Chapter 1

CONIC SECTIONS

1.1 Parabola

1.2 Ellipse

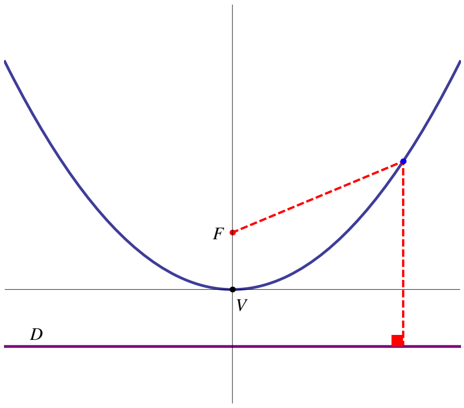
1.3 Hyperbola

1.1 Parabola

Definition: A **parabola** is the set of all points in the plane equidistant from a fixed point F (called the **focus**) and a fixed line D (called the **directrix**) in the same plane.

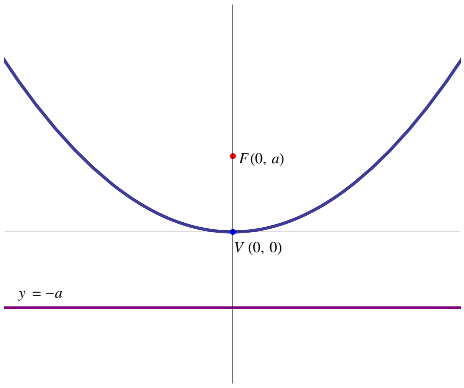
Notes:

1. The line passing through the focus F and perpendicular to the directrix D is called the **axis** of the parabola .
2. The point half-way from the focus F to the directrix D is called the **vertex** of the parabola and is denoted by V .



1.1.1 The vertex of the parabola is the origin :
This section discusses the special case where the vertex of the parabola is $(0, 0)$. There are four different cases :

1) $x^2 = 4ay$, where $a > 0$



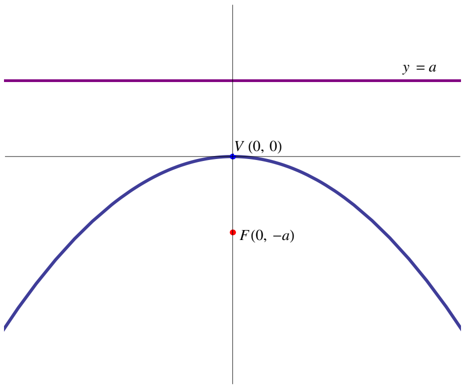
The parabola opens upwards .

The focus is $F(0, a)$.

The equation of the directrix is $y = -a$.

The axis of the parabola is the y-axis .

2) $x^2 = -4ay$, where $a > 0$



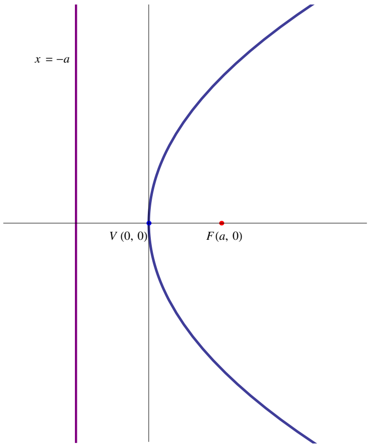
The parabola opens downwards (note the negative sign in the formula).

The focus is $F(0, -a)$.

The equation of the directrix is $y = a$.

The axis of the parabola is the y-axis .

3) $y^2 = 4ax$, where $a > 0$



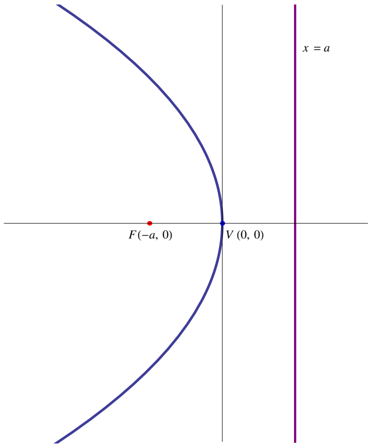
The parabola opens to the right.

The focus is $F(a, 0)$.

The equation of the directrix is $x = -a$.

The axis of the parabola is the x-axis .

4) $y^2 = -4ax$, where $a > 0$



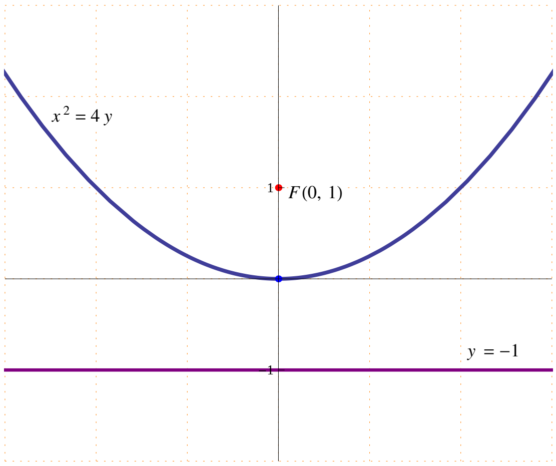
The parabola opens to the left (note the negative sign in the formula) .

The focus is $F(-a, 0)$.

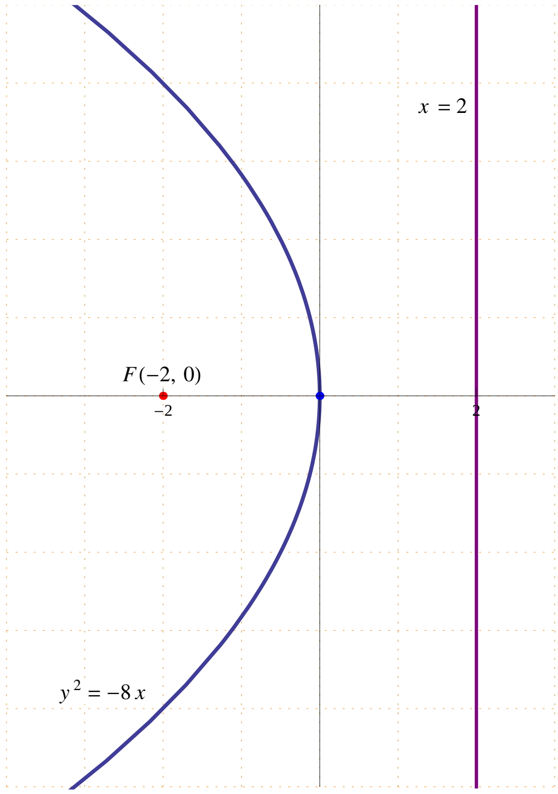
The equation of the directrix is $x = a$.

