ENGINEERING MECHANICS – 1
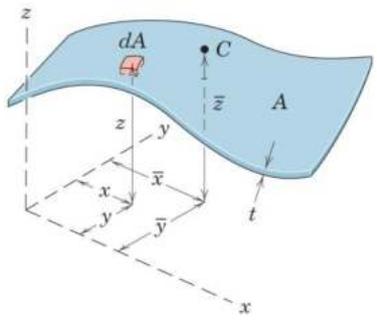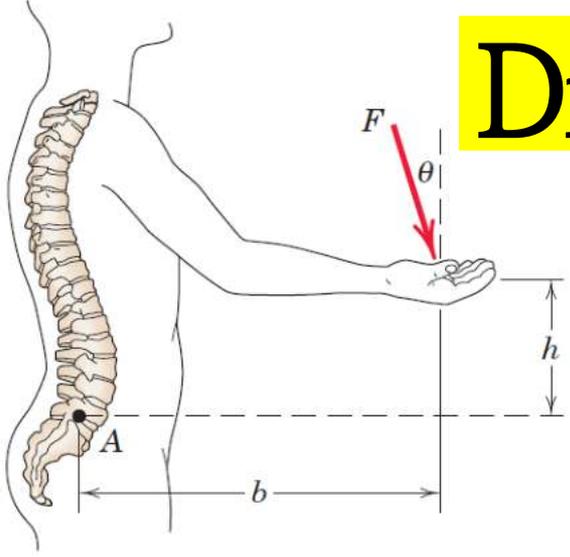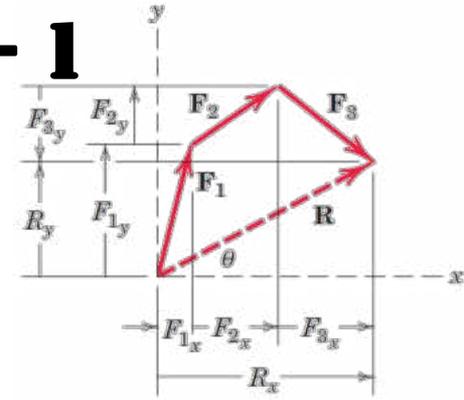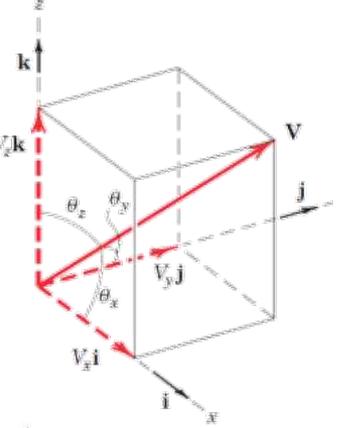# ENG 203

CHAPTER - 5
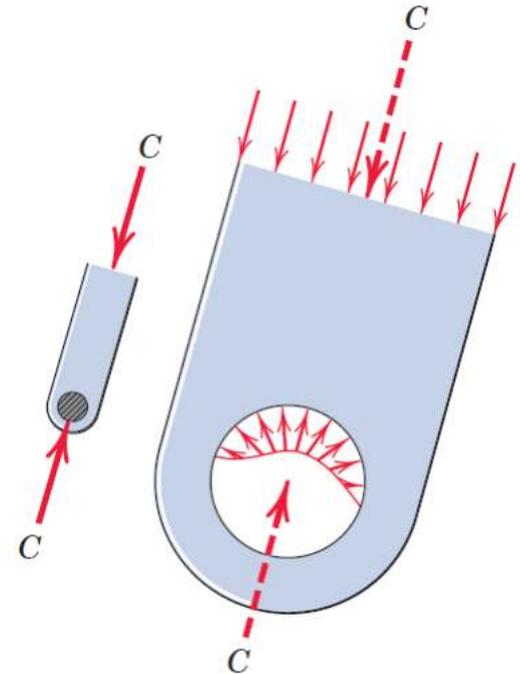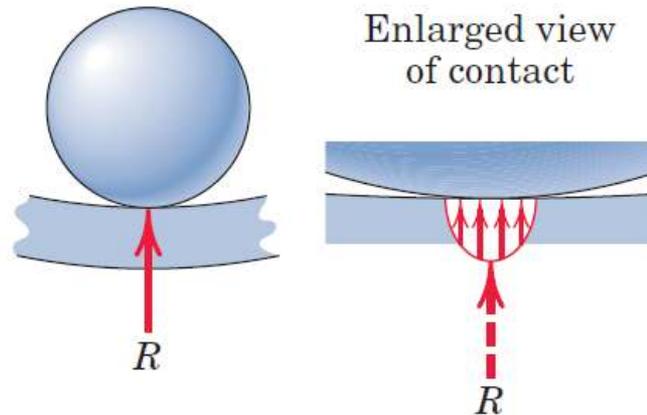# Distributed Forces

جامعة تبوك
University of Tabuk
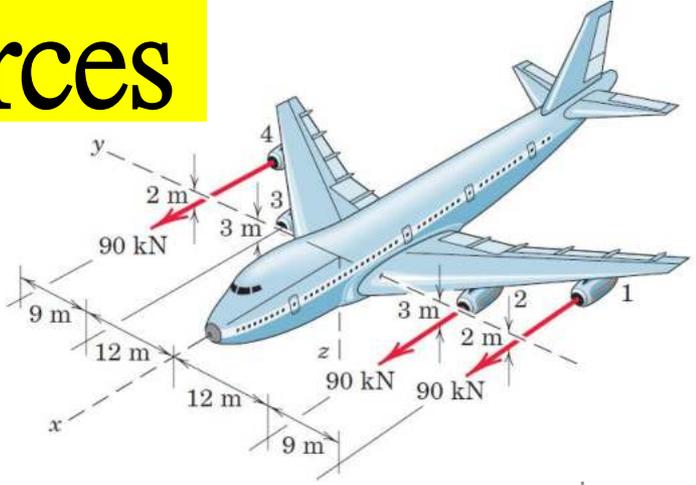
CHAPTER OUTLINE

✓ *Introduction*

✓ *Center of Mass & Centroids*

✓ *Composite Bodies and Figures; Approximations*

✓ *Beams - External & Internal Effects*

✓ *Shear-Force (**S.F.D.**) & Bending-Moment (**B.M.D.**) Diagrams*

Introduction

✓ Actually, "concentrated" forces do not exist in the exact sense, since every external force applied mechanically to a body is distributed over a finite contact area, however small.

✓ In these examples we may treat the forces as concentrated when analyzing their external effects on bodies as a whole.
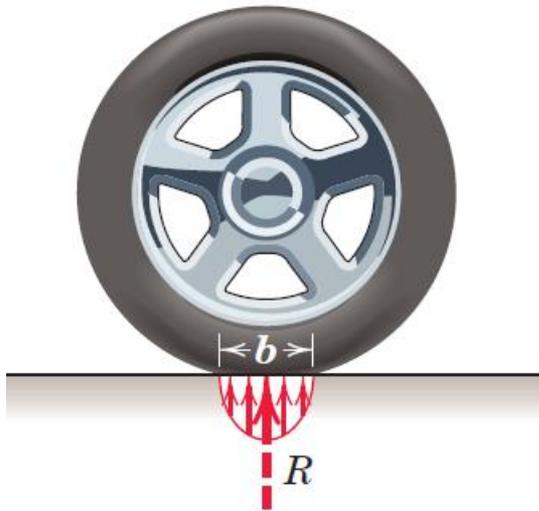
Types of Distributed Forces

**a) *Line Distribution:*** When a force is distributed <u>along a line</u>, the intensity **w** of the loading is expressed as **<u>force per unit length</u>** of line, newtons per meter (**N/m**) or pounds per foot (**lb/ft**).

b) *Area Distribution:* When a force is distributed <u>over an area</u>: the intensity is expressed as **<u>force per unit area</u>**. The basic unit for pressure or stress in **SI** is the newton per square meter (**N/m^2**), which is also called the Pascal (**Pa**).

c) *Volume Distribution:* A force which is distributed <u>over the volume</u> of a body is called a body force, e.g. the force of gravitational attraction. The intensity of gravitational force is the specific weight g, where is the density (mass per unit volume) and **g** is the acceleration due to gravity. The units for **g** are **(kg/m^3)(m/s^2) = N/m^3**. In SI units and **lb/ft^3** or **lb/in^3** in the U.S. customary system.
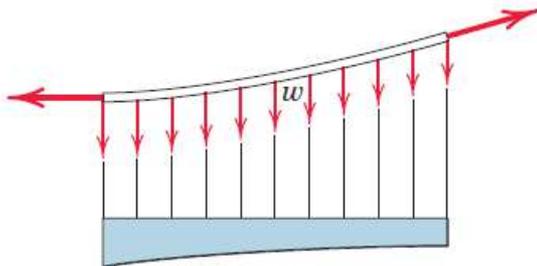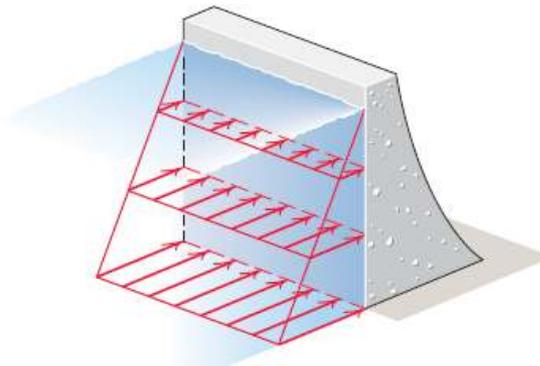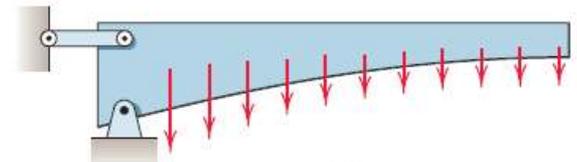
(a)   (b)   (c)

Centers of Mass & Centroids

Center of Mass:
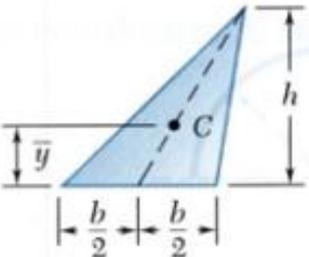
(a)   (b)   (c)

(a)   (b)
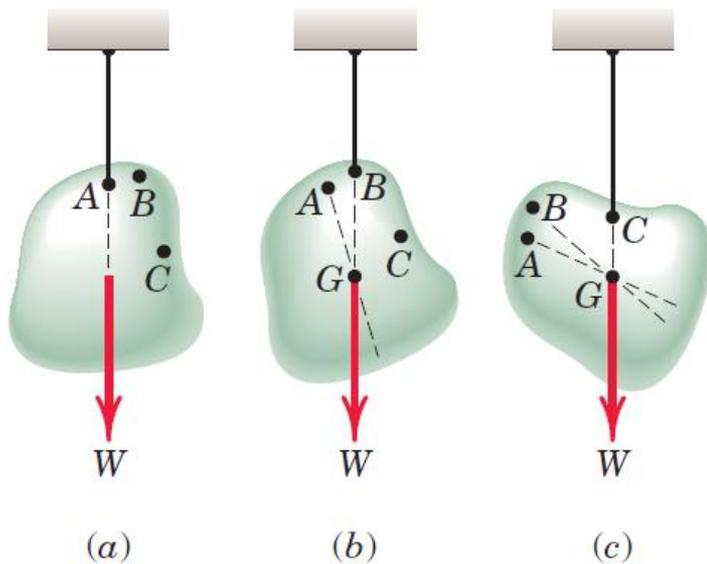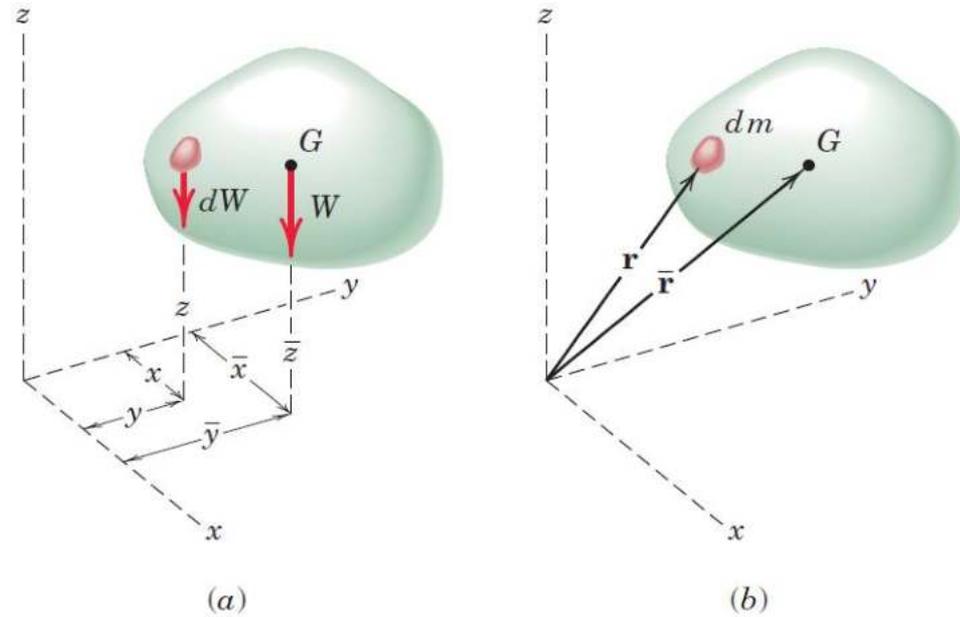
G: *The center of gravity of the body.*

$$\bar{x} = \frac{\int x\, dW}{W} \qquad \bar{y} = \frac{\int y\, dW}{W} \qquad \bar{z} = \frac{\int z\, dW}{W}$$

Centroids of Common Shapes of Lines

Shape		$\overline{x}$	$\overline{y}$	Length
Quarter-circular arc		$\dfrac{2r}{\pi}$	$\dfrac{2r}{\pi}$	$\dfrac{\pi r}{2}$
Semicircular arc		0	$\dfrac{2r}{\pi}$	πr
Arc of circle		$\dfrac{r \sin \alpha}{\alpha}$	0	$2\alpha r$

Centroids of Common Shapes of Areas

Shape		x	y	Area
Triangular area			$\dfrac{h}{3}$	$\dfrac{bh}{2}$
Quarter-circular area		$\dfrac{4r}{3\pi}$	$\dfrac{4r}{3\pi}$	$\dfrac{\pi r^2}{4}$
Semicircular area		0	$\dfrac{4r}{3\pi}$	$\dfrac{\pi r^2}{2}$
Quarter-elliptical area		$\dfrac{4a}{3\pi}$	$\dfrac{4b}{3\pi}$	$\dfrac{\pi ab}{4}$
Semi-elliptical area		0	$\dfrac{4b}{3\pi}$	$\dfrac{\pi ab}{2}$
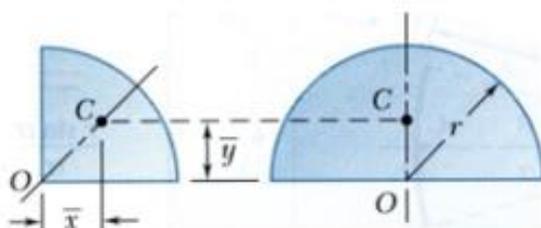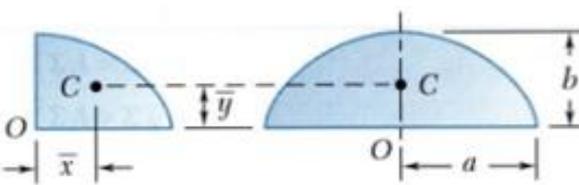
Centroids of Common Shapes of Areas

