertical direction (Eqs. 2.13–2.17 with x changed to y and ay = –g):

$$v_{yf} = v_{yi} - gt$$

$$v_{y,\text{avg}} = \frac{v_{yi} + v_{yf}}{2}$$

$$y_f = y_i + \tfrac{1}{2}(v_{yi} + v_{yf})t$$

$$y_f = y_i + v_{yi}t - \tfrac{1}{2}gt^2$$

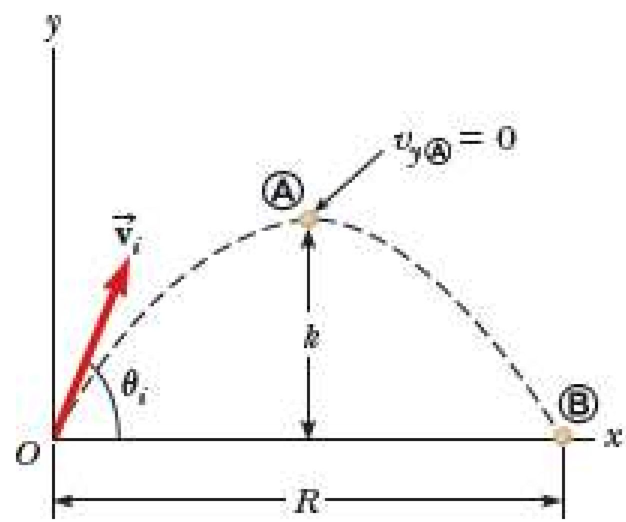
$$v_{yf}^2 = v_{yi}^2 - 2g(y_f - y_i)$$

The horizontal and vertical components of a projectile's motion are completely independent of each other and can be handled separately, with time t as the common variable for both components.

Horizontal Range and Maximum Height of a Projectile

Before embarking on some examples, let us consider a special case of projectile motion that occurs often. Assume a projectile is launched from the origin at $t_i = 0$ with a positive v_{yi} component as shown in Figure 4.9 and returns to *the same horizontal level.* This situation is common in sports, where baseballs, footballs, and golf balls often land at the same level from which they were launched.

Figure 4.9



Two points in this motion are especially interesting to analyze: the peak point Ⓐ, which has Cartesian coordinates $(R/2, h)$, and the point Ⓑ, which has coordinates $(R, 0)$. The distance R is called the *horizontal range* of the projectile, and the distance h is its *maximum height*. Let us find h and R mathematically in terms of v_i, θ_i, and g.

We can determine h by noting that at the peak $v_{y\text{Ⓐ}} = 0$. Therefore, from the particle under constant acceleration model, we can use the y direction version of Equation 2.13 to determine the time $t_{\text{Ⓐ}}$ at which the projectile reaches the peak:

$$v_{yf} = v_{yi} - gt \quad \rightarrow \quad 0 = v_i \sin\theta_i - gt_{\text{Ⓐ}}$$

$$t_{\text{Ⓐ}} = \frac{v_i \sin\theta_i}{g}$$

Substituting this expression for $t_{\text{Ⓐ}}$ into the y direction version of Equation 2.16 and replacing $y_f = y_{\text{Ⓐ}}$ with h, we obtain an expression for h in terms of the magnitude and direction of the initial velocity vector:

$$y_f = y_i + v_{yi}t - \tfrac{1}{2}gt^2 \quad \rightarrow \quad h = (v_i \sin\theta_i)\frac{v_i \sin\theta_i}{g} - \tfrac{1}{2}g\left(\frac{v_i \sin\theta_i}{g}\right)^2$$

$$h = \frac{v_i^2 \sin^2\theta_i}{2g} \qquad \textbf{(4.12)}$$

The range R is the horizontal position of the projectile at a time that is twice the time at which it reaches its peak, that is, at time $t_{\text{Ⓑ}} = 2t_{\text{Ⓐ}}$. Using the particle under constant velocity model, noting that $v_{xi} = v_{x\text{Ⓑ}} = v_i \cos\theta_i$, and setting $x_{\text{Ⓑ}} = R$ at $t = 2t_{\text{Ⓐ}}$, we find that

$$x_f = x_i + v_{xi}t \rightarrow R = v_{xi}t_{Ⓑ} = (v_i \cos\theta_i)2t_{Ⓐ}$$

$$= (v_i \cos\theta_i)\frac{2v_i \sin\theta_i}{g} = \frac{2v_i^2 \sin\theta_i \cos\theta_i}{g}$$