The axis of the parabola is the x-axis .

Example 1: Find the focus and the directrix of the parabola $x^2 = 4y$, and sketch its graph.
Solution: Since the variable x is of degree 2 and the formula contains a positive sign then $x^2 = 4y$ is similar to case(1), where the parabola opens upwards .
$4a = 4 \Rightarrow a = 1$
The focus is F(0,1) , and the equation of the directrix is $y = -1$.



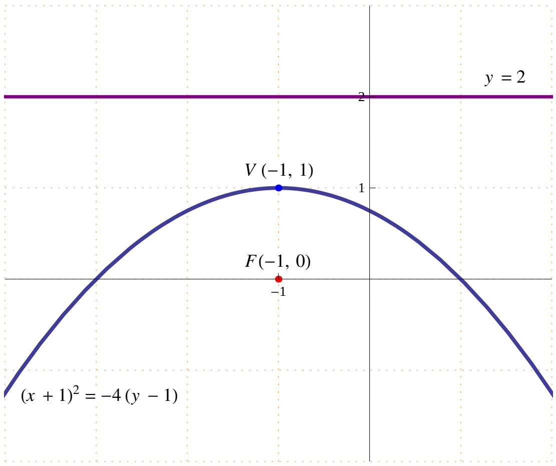
Example 2: Find the focus and the directrix of the parabola $y^2 = -8x$, and sketch its graph.
Solution: Since the variable y is of degree 2 and the formula contains a negative sign then $y^2 = -8x$ is similar to case(4), where the parabola opens to the left .
$-4a = -8 \;\Rightarrow\; a = 2$
The focus is F(-2,0) , and the equation of the directrix is $x = 2$.



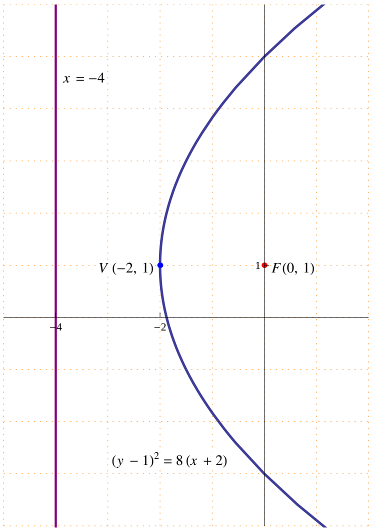
1.1.2 The general formula of a parabola :
This section discusses the general formula of a parabol where the vertex of the parabola is any point $V(h,k)$ in the plane.
There are four different cases :

No.	The general formula	Focus	Directrix	The parabola opens
1	$(x-h)^2 = 4a(y-k)$	$F(h,k+a)$	$y=k-a$	upwards
2	$(x-h)^2 = -4a(y-k)$	$F(h,k-a)$	$y=k+a$	downwards
3	$(y-k)^2 = 4a(x-h)$	$F(h+a,k)$	$x=h-a$	to the right
4	$(y-k)^2 = -4a(x-h)$	$F(h-a,k)$	$x=h+a$	to the left

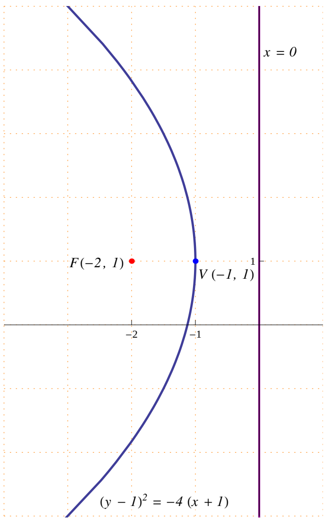
Example 1: Find the focus and the directrix of the parabola $(x+1)^2 = -4(y-1)$, and sketch its graph.
Solution : The equation of the parabola is similar to case (2).
$(x-h)^2 = (x+1)^2 = (x-(-1))^2 \;\Rightarrow\; h=-1$.
$(y-k) = (y-1) \;\Rightarrow\; k=1$.
$-4a = -4 \;\Rightarrow\; a=1$.
The vertex is $V(-1,1)$
The focus is $F(-1,0)$ and the equation of the directrix is $y=2$.
The parabola opens downwards (note the negative sign in the formula).



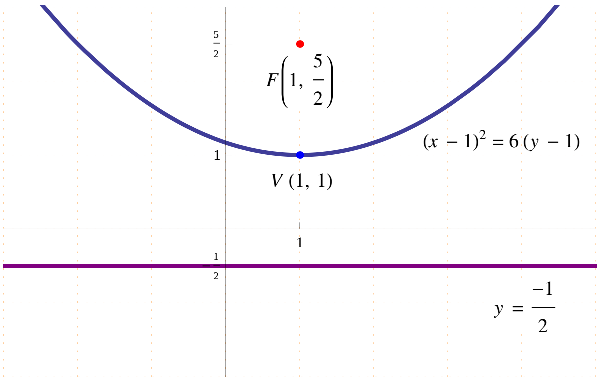
Example 2: Find the focus and the directrix of the parabola $(y-1)^2 = 8(x+2)$, and sketch its graph.
Solution : The equation of the parabola is similar to case (3).
$(y-k)^2 = (y-1)^2 \;\Rightarrow\; k=1$.
$(x-h) = (x+2) = (x-(-2)) \;\Rightarrow\; h=-2$.
$4a = 8 \;\Rightarrow\; a=2$.
The vertex is $V(-2,1)$
The focus is $F(0,1)$ and the equation of the directrix is $x=-4$.
The parabola opens to the right .