Shape		$\overline{x}$	$\overline{y}$	Area
Semi-parabolic area		$\dfrac{3a}{8}$	$\dfrac{3h}{5}$	$\dfrac{2ah}{3}$
parabolic area			$\dfrac{3h}{5}$	$\dfrac{4ah}{3}$
Parabolic spandrel		$\dfrac{3a}{4}$	$\dfrac{3h}{10}$	$\dfrac{ah}{3}$
General spandrel		$\dfrac{n+1}{n+2}a$	$\dfrac{n+1}{4n+2}h$	$\dfrac{ah}{n+1}$
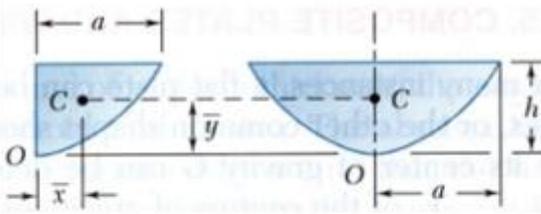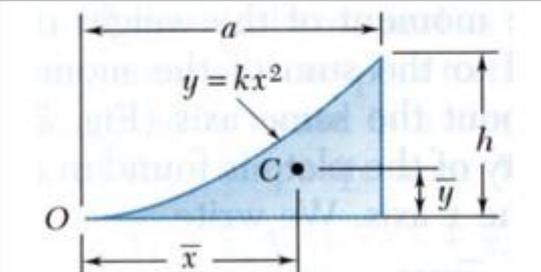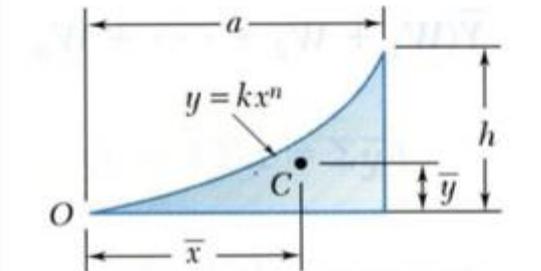
Centroids of Common Shapes of Areas

Shape		$\overline{x}$	$\overline{y}$	Area
Circular sector		$\dfrac{2r \sin \alpha}{3\alpha}$	0	αr^2
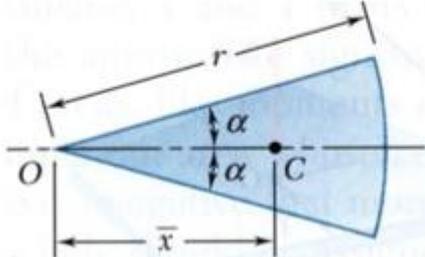
Composite Bodies and Figures

An Approximation Method

"Irregular Lines; Wire; Slender Rod; Segment" - Center – Formula:
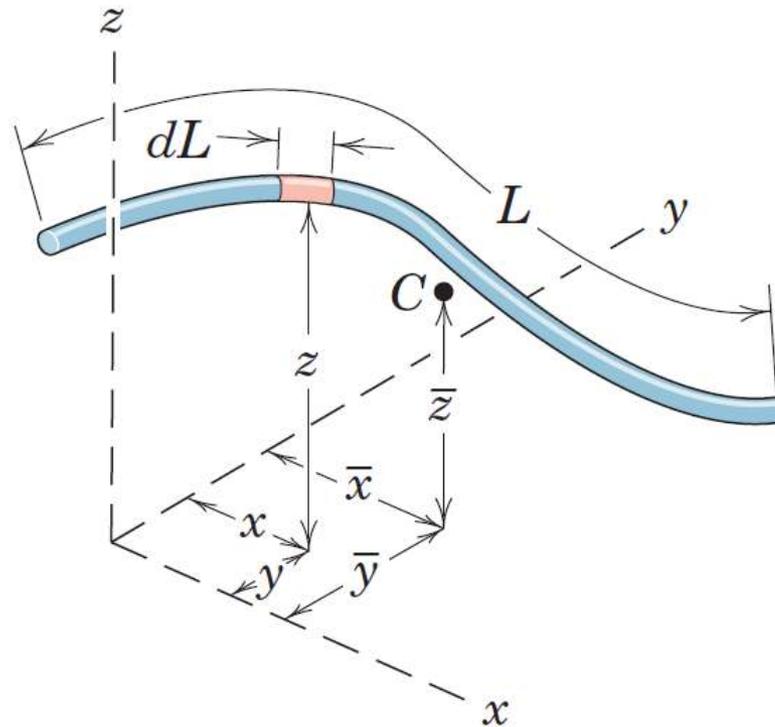
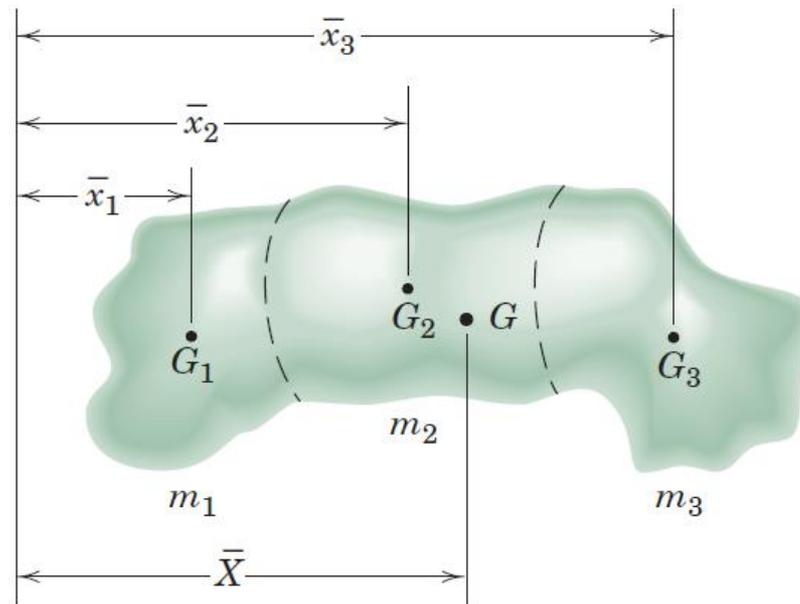$$\overline{x} = \frac{\sum L x_c}{\sum L} \qquad \overline{y} = \frac{\sum L y_c}{\sum L} \qquad \overline{z} = \frac{\sum L z_c}{\sum L}$$

Composite Bodies and Figures
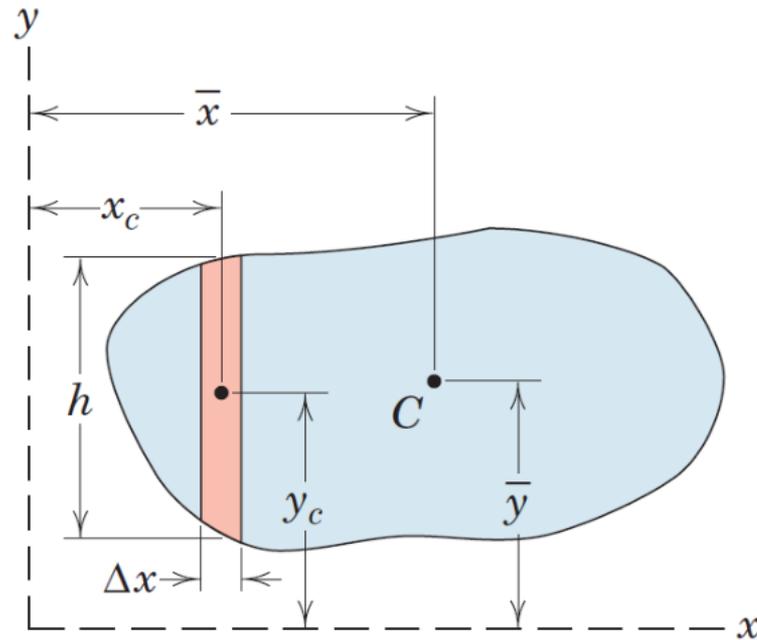
An Approximation Method

Irregular Mass-Center – Formula:

$$\overline{X} = \frac{\Sigma m\overline{x}}{\Sigma m} \qquad \overline{Y} = \frac{\Sigma m\overline{y}}{\Sigma m} \qquad \overline{Z} = \frac{\Sigma m\overline{z}}{\Sigma m}$$

Composite Bodies and Figures

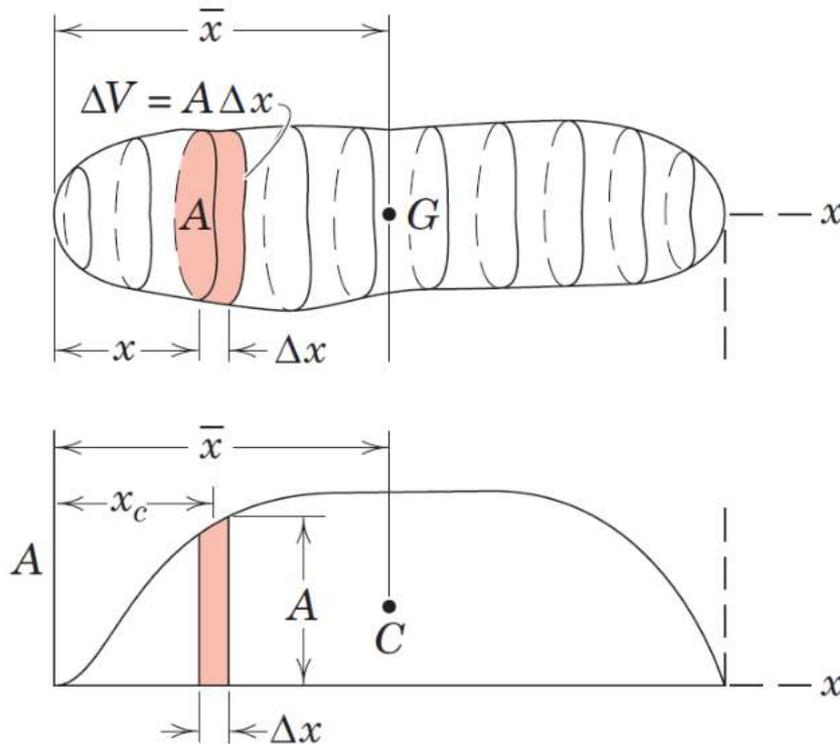## "Irregular Areas" - Center – Formula:

$$\bar{x} = \frac{\Sigma A x_c}{\Sigma A} \qquad \bar{y} = \frac{\Sigma A y_c}{\Sigma A}$$

Composite Bodies and Figures

An Approximation Method
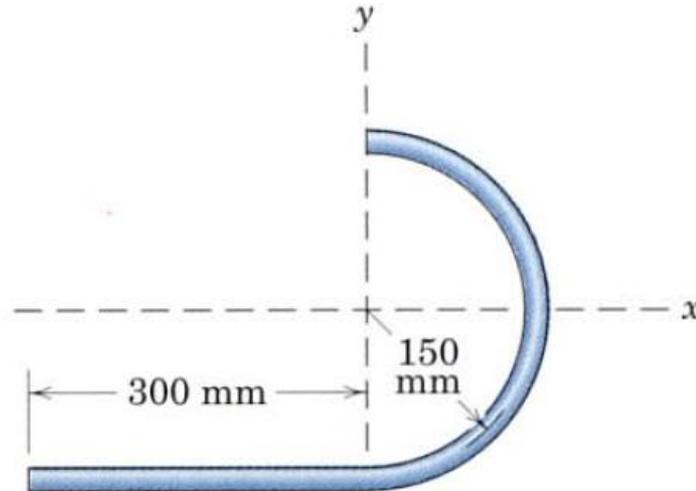
"Irregular Volume" - Center – Formula:

$$\overline{x} = \frac{\sum V x_c}{\sum V} \qquad \overline{y} = \frac{\sum V y_c}{\sum V} \qquad \overline{z} = \frac{\sum V z_c}{\sum V}$$

Exercise – 1

Locate the mass center of the **slender rod** bent into the shape shown.

Solution - Exercise – 1
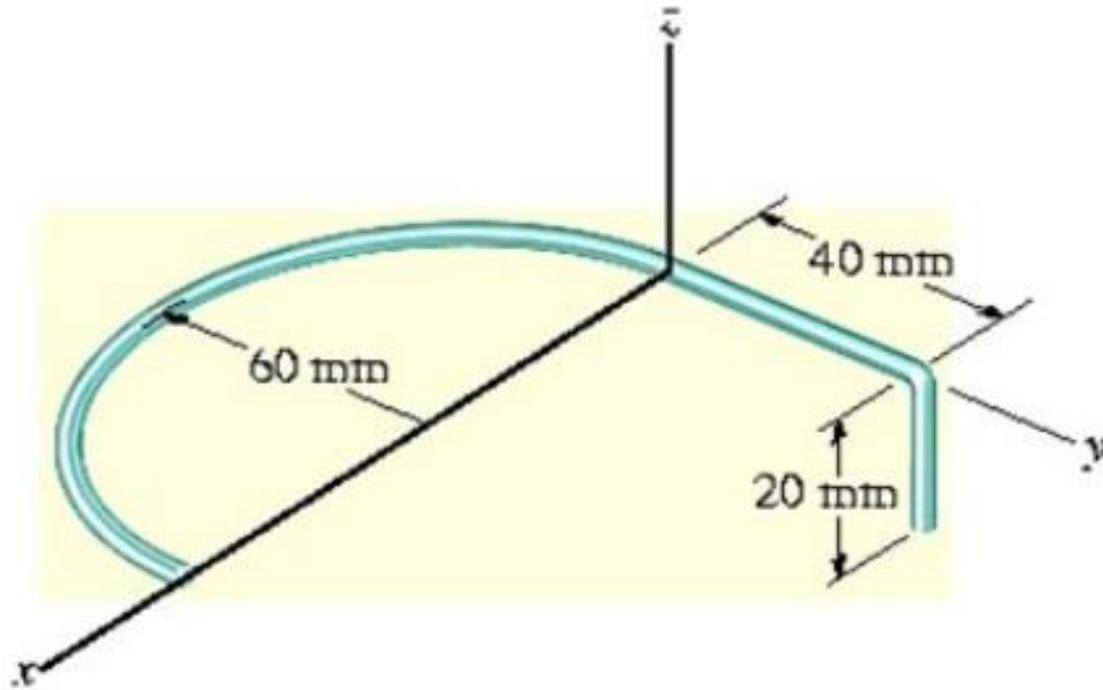
Part	L (mm)	X (mm)	Y (mm)
1			
2			
$\sum L$			

$$\bar{x} = \frac{\sum Lx_c}{\sum L}$$

$$\bar{y} = \frac{\sum Ly_c}{\sum L}$$

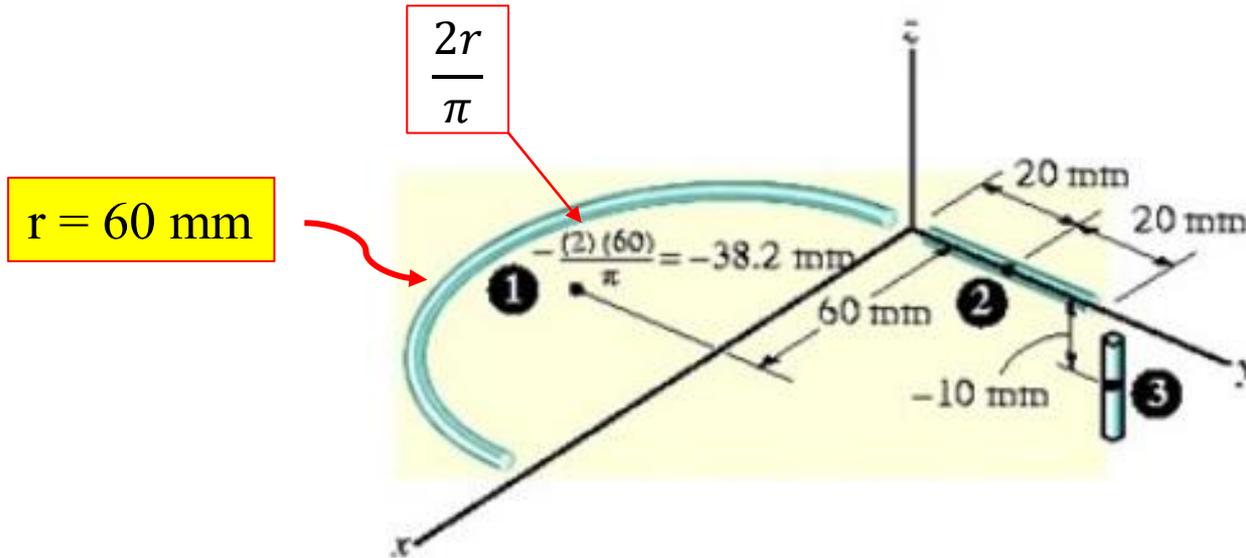$$\bar{z} = \frac{\sum Lz_c}{\sum L}$$

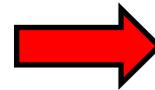# Exercise – 2

Locate the centroid of the **wire** shown.