Using the identity $\sin 2\theta = 2\sin\theta\cos\theta$ (see Appendix B.4), we can write R in the more compact form

$$R = \frac{v_i^2 \sin 2\theta_i}{g} \quad \textbf{(4.13)}$$

The maximum value of R from Equation 4.13 is $R_{max} = v_i^2/g$. This result makes sense because the maximum value of $\sin 2\theta_i$ is 1, which occurs when $2\theta_i = 90°$. Therefore, R is a maximum when $\theta_i = 45°$.

Figure 4.10 illustrates various trajectories for a projectile having a given initial speed but launched at different angles.

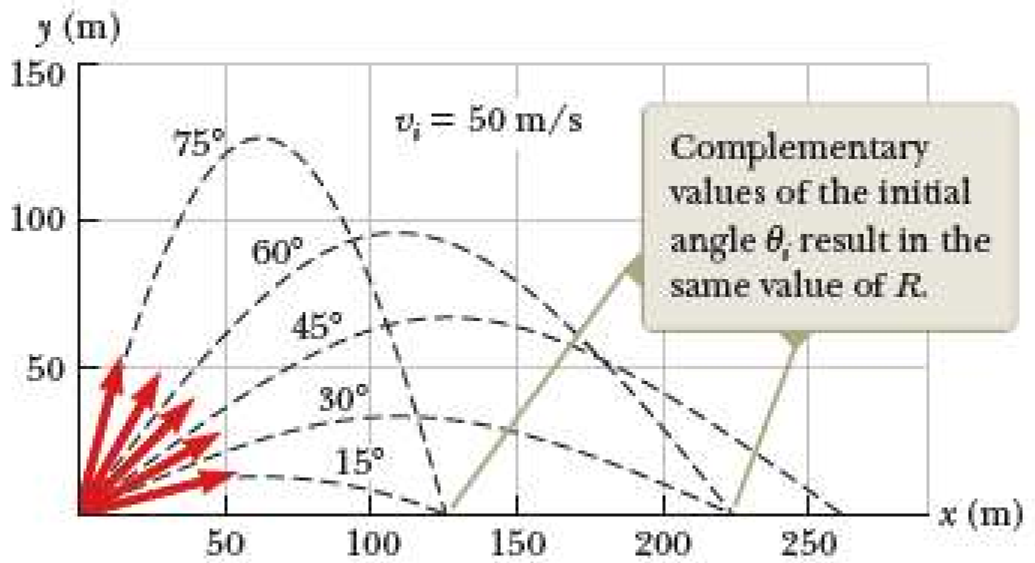


Figure 4.10

Example 4.2 The Long Jump

A long jumper (Fig. 4.11) leaves the ground at an angle of 20.0° above the horizontal and at a speed of 11.0 m/s.

(A) How far does he jump in the horizontal direction?

Figure 4.11

Solution

Use Equation 4.13 to find the range of the jumper:

$$R = \frac{v_i^2 \sin 2\theta_i}{g} = \frac{(11.0\ \text{m/s})^2 \sin 2(20.0°)}{9.80\ \text{m/s}^2} = 7.94\ \text{m}$$

(B) What is the maximum height reached?

Solution

Find the maximum height reached by using Equation 4.12:

$$h = \frac{v_i^2 \sin^2\theta_i}{2g} = \frac{(11.0\ \text{m/s})^2 (\sin 20.0°)^2}{2(9.80\ \text{m/s}^2)} = 0.722\ \text{m}$$

Example 4.4

A stone is thrown from the top of a building upward at an angle of 30.0° to the horizontal with an initial speed of 20.0 m/s as shown in Figure 4.13. The height from which the stone is thrown is 45.0 m above the ground.