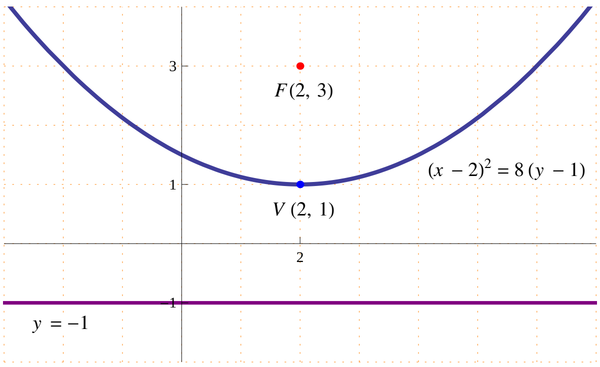
Example 3: Find the focus and the directrix of the parabola $2y^2-4y+8x+10 = 0$, and sketch its graph.
Solution : By completing the square
$2y^2 - 4y + 8x + 10 = 0 \ \Rightarrow\ 2y^2 - 4y = -8x - 10 \ \Rightarrow\ 2(y^2 - 2y) = -8x - 10$
$\Rightarrow\ 2(y^2-2y+1) = -8x-10+2 \ \Rightarrow\ 2(y-1)^2 = -8x-8 \Rightarrow\ 2(y-1)^2 = -8(x+1)$
$\Rightarrow\ (y-1)^2 = -4(x+1)$
The equation of the parabola is similar to case (4).
$(y-k)^2 = (y-1)^2 \ \Rightarrow\ k = 1$.
$(x-h) = (x+1) = (x-(-1)) \ \Rightarrow\ h = -1$.
$-4a = -4 \ \Rightarrow\ a = 1$.
The vertex is $V(-1, 1)$.
The focus is $F(-2, 1)$ and the equation of the directrix is $x = 0$ (the y-axis).
The parabola opens to the left (note the negative sign in the formula)



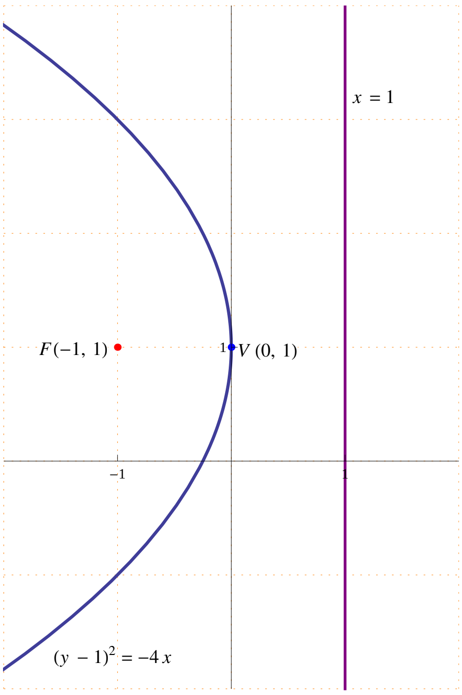
Example 4: Find the focus and the directrix of the parabola $x^2-6y-2x=-7$, and sketch its graph.
Solution : By completing the square
$x^2-6y-2x=-7 \Rightarrow x^2-2x=6y-7 \Rightarrow x^2-2x+1=6y-7+1$
$\Rightarrow (x-1)^2=6y-6 \Rightarrow (x-1)^2=6(y-1)$
The equation of the parabola is similar to case (1).
$(x-h)^2=(x-1)^2 \Rightarrow h=1$.
$(y-k)=(y-1)) \Rightarrow k=1$.
$4a=6 \Rightarrow a=\frac{6}{4}=\frac{3}{2}$.
The vertex is $V(1,1)$
The focus is $F\left(1,\frac{5}{2}\right)$ and the equation of the directrix is $y=-\frac{1}{2}$.
The parabola opens upwards.



Example 5: Find the equation of the parabola with vertex $V(2,1)$ and focus $F(2,3)$ and sketch its graph.
Solution : Since the focus is located upper than the vertex then the parabola opens upwards.
Hence its equation is $(x-h)^2=4a(y-k)$.
Since the vertex is $V(2,1)$ then $h=2$ and $k=1$
a equals the distance between $V(2,1)$ and $F(2,3)$ which equals 2 .
The equation of the parabola with $V(2,1)$ and $F(2,3)$ is $(x-2)^2=8(y-1)$



Example 6: Find the equation of the parabola with focus $F(-1,1)$ and directrix $x=1$ and sketch its graph.
Solution : Since the focus is located to the left of the directrix then the parabola opens to the left.
Hence its equation is $(y-k)^2 = -4a(x-h)$.
The vertex is half-way beween the focus and the directrix , hence $V(0,1)$
a equals the distance between $V(0,1)$ and $F(-1,1)$ which equals 1 .
The equation of the parabola with $F(-1,1)$ and directrix $x=1$ is $(y-1)^2 = -4x$

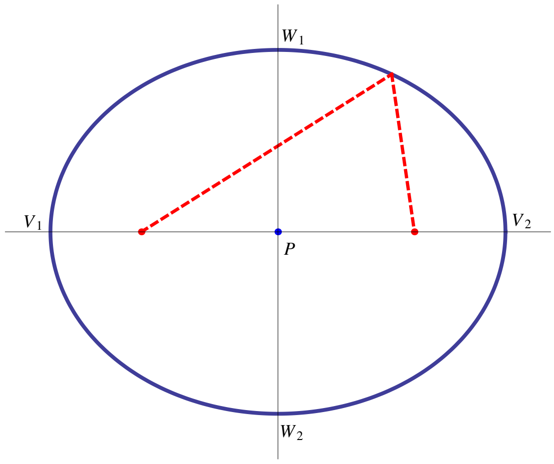


1.2 Ellipse

Definition: An **ellipse** is the set of all points in the plane for which the sum of the distances to two fixed points is constant.

Notes :

1. The two fixed points are called the **foci** of the ellipse and are denoted by F_1 and F_2.
2. The midpoint between F_1 and F_2 is called the **center** of the ellipse and is denoted by P.
3. The endpoints of the **major axis** are called the vertices of the ellipse and are denoted by V_1 and V_2.
4. The endpoints of the **minor axis** are denoted by W_1 and W_2.



1.2.1 The center of the ellipse is the origin :
This section discusses the special case where the center of the ellipse is $(0, 0)$. There are two different cases :

1) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $a > b$:

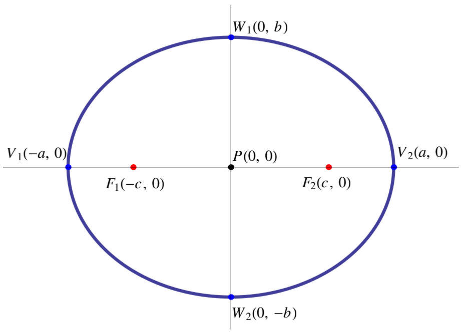
The foci of the ellipse are $F_1(-c, 0)$ and $F_2(c, 0)$, where $c = \sqrt{a^2 - b^2}$.

The vertices of the ellipse are $V_1(-a, 0)$ and $V_2(a, 0)$.

The endpoints of the minor axis are $W_1(0, b)$ and $W_2(0, -b)$.