Solution - Exercise – 2

$\dfrac{2r}{\pi}$

r = 60 mm

$-\dfrac{(2)(60)}{\pi} = -38.2\ mm$

20 mm

20 mm

60 mm

−10 mm

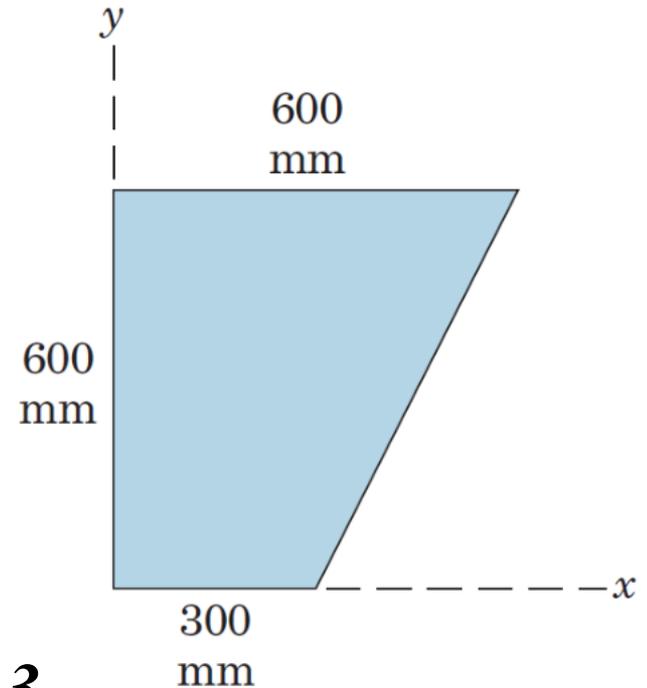| Segment	L (mm)	x (mm)	y (mm)	z (mm)
1	188.5	60	-38.2	0
2	40	0	20	0
3	20	0	40	-10
Sum	248.5			

$$\bar{x} = \frac{\sum \tilde{x}L}{\sum L} = \frac{11310}{248.5} = 45.5 mm$$

$$\bar{y} = \frac{\sum \tilde{y}L}{\sum L} = \frac{-5600}{248.5} = -22.5 mm$$

$$\bar{z} = \frac{\sum \tilde{z}L}{\sum L} = \frac{-200}{248.5} = -0.805 mm$$

Exercise – 3

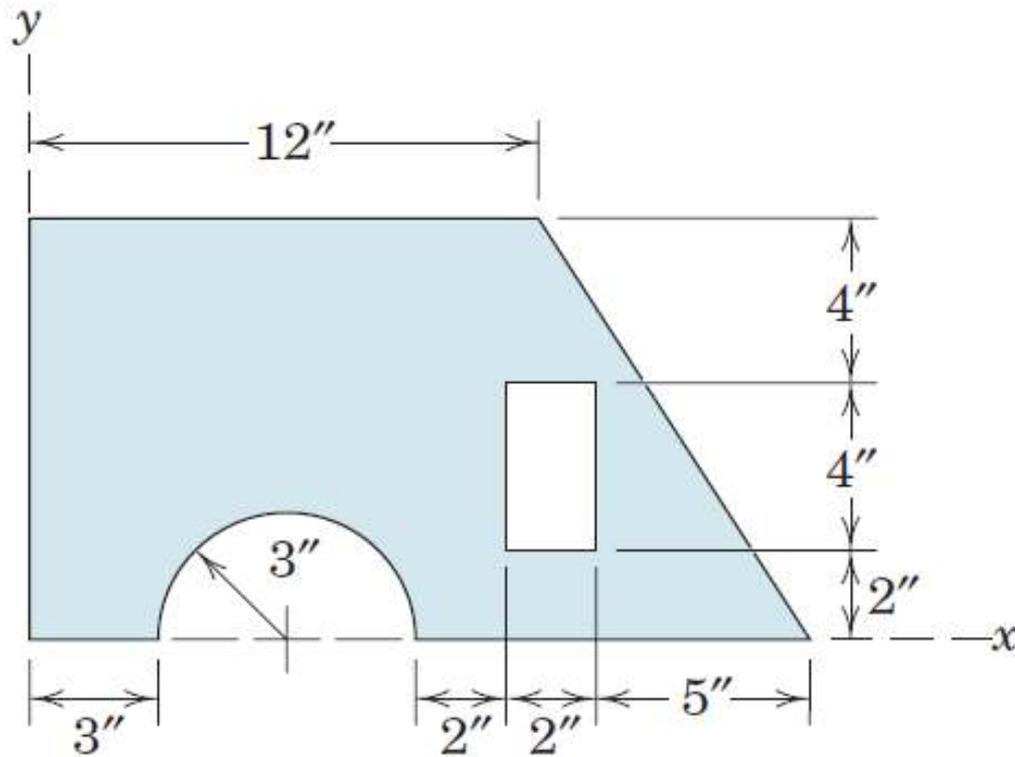Determine the coordinates of the centroid of the trapezoidal **area** shown.

y

600 mm

600 mm

300 mm

x

Solution - Exercise – 3

Part	A (mm^2)	$\bar{x}$ (mm)	$\bar{y}$ (mm)	$\bar{x}.A$ (mm^3)	$\bar{y}.A$ (mm^3)
1					
2					
ΣA			Sum =		

$$\bar{x} = \frac{\Sigma Ax_c}{\Sigma A}$$
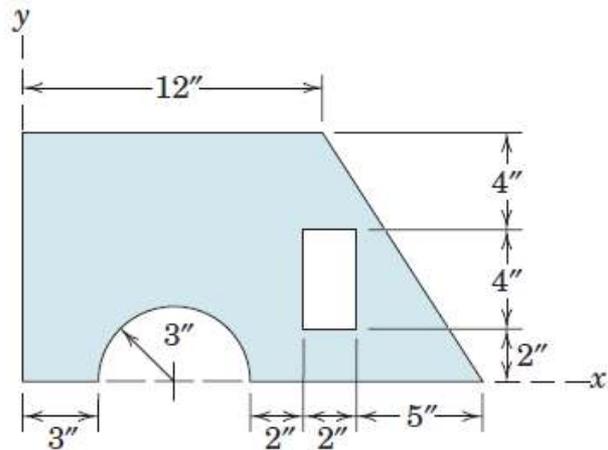
$$\bar{y} = \frac{\Sigma Ay_c}{\Sigma A}$$

Exercise – 4

Locate the centroid of the shaded **area**.

$$\sum A = A_1 + A_2 - A_3 - A_4$$

PART	A in.^{2}	$\bar{x}$ in.	$\bar{y}$ in.	$\bar{x}A$ in.^{3}	$\bar{y}A$ in.^{3}
1	120	6	5	720	600
2	30	14	10/3	420	100
3	−14.14	6	1.273	−84.8	−18
4	−8	12	4	−96	−32
TOTALS	127.9			959	650

The area counterparts to Eqs. 5/7 are now applied and yield

$$\left[\bar{X} = \frac{\Sigma A\bar{x}}{\Sigma A} \right] \qquad \bar{X} = \frac{959}{127.9} = 7.50 \text{ in.} \qquad Ans.$$

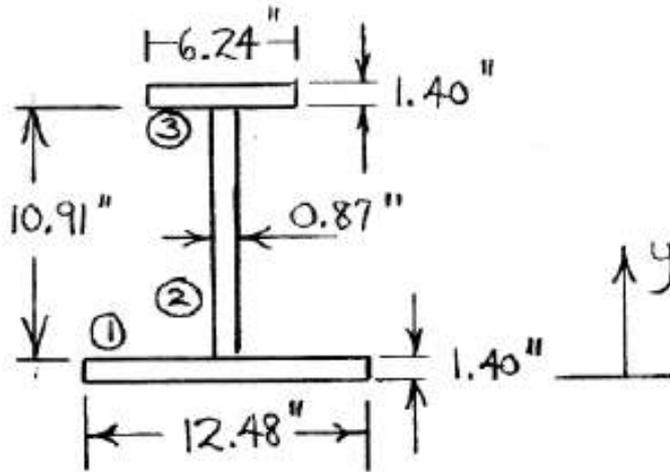$$\left[\bar{Y} = \frac{\Sigma A\bar{y}}{\Sigma A} \right] \qquad \bar{Y} = \frac{650}{127.9} = 5.08 \text{ in.} \qquad Ans.$$

19

Exercise – 5

Determine the **height** (*y*) above the base **of the centroid** of the cross-sectional area of the beam. **Neglect the fillets**.
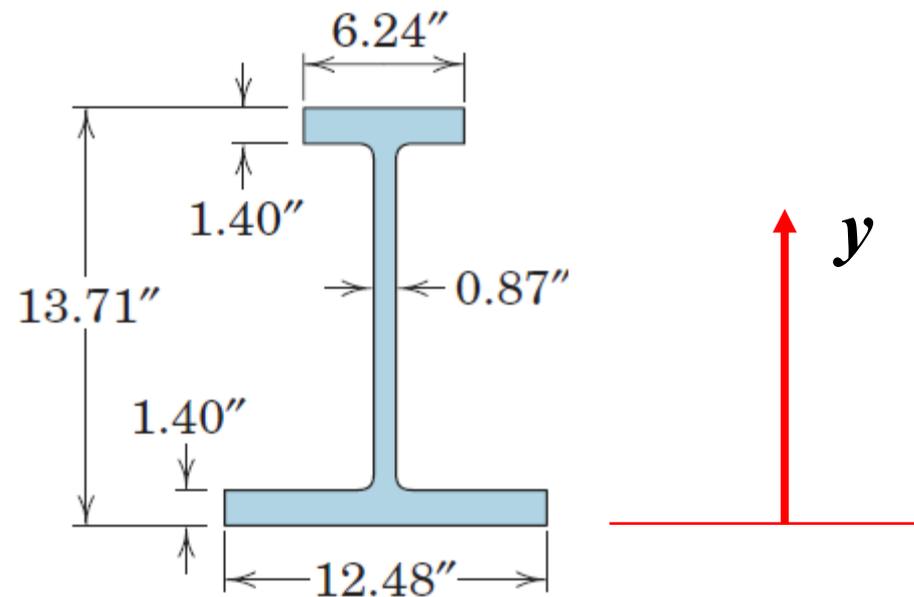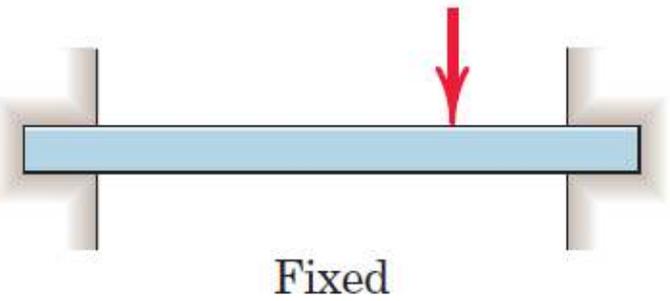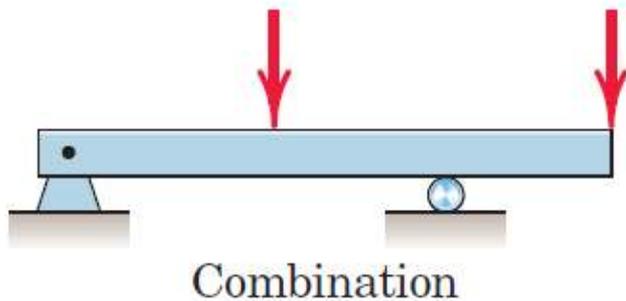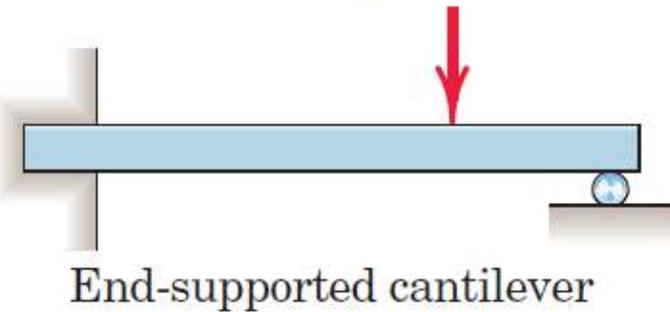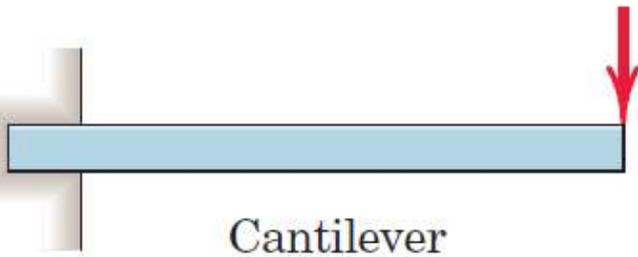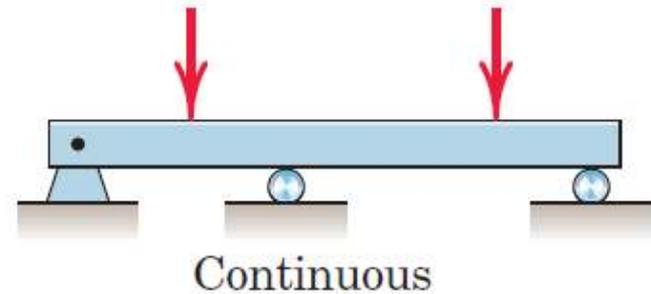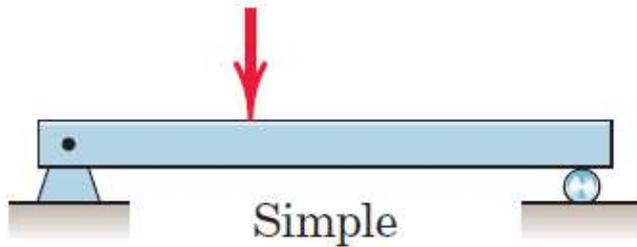
Comp. A (in.²) $\bar{y}$ (in.) $A\bar{y}$ (in.³)