(A) How long does it take the stone to reach the ground?

Solution

Find the initial *x and y components of the stone's* velocity:

$$v_{xi} = v_i \cos\theta_i = (20.0\ \text{m/s}) \cos 30.0^\circ = 17.3\ \text{m/s}$$

$$v_{yi} = v_i \sin\theta_i = (20.0\ \text{m/s}) \sin 30.0^\circ = 10.0\ \text{m/s}$$

Express the vertical position of the stone from the particle under constant acceleration model:

$$y_f = y_i + v_{yi}t - \tfrac{1}{2}gt^2$$

Substitute numerical values:

$$-45.0\ \text{m} = 0 + (10.0\ \text{m/s})t + \tfrac{1}{2}(-9.80\ \text{m/s}^2)t^2$$

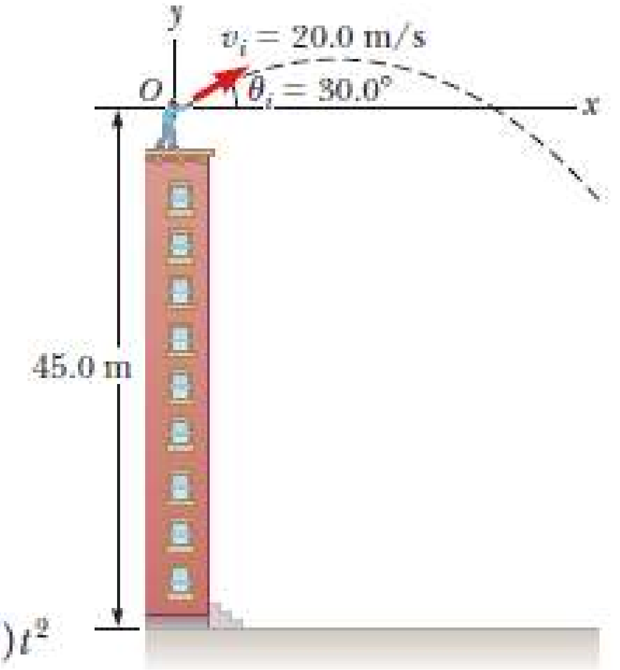


Figure 4.13

Solve the quadratic equation for *t:*

$$t = 4.22\ \text{s}$$

(B) What is the speed of the stone just before it strikes the ground?

Use the velocity equation in the particle under constant acceleration model to obtain the *y* component of the velocity of the stone just before it strikes the ground:

$$v_{yf} = v_{yi} - gt$$

Substitute numerical values, using *t = 4.22 s:*

$$v_{yf} = 10.0\ \text{m/s} + (-9.80\ \text{m/s}^2)(4.22\ \text{s}) = -31.3\ \text{m/s}$$

Use this component with the horizontal component *vxf = vxi = 17.3 m/s to find the speed of the* stone at *t = 4.22 s:*

$$v_f = \sqrt{v_{xf}^2 + v_{yf}^2} = \sqrt{(17.3\ \text{m/s})^2 + (-31.3\ \text{m/s})^2} = 35.8\ \text{m/s}$$

Example 4.5 The End of the Ski Jump

A ski jumper leaves the ski track moving in the horizontal direction with a speed of 25.0 m/s as shown in Figure 4.14. The landing incline below her falls off with a slope of 35.0°. Where does she land on the incline?