The major axis lies on the x-axis , and its length is $2a$.

The minor axis lies on the y-axis , and its length is $2b$.



2) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $b > a$:

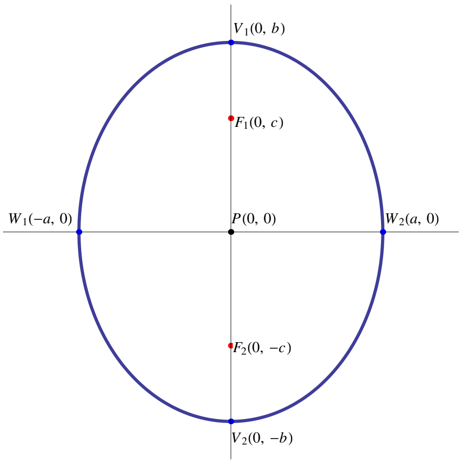
The foci of the ellipse are $F_1(0, c)$ and $F_2(0, -c)$, where $c = \sqrt{b^2 - a^2}$.

The vertices of the ellipse are $V_1(0, b)$ and $V_2(0, -b)$.

The endpoints of the minor axis are $W_1(-a, 0)$ and $W_2(a, 0)$.

The major axis lies on the y-axis , and its length is $2b$.

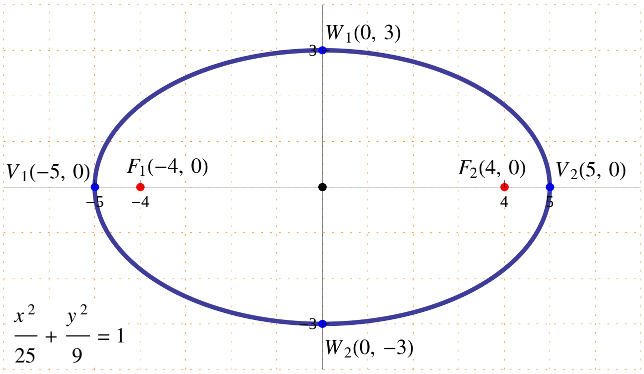
The minor axis lies on the x-axis , and its length is $2a$.



Example 1: Identify the features of the ellipse $9x^2 + 25y^2 = 225$, and sketch its graph.

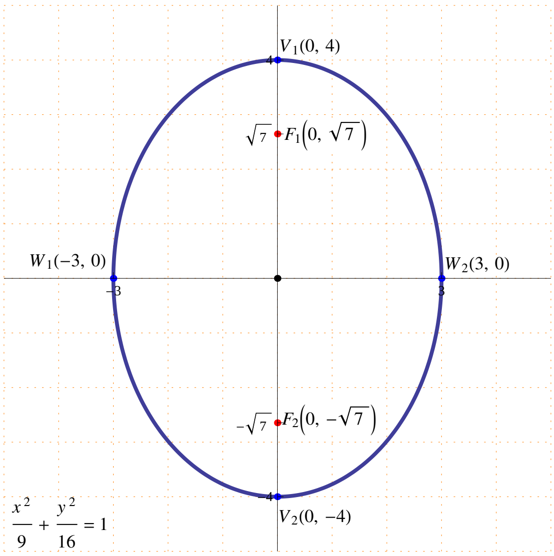
Solution : $9x^2 + 25y^2 = 225 \;\Rightarrow\; \dfrac{9x^2}{225} + \dfrac{25y^2}{225} = 1 \;\Rightarrow\; \dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$
$a^2 = 25 \;\Rightarrow\; a = 5$ and $b^2 = 9 \;\Rightarrow\; b = 3$.
Since $a > b$ then $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$ is similar to case (1).
$c = \sqrt{a^2 - b^2} = \sqrt{25 - 9} = \sqrt{16} = 4$.
The foci are $F_1\,(-4, 0)$ and $F_2\,(4, 0)$.
The vertices are $V_1(-5, 0)$ and $V_2(5, 0)$.

The endpoints of the minor axis are $W_1(0,3)$ and $W_2(0,-3)$.
The length of the major axis is $2a = 10$.
The length of the minor axis is $2b = 6$.



Example 2: Identify the features of the ellipse $16x^2 + 9y^2 = 144$, and sketch its graph.

Solution : $16x^2 + 9y^2 = 144 \Rightarrow \dfrac{16x^2}{144} + \dfrac{9y^2}{144} = 1 \Rightarrow \dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$
$a^2 = 9 \Rightarrow a = 3$ and $b^2 = 16 \Rightarrow b = 4$.
Since $b > a$ then $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$ is similar to case (2).
$c^2 = \sqrt{b^2 - a^2} = \sqrt{16 - 9} = \sqrt{7}$.
The foci are $F_1\left(0, \sqrt{7}\right)$ and $F_2\left(0, -\sqrt{7}\right)$.
The vertices are $V_1(0,4)$ and $V_2(0,-4)$.
The endpoints of the minor axis are $W_1(-3,0)$ and $W_2(3,0)$.
The length of the major axis is $2b = 8$.
The length of the minor axis is $2a = 6$.



1.2.2 The general formula of an ellipse :

This section discusses the general formula of an ellipse where the center of the ellipse is any point $P(h,k)$ in the plane.

There are two different cases :

No.	The general Formula	The Foci	The Vertices	W_1 and W_2
1	$\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$ ($a>b$) and $c=\sqrt{a^2-b^2}$	$F_1(h-c,k)$ $F_2(h+c,k)$	$V_1(h-a,k)$ $V_2(h+a,k)$	$W_1(h,k-b)$ $W_2(h,k+b)$
2	$\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$ ($b>a$) and $c=\sqrt{b^2-a^2}$	$F_1(h,k-c)$ $F_2(h,k+c)$	$V_1(h,k-b)$ $V_2(h,k+b)$	$W_1(h-a,k)$ $W_2(h+a,k)$

Example 1: Find the equation of the ellipse with foci at $(-3,1)$, $(5,1)$, and one of its vertices is $(7,1)$, and sketch its graph.

Solution : The center of the ellipse $P(h,k)$ is located in the middle of the two foci, hence $(h,k)=\left(\dfrac{-3+5}{2},\dfrac{1+1}{2}\right)=(1,1)$.

c is the distance between the center and one of the foci , and it equals to 4 (see the figure).

Since the major axis (where the two foci lie) is parallel to the x-axis , then the general formula of the ellipse is $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$, where $a>b$.