① $(12.48)(1.40)$ $\dfrac{1.40}{2}$ 12.23

② $(10.91)(0.87)$ $1.4 + \dfrac{10.91}{2}$ 65.1

③ $(6.24)(1.40)$ $1.4 + 10.91 + \dfrac{1.4}{2}$ 113.7

$$\Sigma A = 35.7 \qquad\qquad \Sigma A\bar{y} = 191.0$$

$$\bar{Y} = \frac{\Sigma A\bar{y}}{\Sigma A} = 5.35 \text{ in.}$$

21

Beams - External Effects

Types of Beams

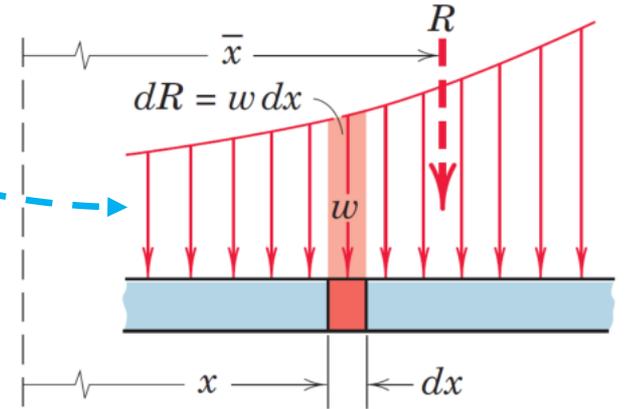Simple

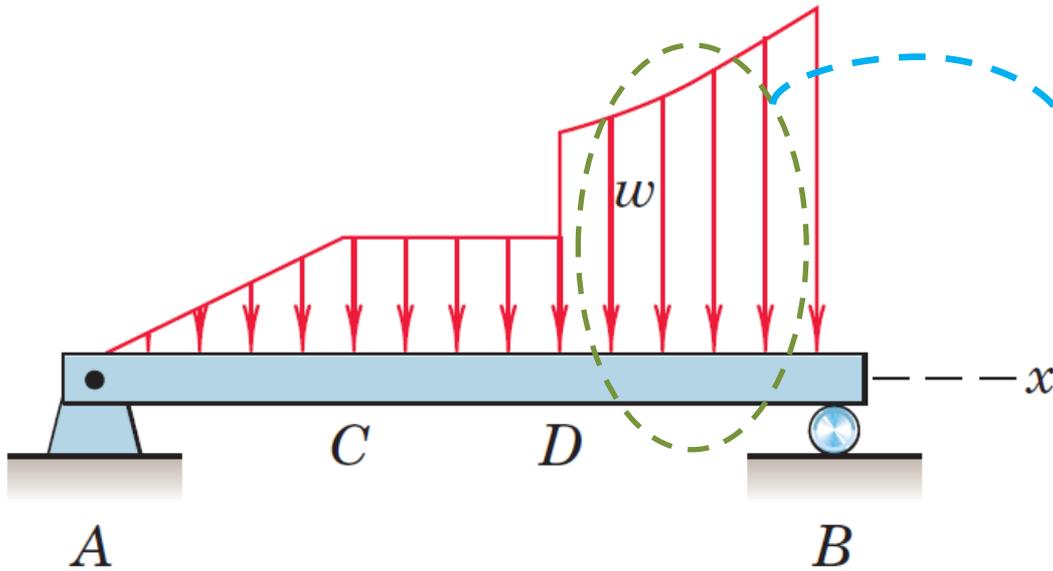Continuous

Cantilever

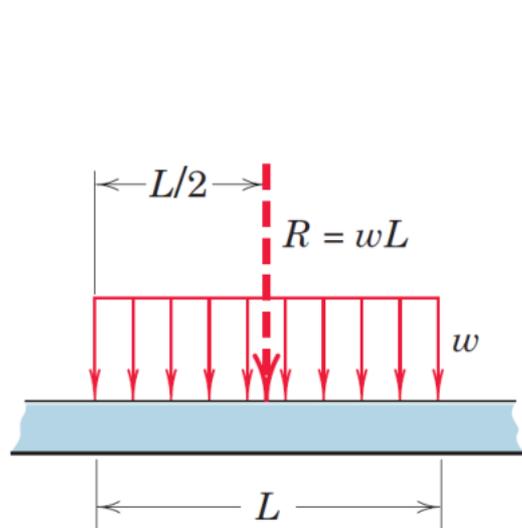End-supported cantilever

Combination

Fixed

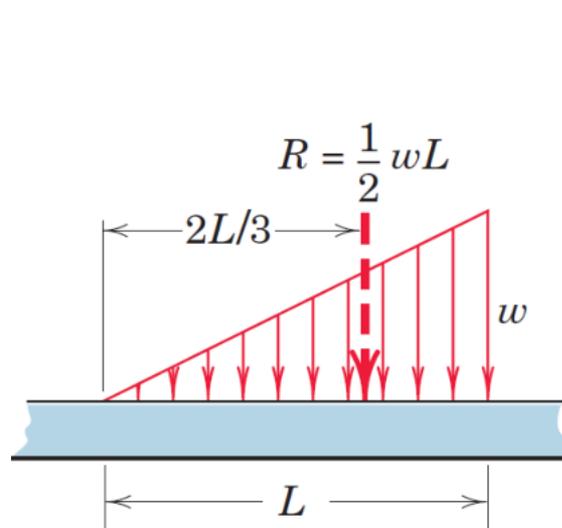Statically determinate beams
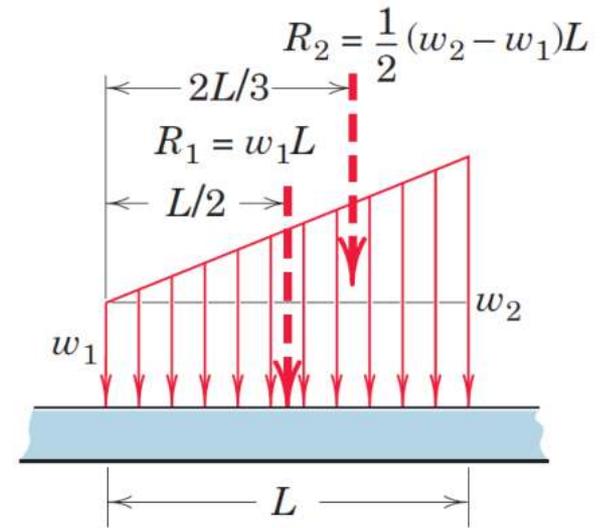
Statically indeterminate beams

Types of Distributed Loads

$$R = \int w\,dx \qquad \bar{x} = \frac{\int xw\,dx}{R}$$

$R = wL$ (a)

$R = \frac{1}{2}wL$ (b)
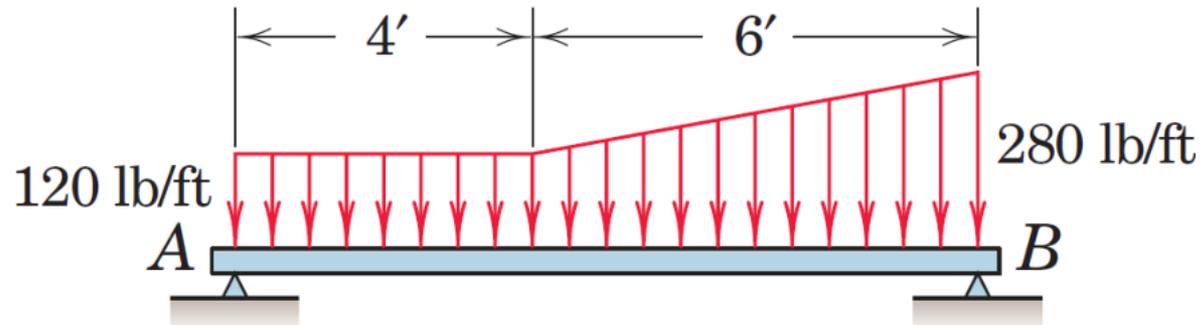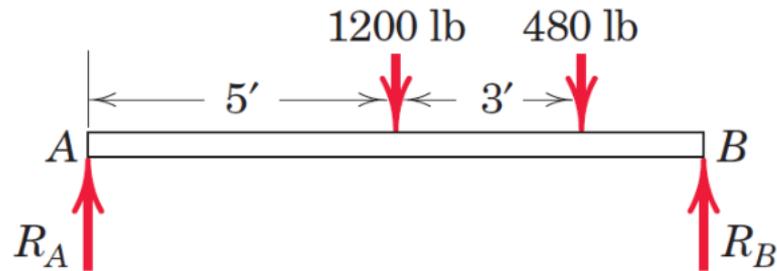
$R_1 = w_1 L$, $R_2 = \frac{1}{2}(w_2 - w_1)L$ (c)

Exercise – 1
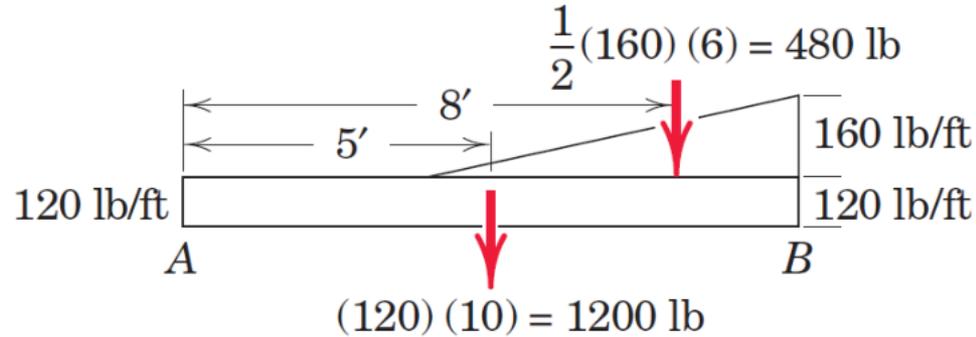
Determine the equivalent **concentrated load** (s) and **external reactions** for the simply supported beam which is subjected to the distributed load shown.

Solution - Exercise – 1

$$[\Sigma M_A = 0] \quad 1200(5) + 480(8) - R_B(10) = 0$$

$$R_B = 984 \text{ lb} \qquad \text{Ans.}$$
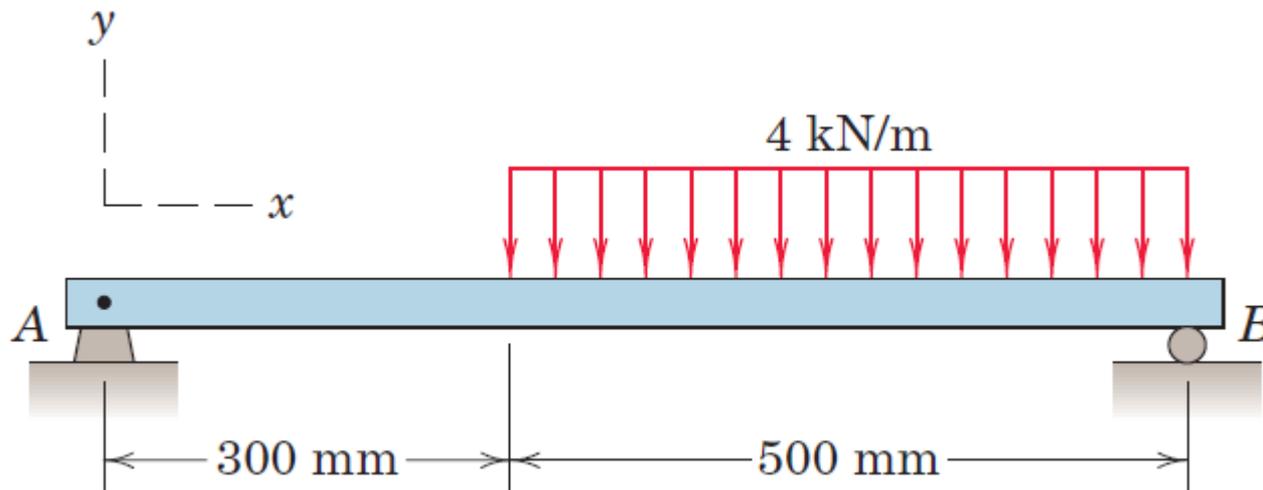
$$[\Sigma M_B = 0] \quad R_A(10) - 1200(5) - 480(2) = 0$$

$$R_A = 696 \text{ lb} \qquad \text{Ans.}$$

Exercise – 2

Determine the reactions at **A** and **B** for the beam subjected to the uniform load distribution.