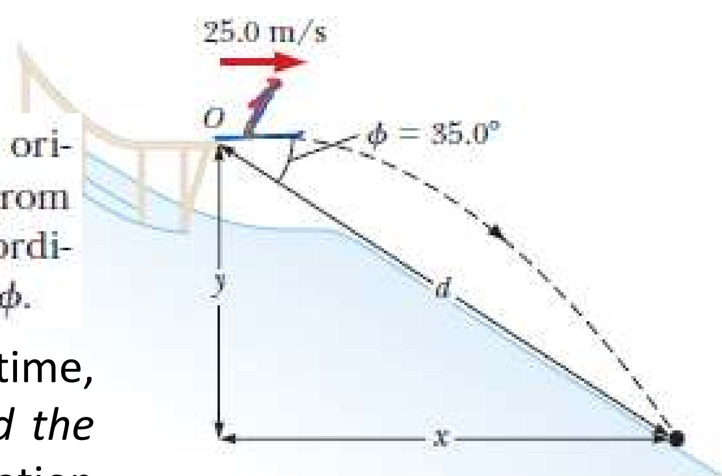


Figure 4.14

Solution

Analyze It is convenient to select the beginning of the jump as the origin. The initial velocity components are $v_{xi} = 25.0$ m/s and $v_{yi} = 0$. From the right triangle in Figure 4.14, we see that the jumper's x and y coordinates at the landing point are given by $x_f = d\cos\phi$ and $y_f = -d\sin\phi$.

Express the coordinates of the jumper as a function of time, using the particle under constant velocity model for *x and the position equation from the particle under* constant acceleration model for *y:*

$$(1)\quad x_f = v_{xi}t$$

$$(2)\quad y_f = v_{yi}t - \tfrac{1}{2}gt^2$$

$$(3)\quad d\cos\phi = v_{xi}t$$

$$(4)\quad -d\sin\phi = -\tfrac{1}{2}gt^2$$

Solve Equation (3) for *t and substitute the result into* Equation (4):

$$-d\sin\phi = -\tfrac{1}{2}g\left(\frac{d\cos\phi}{v_{xi}}\right)^2$$

Solve for *d and substitute numerical values:*

$$d = \frac{2v_{xi}^2\sin\phi}{g\cos^2\phi} = \frac{2(25.0\ \mathrm{m/s})^2\sin 35.0^\circ}{(9.80\ \mathrm{m/s^2})\cos^2 35.0^\circ} = 109\ \mathrm{m}$$

Evaluate the *x and y coordinates of the point at which* the skier lands:

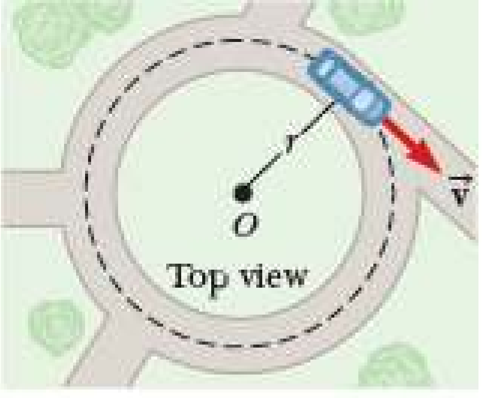
$$x_f = d\cos\phi = (109\ \mathrm{m})\cos 35.0^\circ = 89.3\ \mathrm{m}$$

$$y_f = -d\sin\phi = -(109\ \mathrm{m})\sin 35.0^\circ = -62.5\ \mathrm{m}$$

4.4 Analysis Model: Particle in Uniform Circular Motion

Figure 4.15a shows a car moving in a circular path; we describe this motion by calling it **circular motion**. If the car is moving on this path with constant speed v, we call it **uniform circular motion**. Because it occurs so often, this type of motion is recognized as an analysis model called the **particle in uniform circular motion.**

It is often surprising to students to find that even though an object moves at a constant speed in a circular path, it still has an acceleration. To see why, consider the defining equation for acceleration, $\vec{a} = d\vec{v}/dt$ (Eq. 4.5). Notice that the acceleration depends on the change in the velocity. Because velocity is a vector quantity, an acceleration can occur in two ways as mentioned in Section 4.1: by a change in the magnitude of the velocity and by a change in the direction of the velocity. The latter situation occurs for an object moving with constant speed in a circular path.



a

Figure 4.15

The magnitude of the acceleration is:

$$a_c = \frac{v^2}{r} \qquad (4.14)$$

An acceleration of this nature is called a **centripetal acceleration** (centripetal means center-seeking). The subscript on the acceleration symbol reminds us that the acceleration is centripetal.