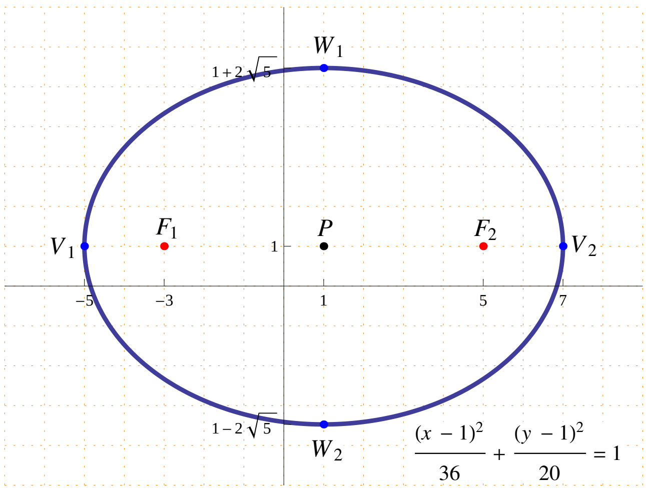
a is the distance between the center and one of the vertices, and it equals 6 (see the figure).

$c^2=a^2-b^2 \Rightarrow (4)^2=(6)^2-b^2 \Rightarrow b^2=36-16=20 \Rightarrow b=2\sqrt{5}$.

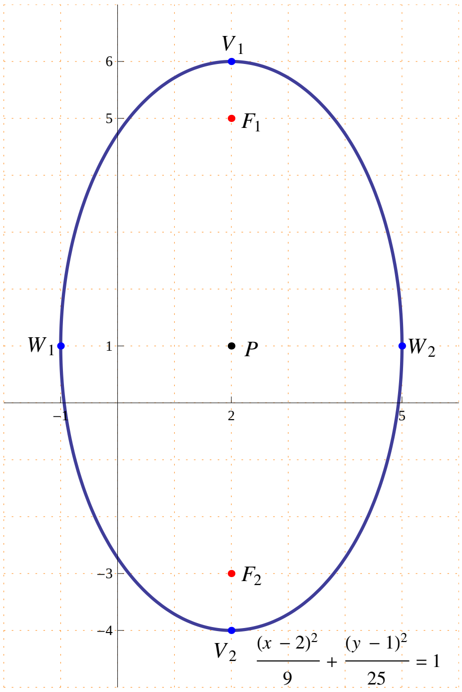
The equation of the ellipse is $\dfrac{(x-1)^2}{36}+\dfrac{(y-1)^2}{20}=1$.

The vertices of the ellipse are $V_1(-5,1)$ and $V_2(7,1)$.

The endpoints of the minor axis are $W_1\left(1,1+2\sqrt{5}\right)$ and $W_2\left(1,1-2\sqrt{5}\right)$.

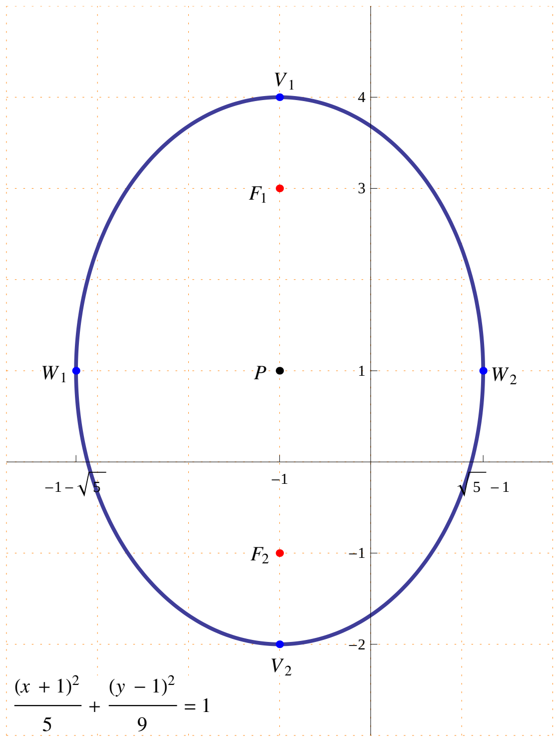


Example 2: Find the equation of the ellipse with foci at $(2,5)$, $(2,-3)$, and the length of its minor axis equals 6 , and sketch its graph.
Solution : The center of the ellipse $P(h,k)$ is located in the middle of the two foci, hence $(h,k)=\left(\dfrac{2+2}{2},\dfrac{-3+5}{2}\right)=(2,1)$.
c is the distance between the center and one of the foci , and it equals to 4 (see the figure).
Since the major axis (where the two foci lie) is parallel to the y-axis , then the general formula of the ellipse is $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$, where $b>a$.
The length of the minor axis is 6 means that $2a=6 \Rightarrow a=3$.
$c^2=b^2-a^2 \Rightarrow (4)^2=b^2-(3)^2 \Rightarrow b^2=16+9=25 \Rightarrow b=5$.
The equation of the ellipse is $\dfrac{(x-2)^2}{9}+\dfrac{(y-1)^2}{25}=1$.
The vertices of the ellipse are $V_1(2,6)$ and $V_2(2,-4)$.
The endpoints of the minor axis are $W_1(-1,1)$ and $W_2(5,1)$.

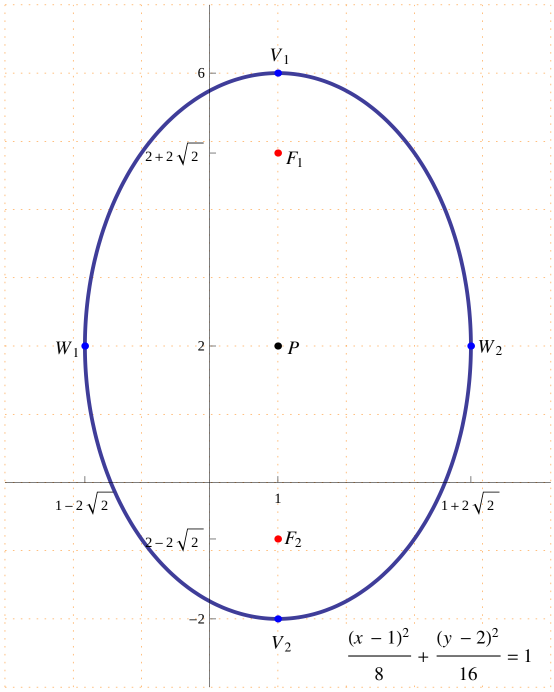


Example 3: Find the equation of the ellipse with vertices at $(-1,4)$, $(-1,-2)$ and the distance between its two foci equals 4 , and sketch its graph.
Solution : The center of the ellipse $P(h,k)$ is located in the middle of the two vertices, hence $(h,k)=\left(\dfrac{-1-1}{2},\dfrac{-2+4}{2}\right)=(-1,1)$.
The distance between the two foci equals 4 means that $2c=4 \Rightarrow c=2$.
Since the major axis (where the two vertices lie) is parallel to the y-axis , then the general formula of the ellipse is $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$, where $b>a$.
The length of the major axis (the distance between the two vertices) equals 6 ,