$$\sum M_A = 0$$

$$-2 * (0.250 + 0.300) + B_y * 0.800 = 0$$

B_y = 1.375 kN

$$\sum F_x = 0$$

A_x=0
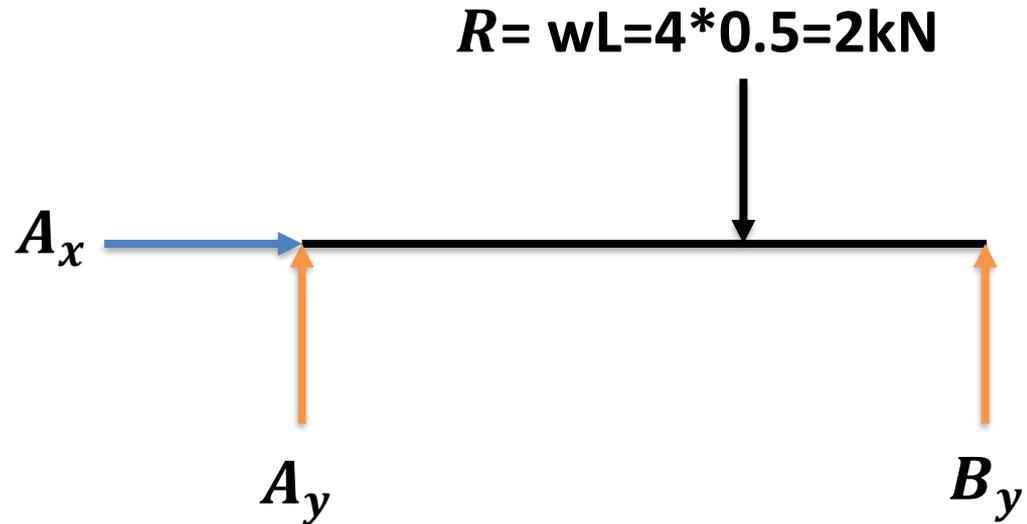
$$\sum F_y = 0$$

$A_y + B_y$-R=0

$$A_y = 2 - 1.375$$
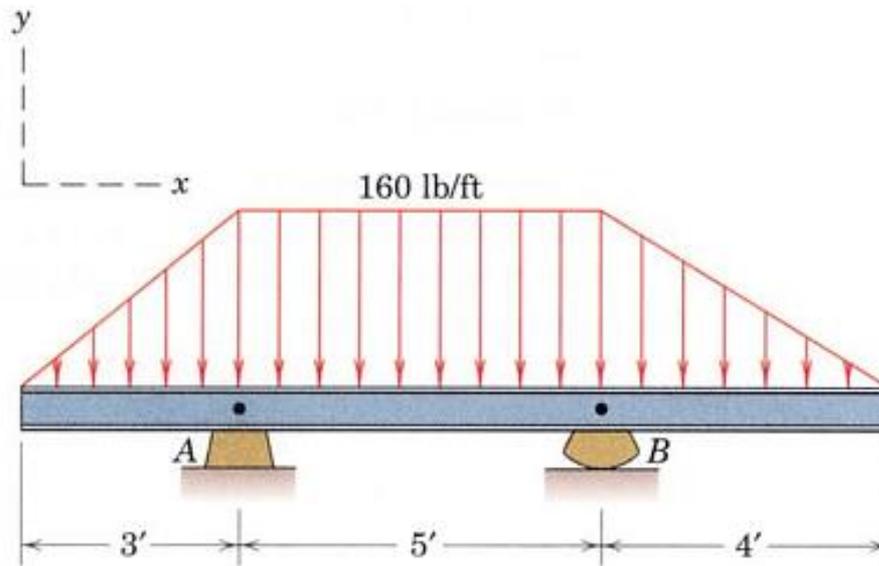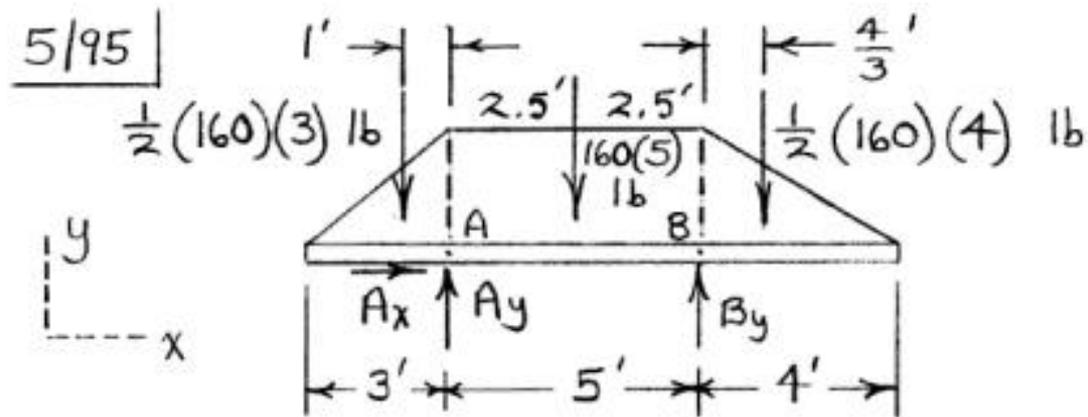
$$A_y = 0.625 \, kN$$

R= wL=4*0.5=2kN

A_x

A_y

B_y

Exercise – 3

Determine the reactions at **A** and **B** for the beam subjected to the uniform load distribution.

$\frac{1}{2}(160)(3)$ lb

$160(5)$ lb

$\frac{1}{2}(160)(4)$ lb

$\overset{\curvearrowleft}{+}\ \Sigma M_A = 0: \quad 240(1) - 800(2.5) - 320(6.33)$

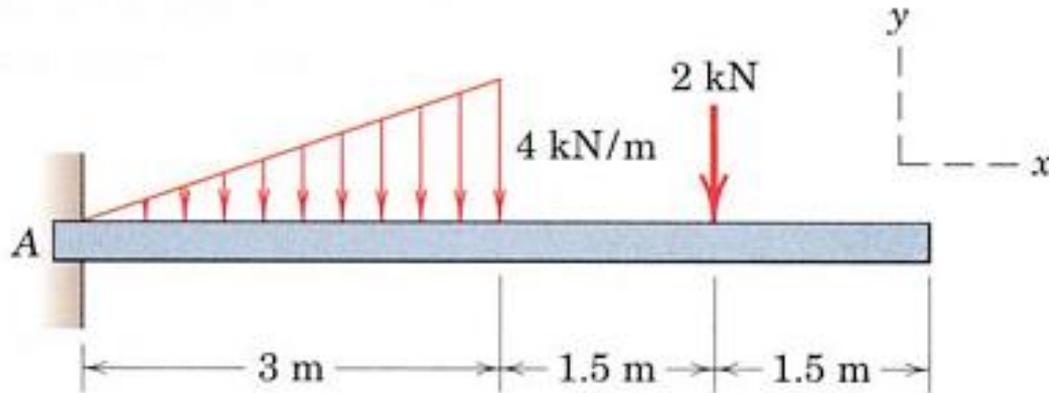$$+ B_y(5) = 0$$

$$\underline{B_y = 757\ lb}$$

$\Sigma F_y = 0: \quad A_y + 757 - 240 - 800 - 320 = 0$
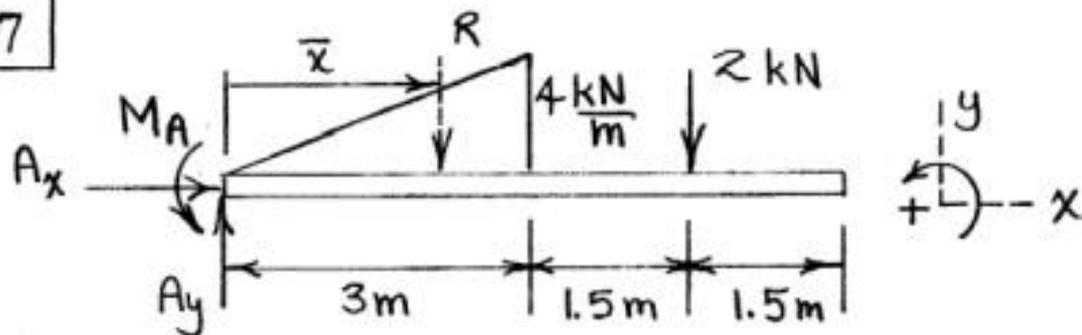
$$\underline{A_y = 603\ lb}$$

$\Sigma F_x = 0 \implies \underline{A_x = 0}$

Exercise – 4

Determine the reactions at **A** for the cantilever beam subjected to the distributed and concentrated loads.

$$R = \tfrac{1}{2}(3)(4) = 6 \text{ kN} \ @ \ \bar{x} = \tfrac{2}{3}(3) = 2m$$
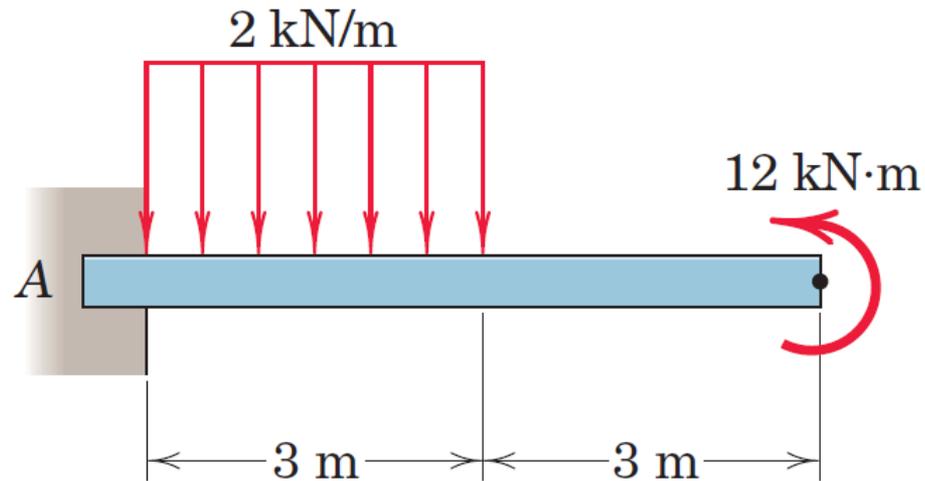
$$\Sigma M_A = 0: \quad M_A - 6(2) - 2(4.5) = 0, \quad \underline{M_A = 21 \text{ kN·m}}$$

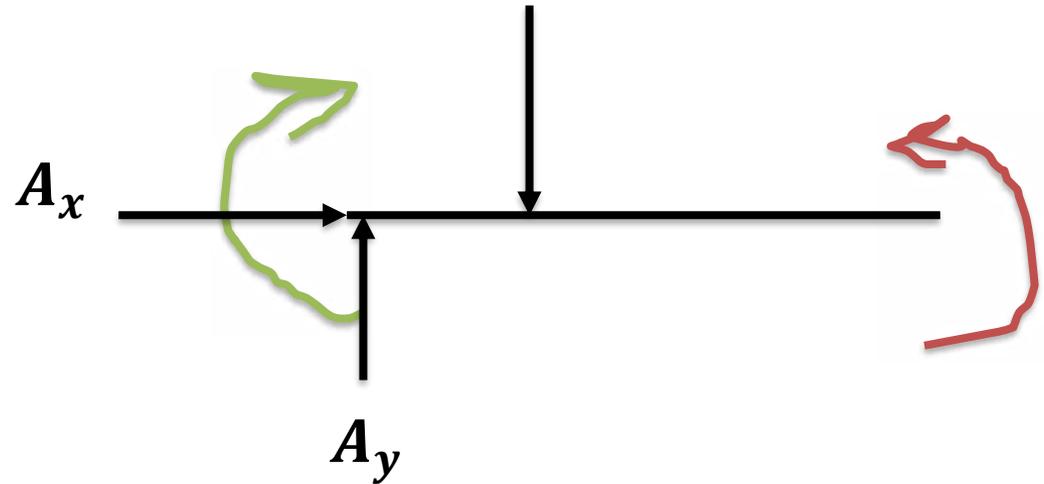$$\Sigma F_y = 0: \quad A_y - 6 - 2 = 0, \qquad \underline{A_y = 8 \text{ kN}}$$

$$\Sigma F_x = 0: \quad \underline{A_x = 0}$$

Exercise – 5

Find the reaction at A due to the uniform loading and the applied couple.

R= wL=2*3=6kN

A_x
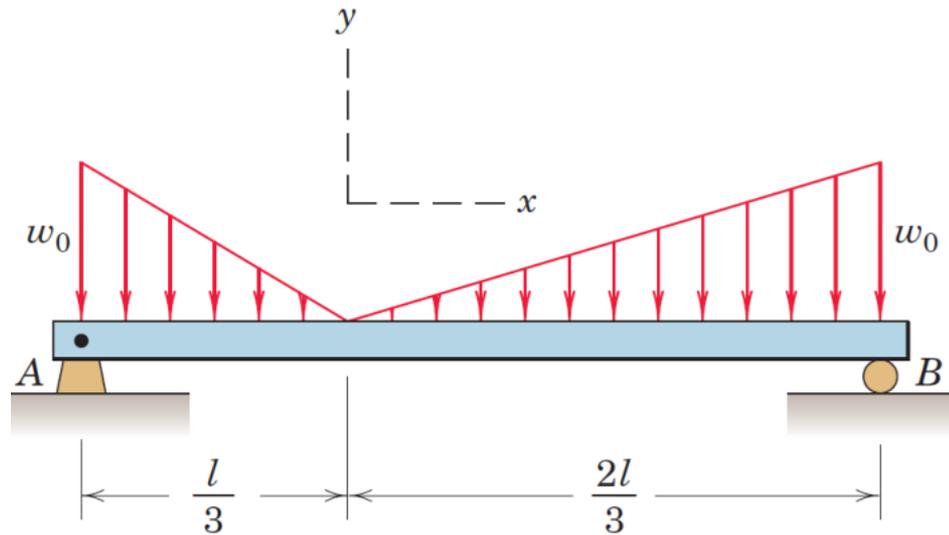
A_y

$$\sum M_A = 0$$

M_A+ 6*1.5-12=0

M_A= 3 kN.m

Exercise – 6

Calculate the support reactions at A and B for the loaded beam.
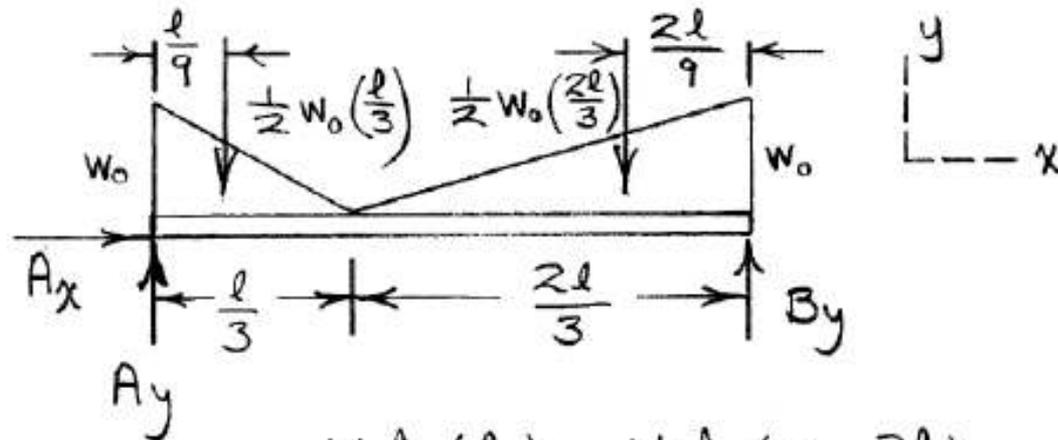
$$\curvearrowleft + \Sigma M_A = 0 : - \frac{W_0 l}{6}\left(\frac{l}{9}\right) - \frac{W_0 l}{3}\left(l - \frac{2l}{9}\right)$$

$$+ B_y l = 0 \; , \quad \underline{B_y = \frac{5}{18} W_0 l}$$

$$\Sigma F_y = 0 : \quad A_y + \frac{5}{18} W_0 l - \frac{W_0 l}{6} - \frac{W_0 l}{3} = 0$$

$$\underline{A_y = \frac{2}{9} W_0 l}$$

$$\Sigma F_x = 0 \Rightarrow \underline{A_x = 0}$$

Exercise – 7

Calculate the support reactions at **A** and **B** for the beam subjected to the two linearly varying load distributions.
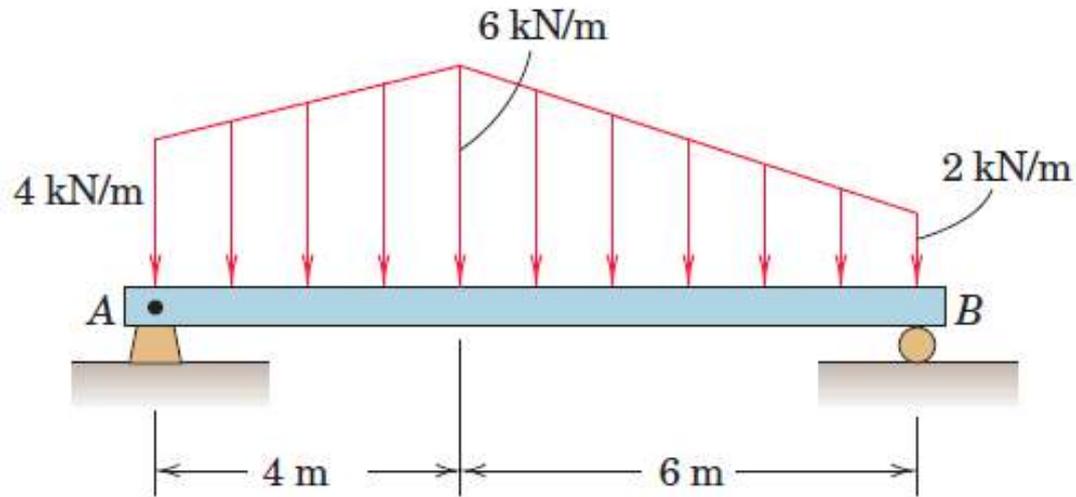
$$R_1 = 4(4) = 16 \text{ kN}, \quad R_2 = \tfrac{1}{2}(2)(4) = 4 \text{ kN}$$