In many situations, it is convenient to describe the motion of a particle moving with constant speed in a circle of radius r in terms of the **period T,** which is defined as the time interval required for one complete revolution of the particle. In the time interval T, the particle moves a distance of 2pr, which is equal to the circumference of the particle's circular path. Therefore, because its speed is equal to the circumference of the circular path divided by the period, or $v = 2\pi r/T$, it follows that:

$$T = \frac{2\pi r}{v} \quad (4.15)$$

The period of a particle in uniform circular motion is a measure of the number of seconds for one revolution of the particle around the circle. The inverse of the period is the rotation rate and is measured in revolutions per second. Because one full revolution of the particle around the circle corresponds to an angle of 2 π radians, the product of 2π and the rotation rate gives the **angular speed** ω of the particle, measured in radians/s or s-1:

$$\omega = \frac{2\pi}{T} \quad (4.16)$$

Combining this equation with Equation 4.15, we find a relationship between angular speed and the translational speed with which the particle travels in the circular path:

$$\omega = 2\pi\left(\frac{v}{2\pi r}\right) = \frac{v}{r} \quad \rightarrow \quad v = r\omega \quad (4.17)$$

We can express the centripetal acceleration of a particle in uniform circular motion in terms of angular speed by combining Equations 4.14 and 4.17:

$$a_c = \frac{(r\omega)^2}{r}$$

$$a_c = r\omega^2 \quad (4.18)$$

Example 4.6 The Centripetal Acceleration of the Earth

(A) What is the centripetal acceleration of the Earth as it moves in its orbit around the Sun?

Solution

We do not know the orbital speed of the Earth to substitute into Equation 4.14. With the help of Equation 4.15, however, we can recast Equation 4.14 in terms of the period of the Earth's orbit, which we know is one year, and the radius of the Earth's orbit around the Sun, which is 1.496×10^{11} m.

Combine Equations 4.14 and 4.15:

$$a_c = \frac{v^2}{r} = \frac{\left(\frac{2\pi r}{T}\right)^2}{r} = \frac{4\pi^2 r}{T^2}$$

Substitute numerical values:

$$a_c = \frac{4\pi^2(1.496 \times 10^{11}\text{ m})}{(1\text{ yr})^2}\left(\frac{1\text{ yr}}{3.156 \times 10^7\text{ s}}\right)^2 = 5.93 \times 10^{-3}\text{ m/s}^2$$

(B) What is the angular speed of the Earth in its orbit around the Sun?

Substitute numerical values into Equation 4.16:

$$\omega = \frac{2\pi}{1\text{ yr}}\left(\frac{1\text{ yr}}{3.156 \times 10^7\text{ s}}\right) = 1.99 \times 10^{-7}\text{s}^{-1}$$

4.5 Tangential and Radial Acceleration

Let us consider a more general motion than that presented in Section 4.4. A particle moves to the right along a curved path, and its velocity changes both in direction and in magnitude as described in Figure 4.16.

The total acceleration vector $\vec{a}$ can be written as the vector sum of the component vectors:

$$\vec{a} = \vec{a}_r + \vec{a}_t \qquad (4.19)$$

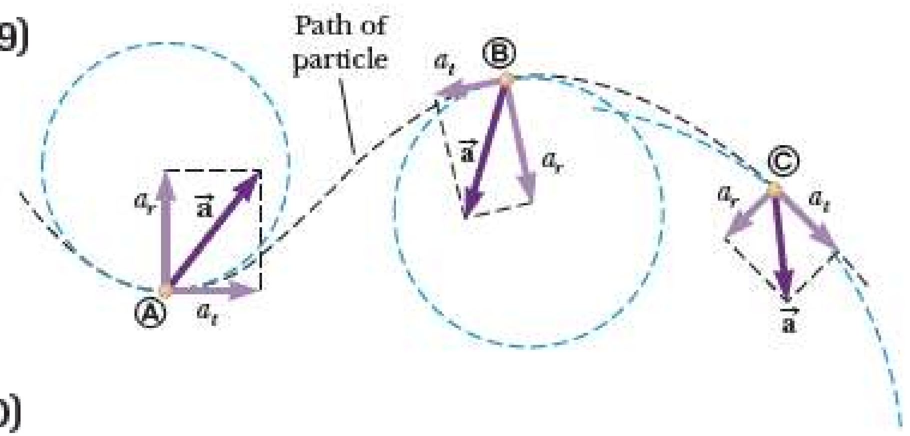


Figure 4.16

The tangential acceleration component causes a change in the speed v of the particle. This component is parallel to the instantaneous velocity, and its magnitude is given by