this means $2b = 6 \Rightarrow b = 3$.
$c^2 = b^2 - a^2 \Rightarrow (2)^2 = (3)^2 - a^2 \Rightarrow a^2 = 9 - 4 = 5 \Rightarrow a = \sqrt{5}$.
The equation of the ellipse is $\dfrac{(x+1)^2}{5} + \dfrac{(y-1)^2}{9} = 1$.
The foci of the ellipse are $F_1(-1, 3)$ and $F_2(-1, -1)$.
The endpoints of the minor axis are $W_1\left(-1-\sqrt{5}, 1\right)$ and $W_2\left(-1+\sqrt{5}, 1\right)$.



Example 4: Identify the features of the ellipse $4x^2 + 2y^2 - 8x - 8y - 20 = 0$, and sketch its graph.
Solution :
$4x^2 + 2y^2 - 8x - 8y - 20 = 0 \Rightarrow (4x^2 - 8x) + (2y^2 - 8y) = 20$
$\Rightarrow 4(x^2 - 2x) + 2(y^2 - 4y) = 20$
By completing the square
$4(x^2 - 2x) + 2(y^2 - 4y) = 20 \Rightarrow 4(x^2 - 2x + 1) + 2(y^2 - 4y + 4) = 20 + 12$
$\Rightarrow 4(x-1)^2 + 2(y-2)^2 = 32$
$\Rightarrow \dfrac{4(x-1)^2}{32} + \dfrac{2(y-2)^2}{32} = 1 \Rightarrow \dfrac{(x-1)^2}{8} + \dfrac{(y-2)^2}{16} = 1$
$b^2 = 16 \Rightarrow b = 4$ and $a^2 = 8 \Rightarrow b = \sqrt{8} = 2\sqrt{2}$.
$c^2 = b^2 - a^2 \Rightarrow c^2 = 16 - 8 = 8 \Rightarrow c = \sqrt{8} = 2\sqrt{2}$.
The center of the ellipse is $(1, 2)$.
The foci of the ellipse are $F_1\left(1, 2 + 2\sqrt{2}\right)$ and $F_2\left(1, 2 - 2\sqrt{2}\right)$.
The vertices of the ellipse are $V_1(1, 6)$ and $V_2(1, -2)$.
The endpoints of the minor axis are $W_1\left(1 - 2\sqrt{2}, 2\right)$ and $W_2\left(1 + 2\sqrt{2}, 2\right)$
The length of the major axis is 8 and the length of the minor axis is $2\sqrt{8} = 4\sqrt{2}$.



V1
6
2+2√2
F1
W1
2
P
W2
1−2√2
1
1+2√2
2−2√2
F2
−2
V2
(x − 1)^2/8 + (y − 2)^2/16 = 1

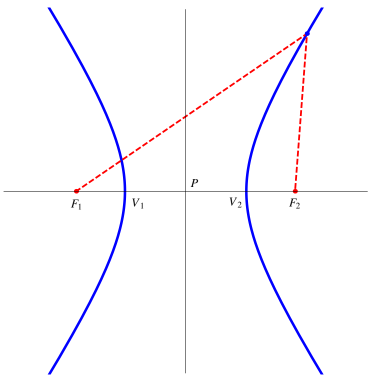
1.3 Hyperbola

Definition: A **hyperbola** is the set of all points in the plane for which the difference of the distances between two fixed points is constant.

Notes :

1. The two fixed points are called the **foci** of the hyperbola and are denoted by F_1 and F_2.

2. The midpoint between F_1 and F_2 is called the **center** of the hyperbola and is denoted by P.



1.3.1 The center of the hyperbola is the origin :
This section discusses the special case where the center of the hyperbola is $(0, 0)$. There are two different cases :

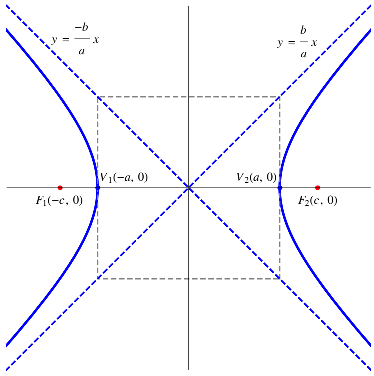
1) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, where $a > 0$ and $b > 0$:

The foci of the hyperbola are $F_1(-c, 0)$ and $F_2(c, 0)$, where $c = \sqrt{a^2 + b^2}$.

The vertices of the hyperbola are $V_1(-a, 0)$ and $V_2(a, 0)$.

The line segment between V_1 and V_2 is the **transverse axis** , it lies on the x-axis and its length is $2a$.

The equations of the asymptotes are $y = \dfrac{b}{a}x$ and $y = -\dfrac{b}{a}x$.



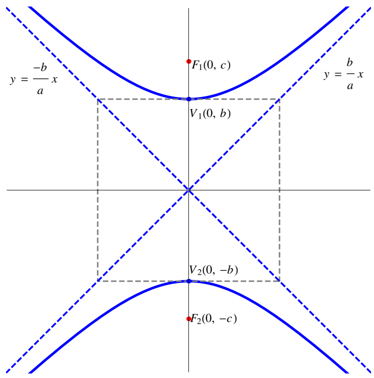
2) $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$, where $a > 0$ and $b > 0$:

The foci of the hyperbola are $F_1(0, c)$ and $F_2(0, -c)$, where $c = \sqrt{a^2 + b^2}$.

The vertices of the hyperbola are $V_1(0, b)$ and $V_2(0, -b)$.

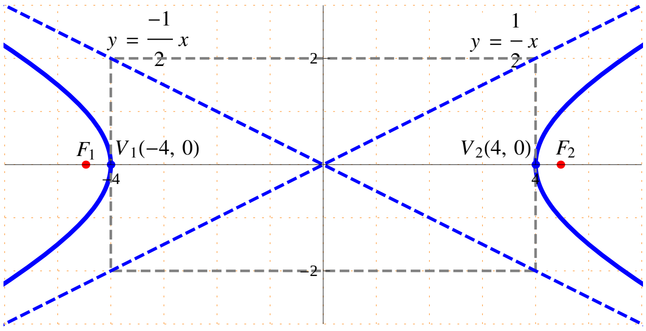
The line segment between V_1 and V_2 is the **transverse axis** , it lies on the y-axis and its length is $2b$.