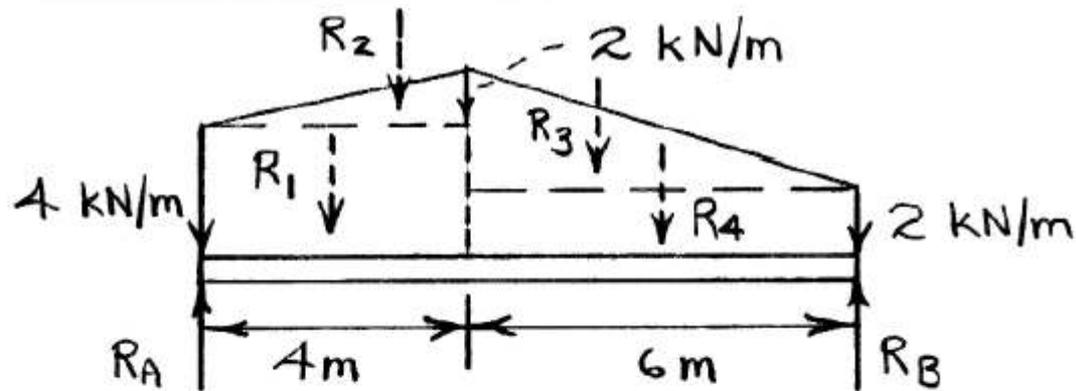$$R_3 = \tfrac{1}{2}(4)(6) = 12 \text{ kN}, \quad R_4 = 2(6) = 12 \text{ kN}$$

$$\circlearrowright \Sigma M_A = 0: \quad 16(2) + 4\left(\tfrac{2}{3}4\right) + 12\left(4+\tfrac{1}{3}6\right)$$

$$+ 12(4+3) - 10\,R_B = 0, \quad \underline{R_B = 19.87 \text{ kN}}$$

$$+\uparrow \Sigma F = 0: \quad R_A + 19.87 - (16+4+12+12) = 0$$

$$\underline{R_A = 24.1 \text{ kN}}$$

Shear, Bending, & Torsion

V

Shear

V

Shear

M

Bending

M

T

Torsion

T

T

M

V

Combined loading

+M

+M

I

+M

+V

+M

+V
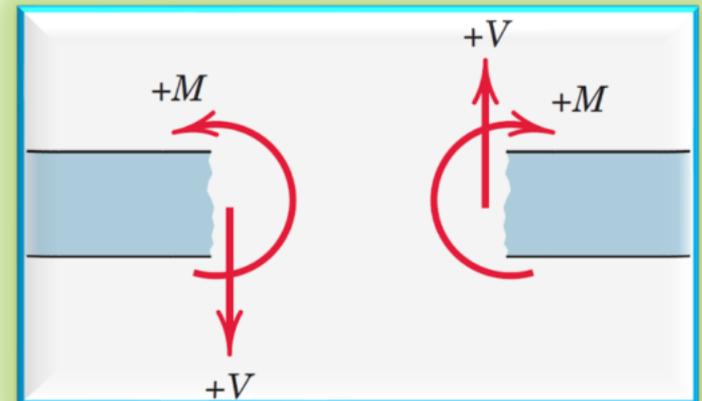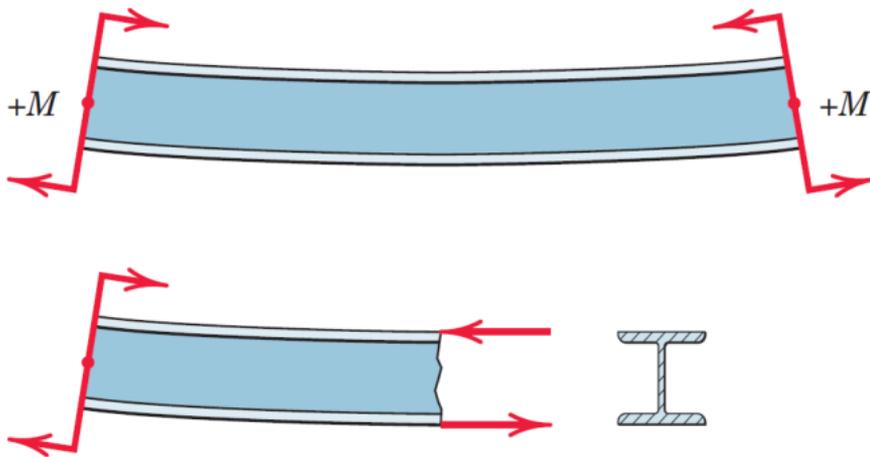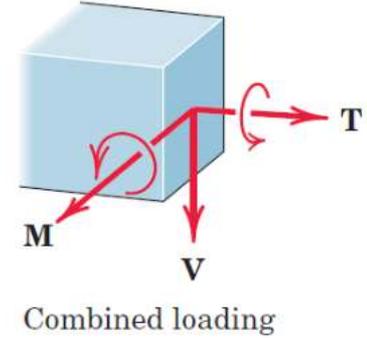
Shear-Force (S.F.D) & Bending-Moment (B.M.D.) Diagrams

Steps *in the determination of the* **Shear** *and* **Moment** *Relations:*

1. Establish the values of all **External Reactions** on the beam by applying the equations of equilibrium to a ***free-body diagram*** of the beam **as a whole**.

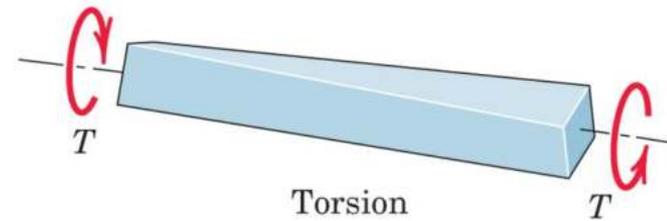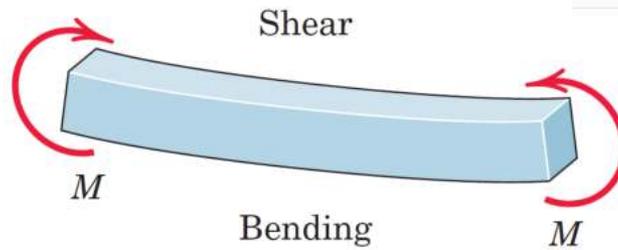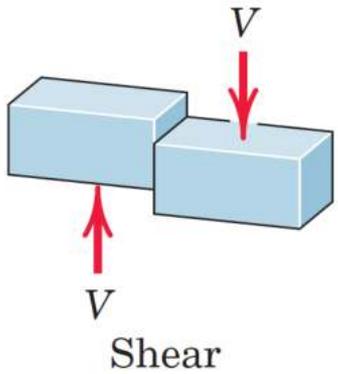2. Next, we **isolate a portion of the beam**, either to the right or to the left of an arbitrary transverse section, with a free-body diagram, and **apply the equations of equilibrium to this isolated portion** of the beam.

3. We should **avoid using a transverse section** **which coincides with the location of a concentrated load or couple**, as such a position represents a point of discontinuity in the variation of shear or bending moment.

4. Finally, it is important to note that the calculations for *V and M on each section chosen should be consistent with the **positive convention***.

Exercise – 1

Determine the **shear** and **moment** distributions produced in the simple beam by the **4-kN** concentrated load.
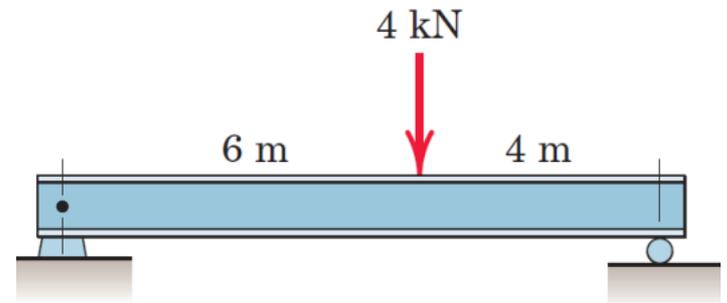
Solution - Exercise – 1

$R_1 = 1.6 \text{ kN}$ $R_2 = 2.4 \text{ kN}$

A section of the beam of length x is next isolated with its free-body diagram on which we show the shear V and the bending moment M in their positive directions. Equilibrium gives

$[\Sigma F_y = 0]$ $1.6 - V = 0$ $V = 1.6 \text{ kN}$

$[\Sigma M_{R_1} = 0]$ $M - 1.6x = 0$ $M = 1.6x$

These values of V and M apply to all sections of the beam to the left of the 4-kN load.

A section of the beam to the right of the 4-kN load is next isolated with its free-body diagram on which V and M are shown in their positive directions. Equilibrium requires

$[\Sigma F_y = 0]$ $V + 2.4 = 0$ $V = -2.4 \text{ kN}$

$[\Sigma M_{R_2} = 0]$ $-(2.4)(10 - x) + M = 0$ $M = 2.4(10 - x)$

These results apply only to sections of the beam to the right of the 4-kN load.

The values of V and M are plotted as shown. The maximum bending moment occurs where the shear changes direction. As we move in the positive x-direction starting with $x = 0$, we see that the moment M is merely the accumulated area under the shear diagram.
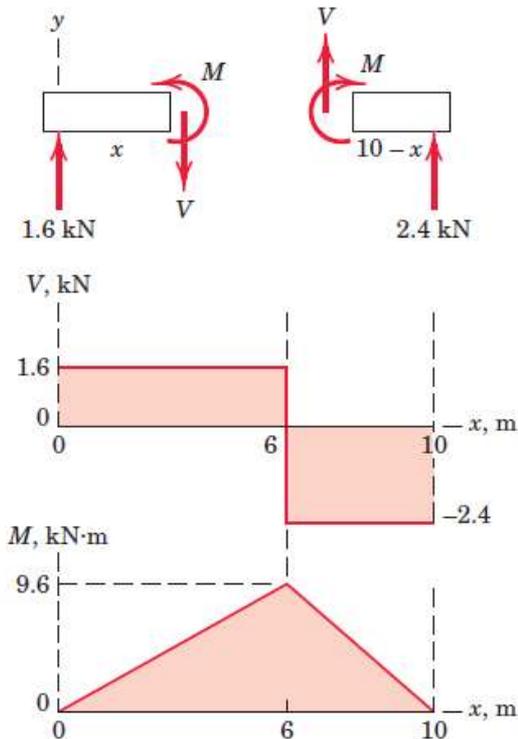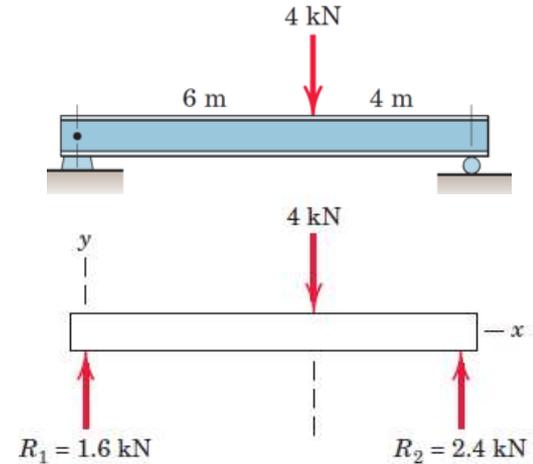
Draw the **shear** and **moment** diagrams for the loaded beam and **determine the distance "_d_"** to the **right of _A_** where the **_moment_ is _zero_**.
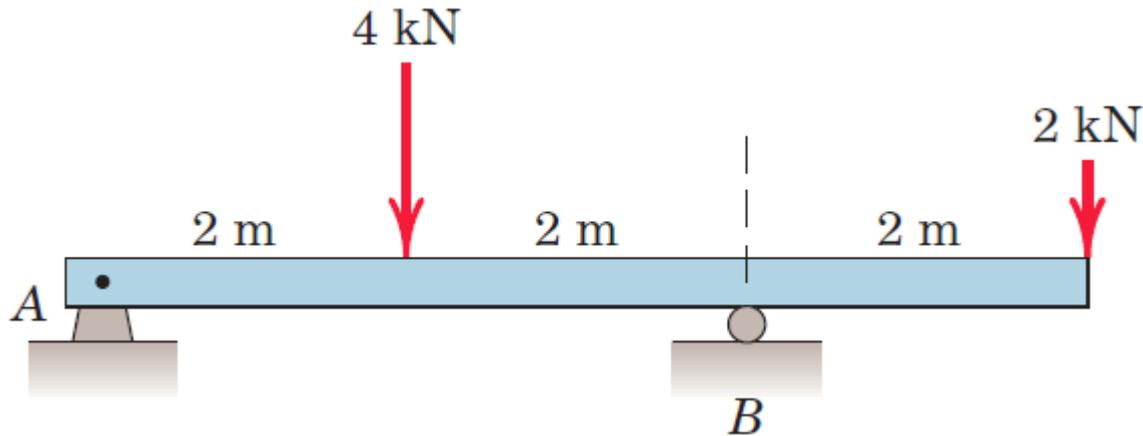
Solution - Exercise – 2

$\oplus \sum M_A = 0: \ 4(2) + 2(6) - 4R_B = 0$

$\qquad R_B = 5 \ kN$

$+\!\uparrow \sum F = 0: \ R_A + 5 - 6 = 0$

$\qquad R_A = 1 \ kN$

$0 < x < 2m:$

$V = 1 \ kN, \quad M = 1x$

$1 \ kN$ ... x ... M ... V
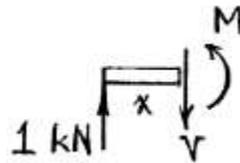
$2 < x < 4 \ m:$

$V = -3 \ kN$

$\curvearrowleft + \sum M_A = 0: -8 + 3x + M = 0, \ M = 8 - 3x$

$2m$ ... $4 \ kN$ ... x ... M ... $1 \ kN$ ... V
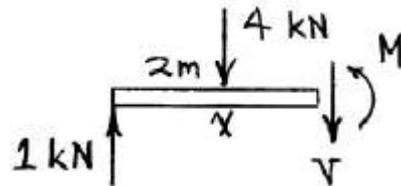
$4 < x < 6 \ m:$

$V = 2 \ kN$

$\curvearrowleft + \sum M = 0: -M - 2(6-x) = 0$

$\qquad M = -2(6-x)$

M ... V ... $2 kN$ ... $6-x$
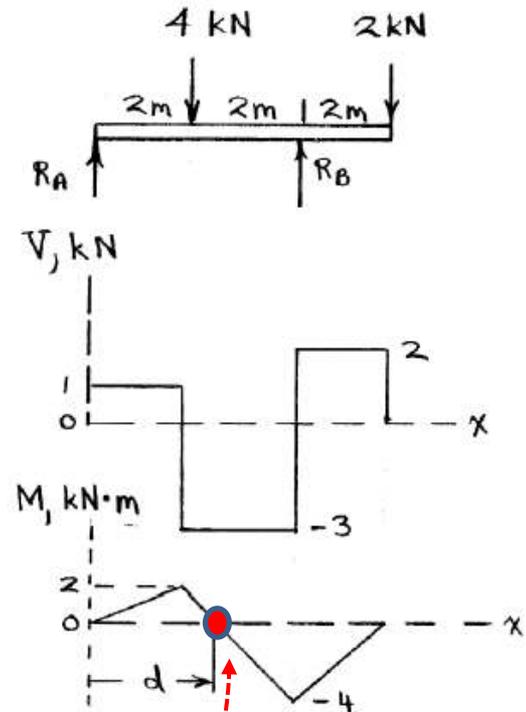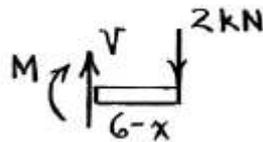
M=0 (Range: **2<x<4**)