$$a_t = \left|\frac{dv}{dt}\right| \qquad (4.20)$$

The radial acceleration component arises from a change in direction of the velocity vector and is given by

$$a_r = -a_c = -\frac{v^2}{r} \qquad (4.21)$$

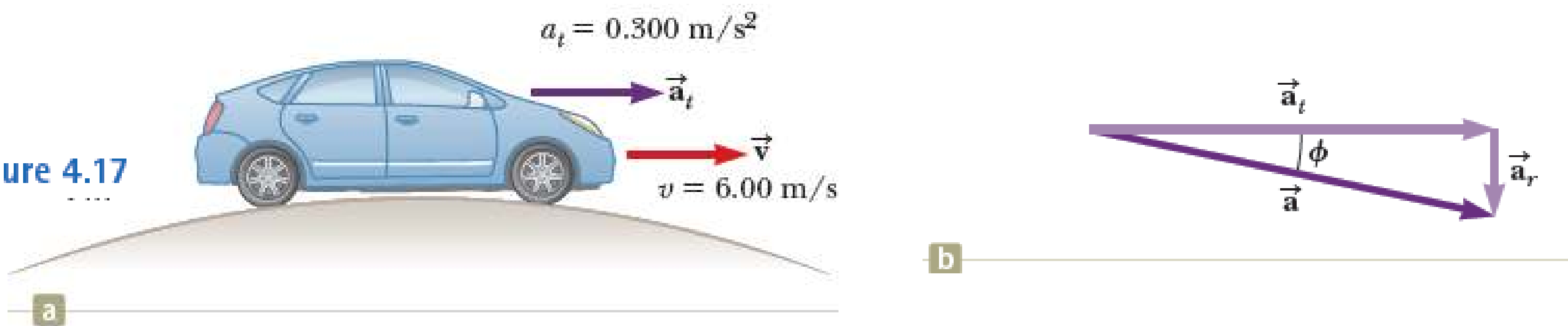
Because $\vec{a}_r$ and $\vec{a}_t$ are perpendicular component vectors of $\vec{a}$, it follows that the magnitude of $\vec{a}$ is $a = \sqrt{a_r^2 + a_t^2}$

Example 4.7 Over the Rise

A car leaves a stop sign and exhibits a constant acceleration of 0.300 m/s2 parallel to the roadway. The car passes over a rise in the roadway such that the top of the rise is shaped like an arc of a circle of radius 500 m. At the moment the car is at the top of the rise, its velocity vector is horizontal and has a magnitude of 6.00 m/s. What are the magnitude and direction of the total acceleration vector for the car at this instant?

Solution

Figure 4.17



The tangential acceleration vector has magnitude 0.300 m/s2 and is horizontal. The radial acceleration is given by Equation 4.21, with v = 6.00 m/s and r = 500 m. The radial acceleration vector is directed straight downward.

Evaluate the radial acceleration:

$$a_r = -\frac{v^2}{r} = -\frac{(6.00 \text{ m/s})^2}{500 \text{ m}} = -0.072\,0 \text{ m/s}^2$$

Find the magnitude of $\vec{a}$:

$$\sqrt{a_r^2 + a_t^2} = \sqrt{(-0.072\,0 \text{ m/s}^2)^2 + (0.300 \text{ m/s}^2)^2}$$

$$= 0.309 \text{ m/s}^2$$

Find the angle ϕ (see Fig. 4.17b) between $\vec{a}$ and the horizontal:

$$\phi = \tan^{-1}\frac{a_r}{a_t} = \tan^{-1}\left(\frac{-0.072\,0 \text{ m/s}^2}{0.300 \text{ m/s}^2}\right) = -13.5°$$