The equations of the asymptotes are $y = \dfrac{b}{a}x$ and $y = -\dfrac{b}{a}x$.

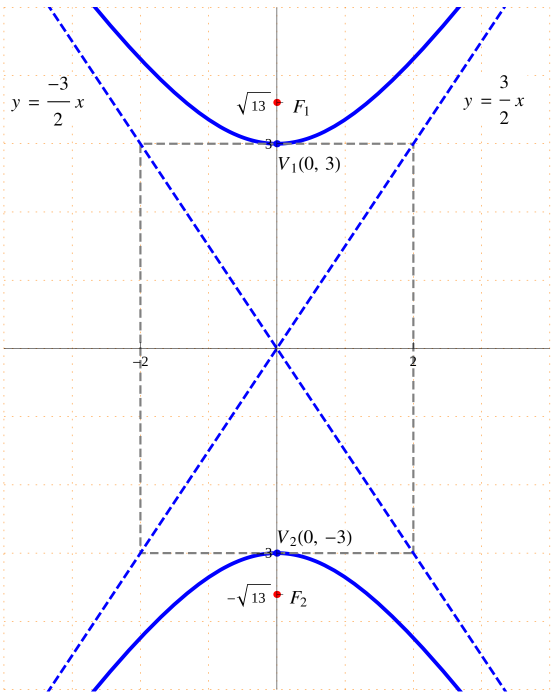


Example 1: Identify the features of the hyperbola $4x^2 - 16y^2 = 64$, and sketch its graph.
Solution :
$4x^2 - 16y^2 = 64 \;\Rightarrow\; \dfrac{4x^2}{64} - \dfrac{16y^2}{64} = 1 \;\Rightarrow\; \dfrac{x^2}{16} - \dfrac{y^2}{4} = 1$
This form is similar to case (1).
$a^2 = 16 \;\Rightarrow\; a = 4$ and $b^2 = 4 \;\Rightarrow\; b = 2$
$c = \sqrt{a^2 + b^2} = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$
The foci of the hyperbola are $F_1\left(-2\sqrt{5}, 0\right)$ and $F_2\left(2\sqrt{5}, 0\right)$.
The vertices are $V_1(-4, 0)$ and $V_2(4, 0)$.
The transverse axis lies on the x-axis and its length is $2a = 8$.

The equations of the asymptotes are $y = \frac{2}{4}x = \frac{1}{2}x$ and $y = -\frac{2}{4}x = -\frac{1}{2}x$



Example 2: Identify the features of the hyperbola $4y^2 - 9x^2 = 36$, and sketch its graph.
Solution :
$4y^2 - 9x^2 = 36 \Rightarrow \frac{4y^2}{36} - \frac{9x^2}{36} = 1 \Rightarrow \frac{y^2}{9} - \frac{x^2}{4} = 1$
This form is similar to case (2).
$a^2 = 4 \Rightarrow a = 2$ and $b^2 = 9 \Rightarrow b = 3$
$c = \sqrt{a^2 + b^2} = \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13}$
The foci of the hyperbola are $F_1 \left(0, \sqrt{13}\right)$ and $F_2 \left(0, -\sqrt{13}, \right)$.
The vertices are $V_1(0, 3)$ and $V_2(0, -3)$.
The transverse axis lies on the y-axis and its length is $2b = 6$.
The equations of the asymptotes are $y = \frac{3}{2}x$ and $y = -\frac{3}{2}x$



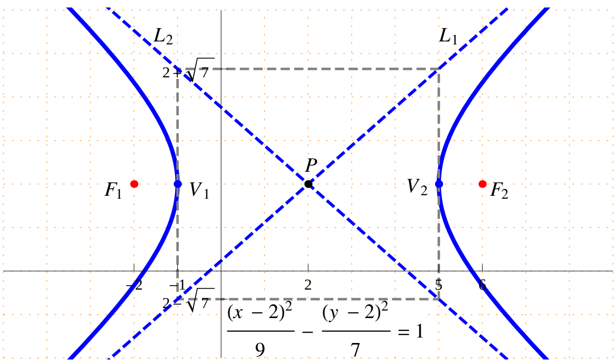
1.3.2 The general formula of a hyperbola :

This section discusses the general formula of a hyperbola where the center of the hyperbola is any point $P(h,k)$ in the plane.
There are two different cases :

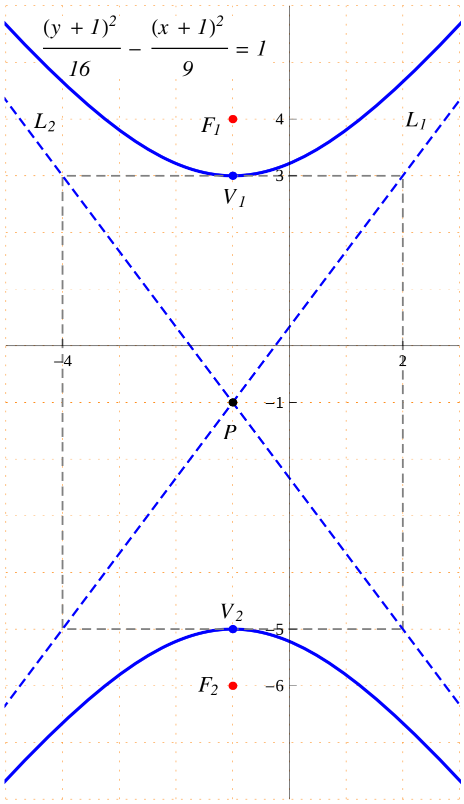
No.	The general Formula	The Foci	The Vertices	Transverse axis
1	$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ $(\ c^2=a^2+b^2\ )$	$F_1(h-c,k)$ $F_2(h+c,k)$	$V_1(h-a,k)$ $V_2(h+a,k)$	parallel to the x-axis
2	$\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$ $(\ c^2=a^2+b^2\ )$	$F_1(h,k+c)$ $F_2(h,k-c))$	$V_1(h,k+b)$ $V_2(h,k-b)$	parallel to the y-axis

The equations of the asymptotes are $y=\frac{b}{a}(x-h)+k$ and $y=-\frac{b}{a}(x-h)+k$