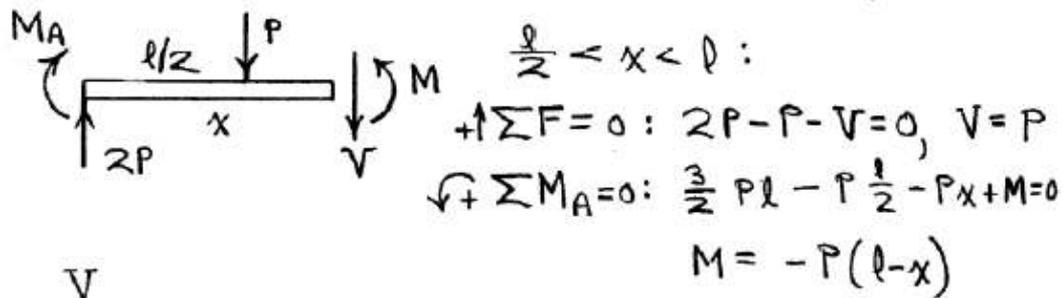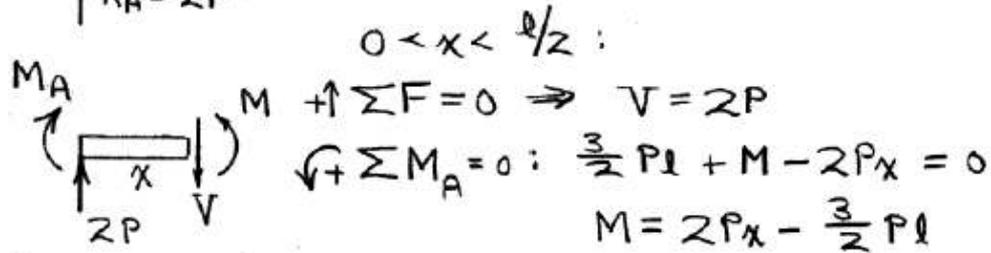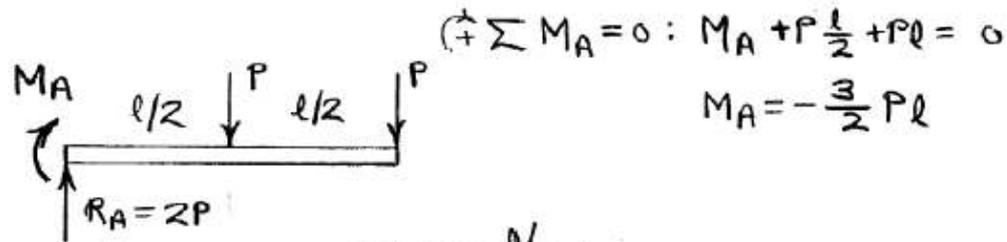**M=0 → x=d=8/3=2.67 m**

Exercise – 3

Draw the shear and moment diagrams for the loaded cantilever beam. State the value of the bending moment at midbeam.

Solution - Exercise – 3

$$\circlearrowright^+ \Sigma M_A = 0: \quad M_A + P\frac{l}{2} + Pl = 0$$

$$M_A = -\frac{3}{2}Pl$$

M_A ... $l/2$ ... P ... $l/2$ ... P

$R_A = 2P$

$$0 < x < \frac{l}{2}:$$

M_A ... x ... M, V, $2P$

$$+\uparrow \Sigma F = 0 \Rightarrow V = 2P$$

$$\circlearrowright^+ \Sigma M_A = 0: \quad \frac{3}{2}Pl + M - 2Px = 0$$

$$M = 2Px - \frac{3}{2}Pl$$

M_A ... $l/2$ ... P ... x ... M, V, $2P$

$$\frac{l}{2} < x < l:$$

$$+\uparrow \Sigma F = 0: \quad 2P - P - V = 0, \quad V = P$$

$$\circlearrowright^+ \Sigma M_A = 0: \quad \frac{3}{2}Pl - P\frac{l}{2} - Px + M = 0$$

$$M = -P(l-x)$$

V

$2P$

P

O ... $l/2$ ... l ... x

M

x

$-\frac{Pl}{2}$

$-\frac{3Pl}{2}$

45
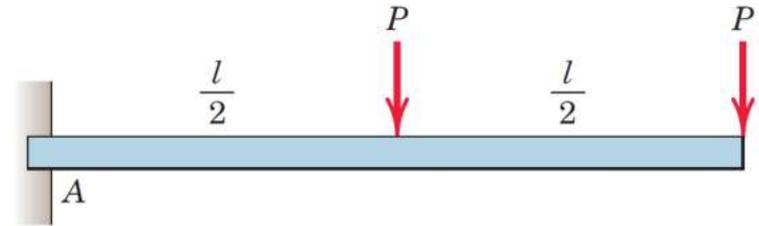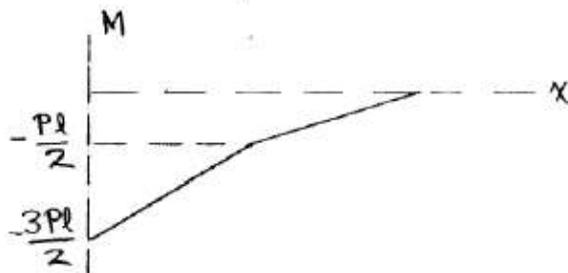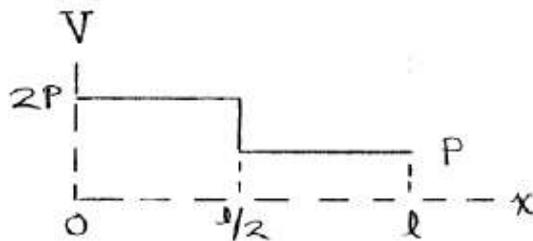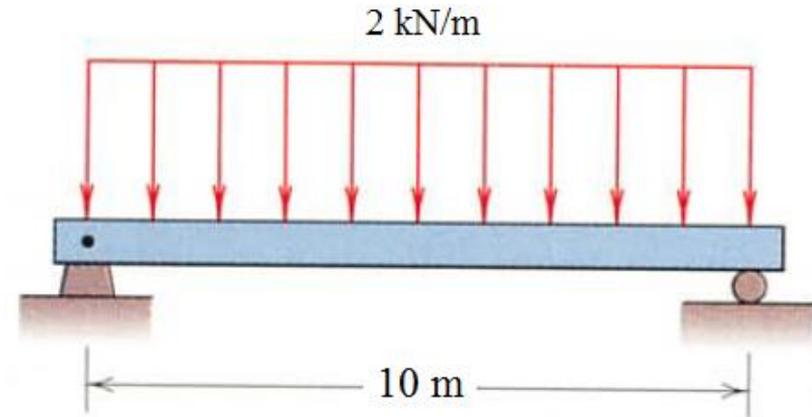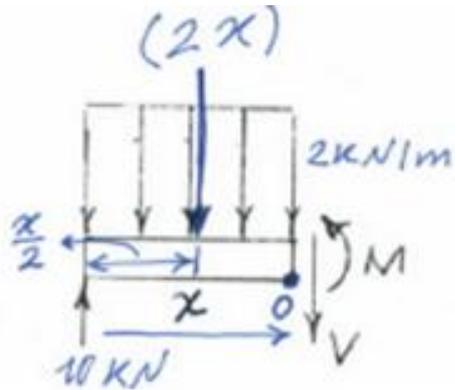
Exercise – 4

Draw the **shear** and **moment diagrams** for the uniformly loaded beam and find the maximum bending moment M_{max}.

2 kN/m

10 m

$F = 2 \times 10 = 20\,KN$

$R_1 = R_2 = 10\,KN$

$2\,kN/m$

$10\,m$

$10\,KN$ $10\,KN$

$(2x)$

$\dfrac{x}{2}$ x $2kN/m$ M
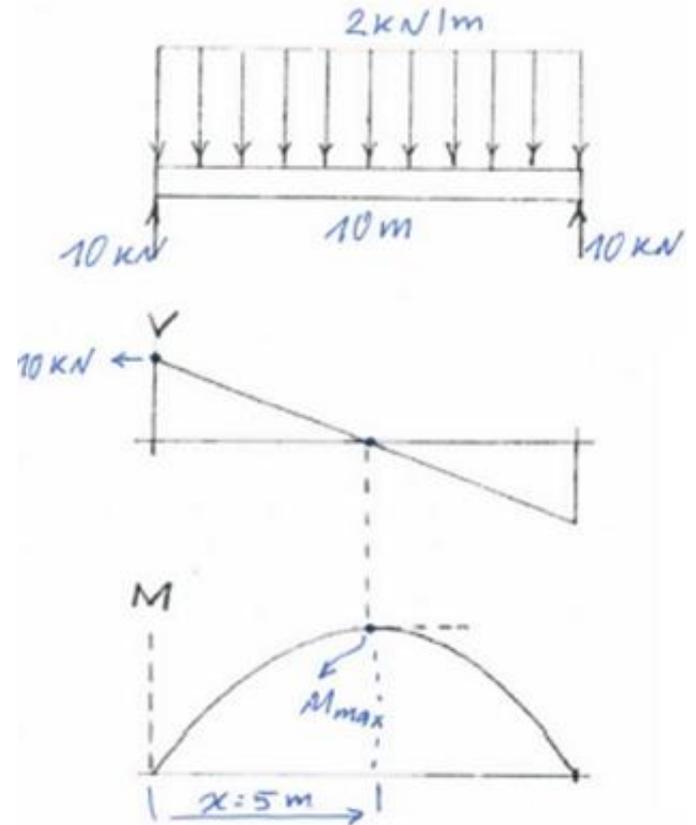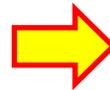
$10\,KN$ V

$\Sigma F_y = 0 \Rightarrow$

$10 - V - 2x = 0 \Rightarrow$

$\boxed{V = 10 - 2x}$

$\Sigma M_o = 0 \Rightarrow$

$M + 2x\left(\dfrac{x}{2}\right) - 10 \cdot x = 0$
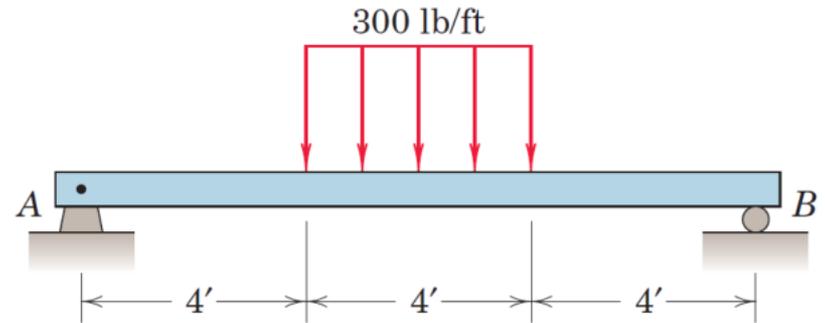
$\Rightarrow \boxed{M = -x^2 + 10x}$

V

$10\,KN \leftarrow$

M

M_{max}

$x = 5\,m$

$\dfrac{dM}{dx} = 0 \Rightarrow -2x + 10 = 0 \Rightarrow$
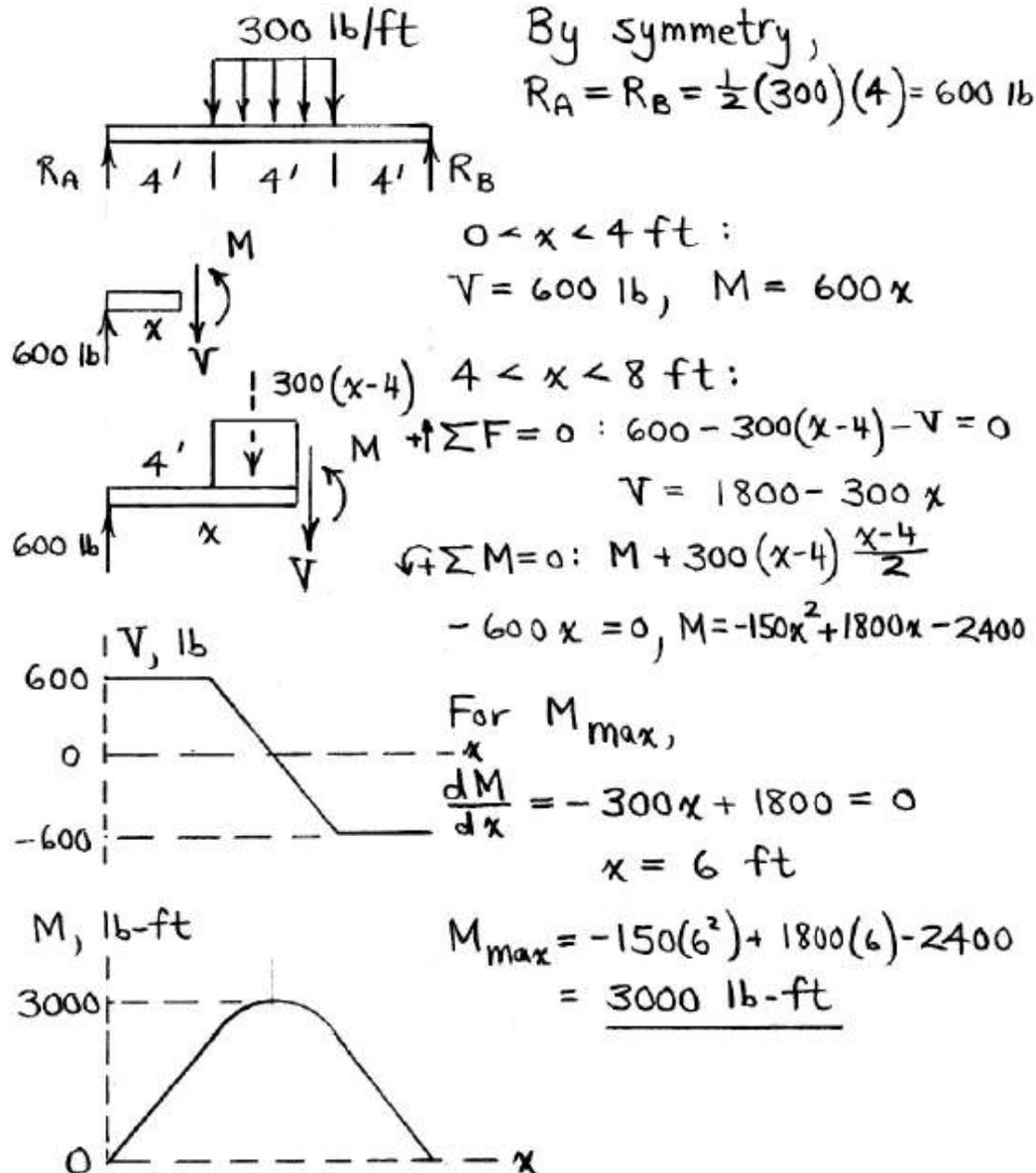
$\boxed{x = 5\,m} \Rightarrow$

$M_{max} = -(5)^2 + 10\,(5) = 25\,kN.m$

Exercise – 5

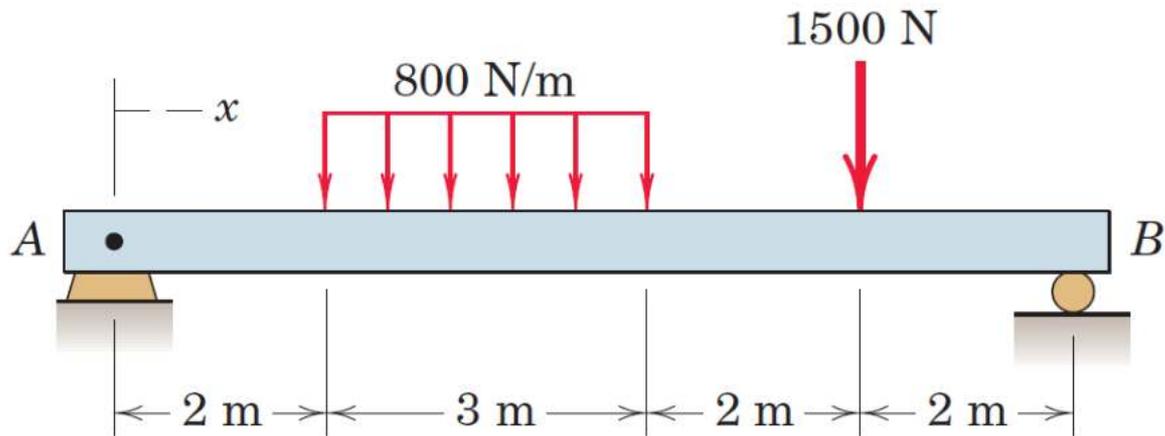Draw the shear and moment diagrams for the loaded beam and determine the maximum value M_{max} of the moment.