Example 1: Find the equation of the hyperbola with foci at $(-2,2)$, $(6,2)$ and one of its vertices is $(5,2)$, and sketch its graph.
Solution :
The center of the hyperbola $P(h,k)$ is located in the middle of the two foci , hence $(h,k)=\left(\frac{-2+6}{2},\frac{2+2}{2}\right)=(2,2)$
Note that the two foci lie on a line parallel to the x-axis , hence the general formula of the hyperbola is $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$.
$2c$ is the distance between the two foci , hence $2c=8\ \Rightarrow\ c=4$.
a is the distance between the center $(2,2)$ and the vertex $(5,2)$, hence $a=3$, and the other vertex is $(-1,2)$.
$c^2=a^2+b^2\ \Rightarrow\ 4^2=3^2+b^2\ \Rightarrow\ b^2=16-9=7\ \Rightarrow\ c=\sqrt{7}$.
The equation of the hyperbola is $\frac{(x-2)^2}{9}-\frac{(y-2)^2}{7}=1$
The equations of the asymptotes are $L_1: y=\frac{\sqrt{7}}{3}(x-2)+2$ and
$L_2: y=-\frac{\sqrt{7}}{3}(x-2)+2$

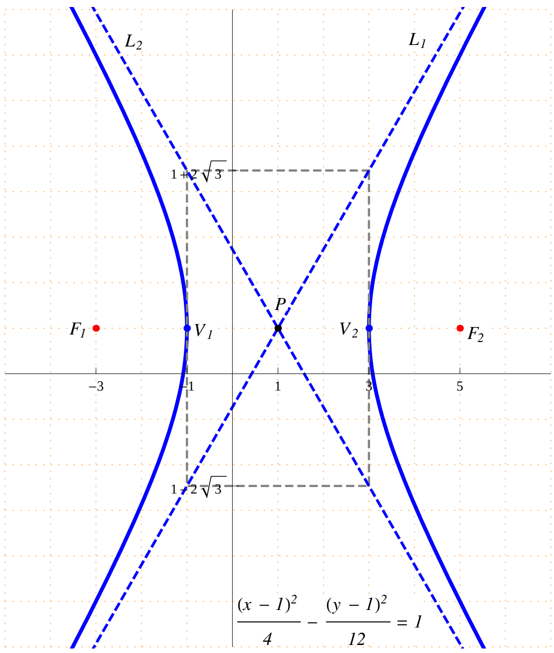


Example 2: Find the equation of the hyperbola with foci at $(-1,-6)$, $(-1,4)$ and the length of its transverse axis is 8 , and sketch its graph.
Solution :
The center of the hyperbola $P(h,k)$ is located in the middle of the two foci , hence $(h,k)=\left(\frac{-1-1}{2},\frac{-6+4}{2}\right)=(-1,-1)$
Note that the two foci lie on a line parallel to the y-axis , hence the general formula of the hyperbola is $\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$.
$2c$ is the distance between the two foci , hence $2c=10 \Rightarrow c=5$.
The length of the transverse axis is 8 , this means $2b=8 \Rightarrow b=4$.
The vertices are $(-1,-5)$ and $(-1,3)$.
$c^2=a^2+b^2 \Rightarrow 5^2=a^2+4^2 \Rightarrow a^2=25-16=9 \Rightarrow a=3$.
The equation of the hyperbola is $\frac{(y+1)^2}{16}-\frac{(x+1)^2}{9}=1$.
The equations of the asymptotes are $L_1: \ y=\frac{4}{3}(x+1)-1$ and
$L_2: \ y=-\frac{4}{3}(x+1)-1$



Example 3: Find the equation of the hyperbola with center at $(1,1)$, one of its foci is $(5,1)$ and one of its vertices is $(-1,1)$, and sketch its graph.
Solution :
Since the center and the focus lie on a line parallel to the x-axis , then the

general formula of the hyperbola is $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$.
c is the distance between the center $(1,1)$ and the focus $(5,1)$, hence $c = 4$, the other foci is $(-3,1)$.
a is the distance between the center $(1,1)$ and the vertex $(-1,1)$, hence $a = 2$, the other vertex is $(3,1)$.
$c^2 = a^2 + b^2 \Rightarrow 4^2 = 2^2 + b^2 \Rightarrow b^2 = 16 - 4 = 12 \Rightarrow b = \sqrt{12} = 2\sqrt{3}$
The equation of the hyperbola is $\dfrac{(x-1)^2}{4} - \dfrac{(y-1)^2}{12} = 1$.
The equations of the asymptotes are
$L_1: \ y = \dfrac{2\sqrt{3}}{2}(x-1)+1 = \sqrt{3}(x-1)+1$ and $L_2: \ y = -\sqrt{3}(x-1)+1$



Example 4: Identify the features of the hyperbola $2y^2 - 4x^2 - 4y - 8x - 34 = 0$, and sketch its graph.
Solution :
$2y^2 - 4x^2 - 4y - 8x - 34 = 0 \Rightarrow (2y^2 - 4y) - (4x^2 + 8x) = 34$
$\Rightarrow 2(y^2 - 2y) - 4(x^2 + 2x) = 34$
$\Rightarrow 2(y^2 - 2y + 1) - 4(x^2 + 2x + 1) = 34 + 2 - 4 \Rightarrow 2(y-1)^2 - 4(x+1)^2 = 32$
$\Rightarrow \dfrac{2(y-1)^2}{32} - \dfrac{4(x+1)^2}{32} = 1 \Rightarrow \dfrac{(y-1)^2}{16} - \dfrac{(x+1)^2}{8} = 1$
$b^2 = 16 \Rightarrow b = 4$ and $a^2 = 8 \Rightarrow a = \sqrt{8} = 2\sqrt{2}$.
$c^2 = a^2 + b^2 \Rightarrow c^2 = 16 + 8 = 24 \Rightarrow c = \sqrt{24} = 2\sqrt{6}$.
The center of the hyperbola is $P(-1,1)$.
The foci of the hyperbola are $F_1\left(-1, 1+2\sqrt{6}\right)$ and $F_2\left(-1, 1-2\sqrt{6}\right)$.
The vertices of the hyperbola are $V_1(-1,5)$ and $V_2(-1,-3)$.
The transverse axis is parallel to the y-axis and its length is $2b = 8$.
The equations of the asymptotes are

$$L_1: \ y = \frac{4}{2\sqrt{2}}(x+1)+1 = \sqrt{2}(x+1)+1 \text{ and } L_2: \ y = -\sqrt{2}(x+1)+1$$