300 lb/ft

R_A 4' 4' 4' R_B

By symmetry,
$R_A = R_B = \frac{1}{2}(300)(4) = 600 \text{ lb}$

M

600 lb x V

$0 < x < 4 \text{ ft}$:
$V = 600 \text{ lb}, \quad M = 600x$

$300(x-4)$

4'

600 lb x V

M

$4 < x < 8 \text{ ft}$:
$+\uparrow\Sigma F = 0 : 600 - 300(x-4) - V = 0$
$V = 1800 - 300x$
$\overset{\curvearrowleft}{+}\Sigma M = 0: M + 300(x-4)\frac{x-4}{2}$
$- 600x = 0, \quad M = -150x^2 + 1800x - 2400$

$V, \text{ lb}$

600

0

-600

For M_{max},

$\frac{dM}{dx} = -300x + 1800 = 0$
$x = 6 \text{ ft}$

$M, \text{ lb-ft}$

3000

0

$M_{max} = -150(6^2) + 1800(6) - 2400$
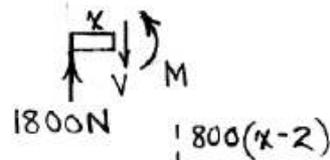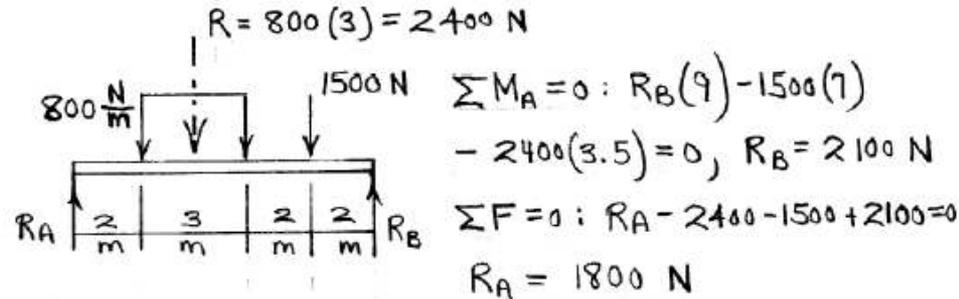$= \underline{3000 \text{ lb-ft}}$

Exercise – 6

Plot the **shear** and **moment** diagrams for the beam loaded with both the distributed and point loads. What are the values of the **shear** and **moment** at $x = 6\ m$? Determine the maximum bending moment M_{max}.
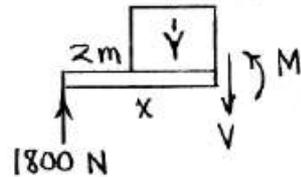
$R = 800\,(3) = 2400\ N$

$800\ \frac{N}{m}$ $1500\ N$

$\sum M_A = 0:\ R_B\left(9\right) - 1500\left(7\right)$

$-\ 2400\left(3.5\right) = 0,\ R_B = 2100\ N$

R_A $\frac{2}{m}$ $\frac{3}{m}$ $\frac{2}{m}$ $\frac{2}{m}$ R_B

$\sum F = 0:\ R_A - 2400 - 1500 + 2100 = 0$

$R_A = 1800\ N$

$0 < x < 2\ m:$

M

$\sum F = 0 \Rightarrow V = 1800\ N$

$1800\ N$ $800(x-2)$

$\sum M = 0 \Rightarrow M = 1800\,x$

$2m$ M

$2 < x < 5\ m:$

x V

$\sum F = 0:\ 1800 - 800\,(x-2) - V = 0$

$1800\ N$

$V = 3400 - 800\,x$

$\sum M = 0:\ M + 800\,(x-2)\ \frac{x-2}{2} + 1800\,x = 0$

$1500\ N$

$M = -400\,x^2 + 3400\,x - 1600$

V

M $2m$ $5 < x < 7:$

$9-x$
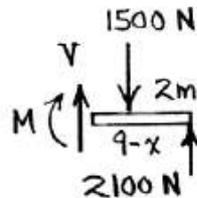
$\sum F = 0:\ 2100 - 1500 + V = 0$

$2100\ N$

$V = -600\ N$

$\sum M = 0:\ -M - 1500\,(7-x) + 2100(9-x)$

V $= 0,\quad M = 8400 - 600\,x$

M $9-x$ $7 < x < 9\ m:$
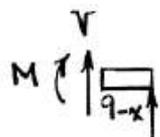
$\sum F = 0 \Rightarrow V = -2100\ N$

$2100\ N$ $\sum M = 0 \Rightarrow M = 18\,900 - 2100\,x$
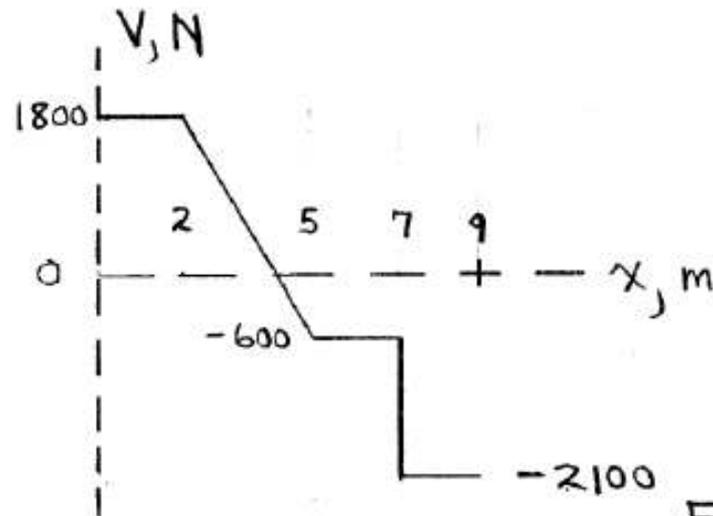
At $x = 6$ m:

$V = -600$ N
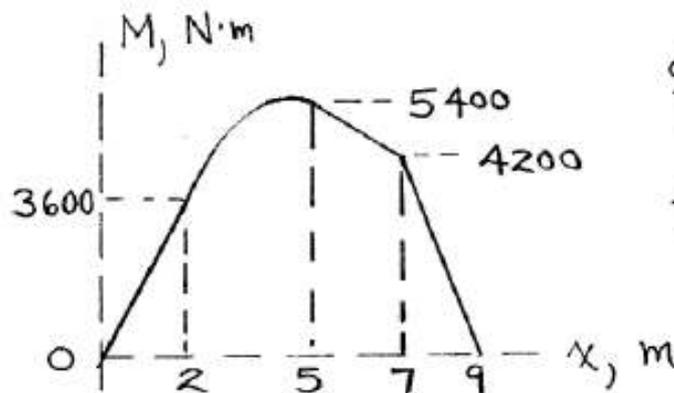
$M = 8400 - 600(6)$

$\quad = 4800$ N·m

For M_{max},

$\dfrac{dM}{dx} = 0$

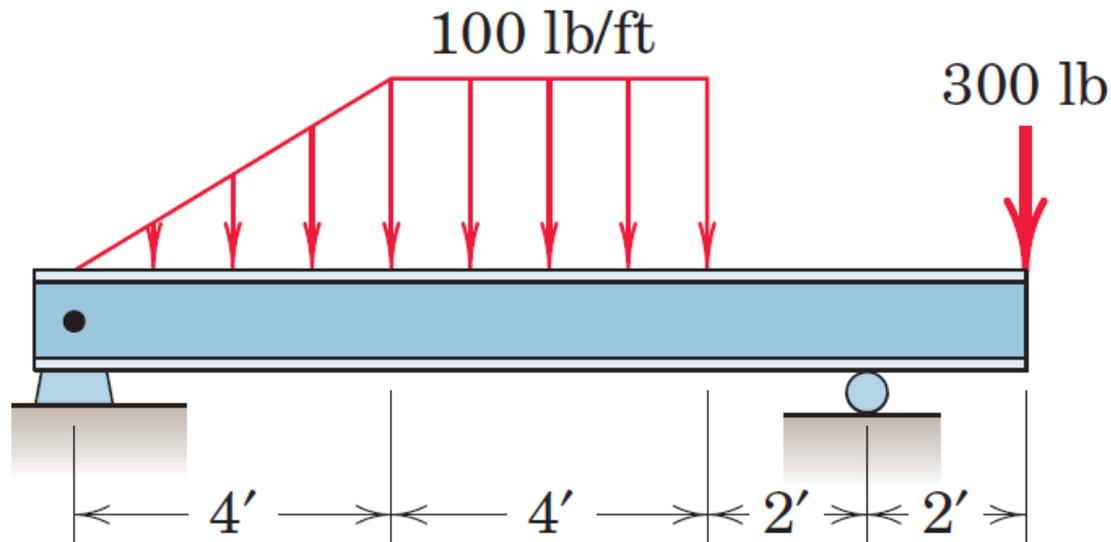$\dfrac{d}{dx}\left(-400x^2 + 3400x - 1600\right)$

$\quad = -800x + 3400 = 0$

$\underline{x = 4.25 \text{ m}}$

$M_{max} = -400(4.25)^2 + 3400(4.25) - 1600 = \underline{5620 \text{ N·m}}$

Exercise – 7

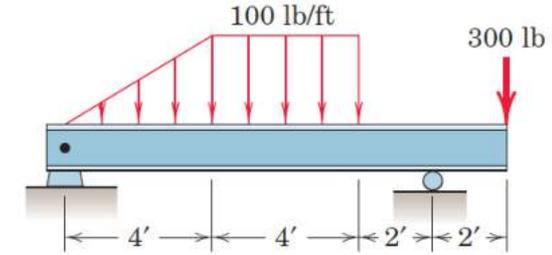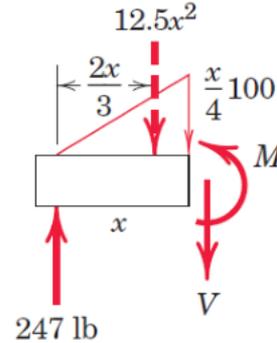Draw the **shear-force** and **bending-moment** diagrams for the loaded beam and determine the **maximum moment** M_{max} and its **location** x from the **left end**.

Solution - Exercise – 7

$[\Sigma F_y = 0] \quad V = 247 - 12.5x^2$

$[\Sigma M = 0]$

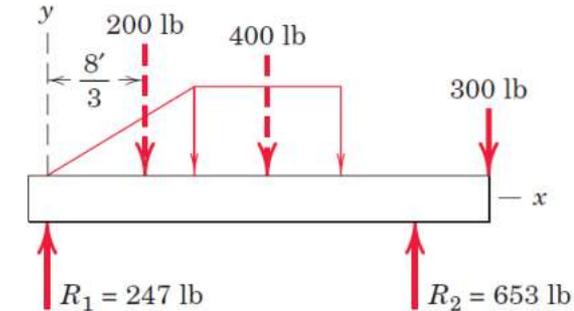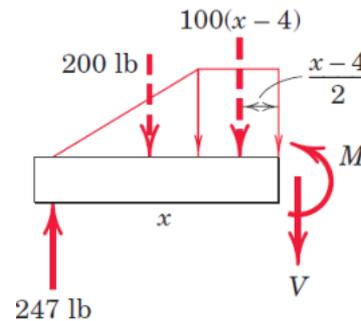$M + (12.5x^2)\dfrac{x}{3} - 247x = 0 \qquad M = 247x - 4.17x^3$

$[\Sigma F_y = 0] \quad V + 100(x - 4) + 200 - 247 = 0$

$$V = 447 - 100x$$
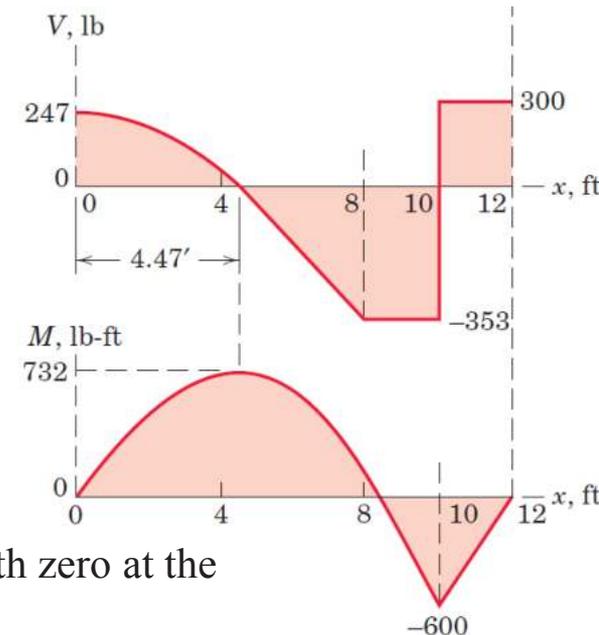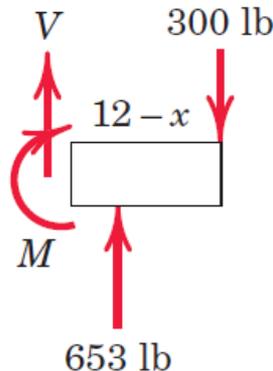
$[\Sigma M = 0] \quad M + 100(x - 4)\dfrac{x - 4}{2} + 200[x - \tfrac{2}{3}(4)] - 247x = 0$

$$M = -267 + 447x - 50x^2$$

$V = -353\,lb \quad \text{and} \quad M = 2930 - 353x$

V= 300 lb, and the moment **M** follows a straight-line relation beginning with zero at the right end of the beam. M_{max}= 732 lb.ft @ x = 4.47 ft.