Differential Equations

MATH 222

L3

First Semester for Second Year

College of Science and Computer Engineering

Taibah University

Yanbu

Saudi Arabian .

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Dr. Essam Othman Abdel-Rahman

Separable Equations

- In this section we examine a subclass of linear and nonlinear first order equations. Consider the first order ordinary differential equation

$$\frac{dy}{dx} = f(x, y)$$

- We can rewrite this in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

- For example, let $M(x,y) = -f(x,y)$ and $N(x,y) = 1$. There may be other ways as well. In differential form,

$$M(x, y)dx + N(x, y)dy = 0$$

- If M is a function of x only and N is a function of y only, then

$$M(x)dx + N(y)dy = 0$$

- In this case, the equation is called **separable**.

Separable Equations

- Solve the following first order nonlinear equation:

$$\frac{dy}{dx} = -\frac{x}{y}$$

- Separating variables, and using calculus, we obtain

$$ydy = -xdx$$

$$\int ydy = -\int xdx$$

$$\frac{y^2}{2} = \frac{-x^2}{2} + C$$

Separable Equations

- Solve the following first order nonlinear equation:

$$\frac{dy}{dx} = \frac{x^2 + 1}{y^2 - 1}$$

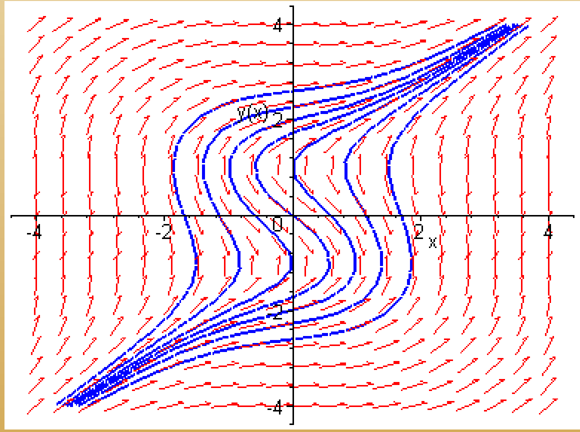
- Separating variables, and using calculus, we obtain

$$\left(y^2 - 1\right)dy = \left(x^2 + 1\right)dx$$

$$\int \left(y^2 - 1\right)dy = \int \left(x^2 + 1\right)dx$$

$$\frac{1}{3}y^3 - y = \frac{1}{3}x^3 + x + C$$

$$y^3 - 3y = x^3 + 3x + C$$



- The equation above defines the solution *y* implicitly.

A graph showing the direction field and implicit plots of several integral curves for the differential equation is given above.

Implicit and Explicit Solutions

- Solve the following first order nonlinear equation:

$$\frac{dy}{dx}=\frac{3x^2+4x+2}{2(y-1)}$$

- Separating variables and using calculus, we obtain

$$2(y-1)dy=\left(3x^2+4x+2\right)dx$$

$$2\int(y-1)dy=\int\left(3x^2+4x+2\right)dx$$

$$y^2-2y=x^3+2x^2+2x+C$$

- The equation above defines the solution *y* implicitly. An explicit expression for the solution can be found in this case:

$$y^2-2y-\left(x^3+2x^2+2x+C\right)=0 \;\Rightarrow\; y=\frac{2\pm\sqrt{4+4\left(x^3+2x^2+2x+C\right)}}{2}$$

$$y=1\pm\sqrt{x^3+2x^2+2x+C1}$$

Initial Value Problem

- Suppose we seek a solution satisfying $y(0) = -1$. Using the implicit expression of y, we obtain

$$y^2 - 2y = x^3 + 2x^2 + 2x + C$$

$$(-1)^2 - 2(-1) = C \quad \Rightarrow \quad C = 3$$

- Thus the implicit equation defining y is

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3$$

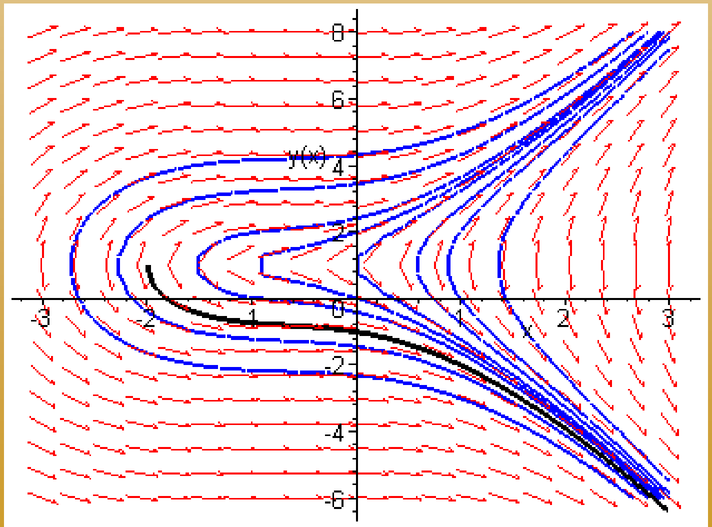
- Using explicit expression of y,

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + C1}$$

$$-1 = 1 \pm \sqrt{C1} \quad \Rightarrow \quad C1 = 4$$

- It follows that

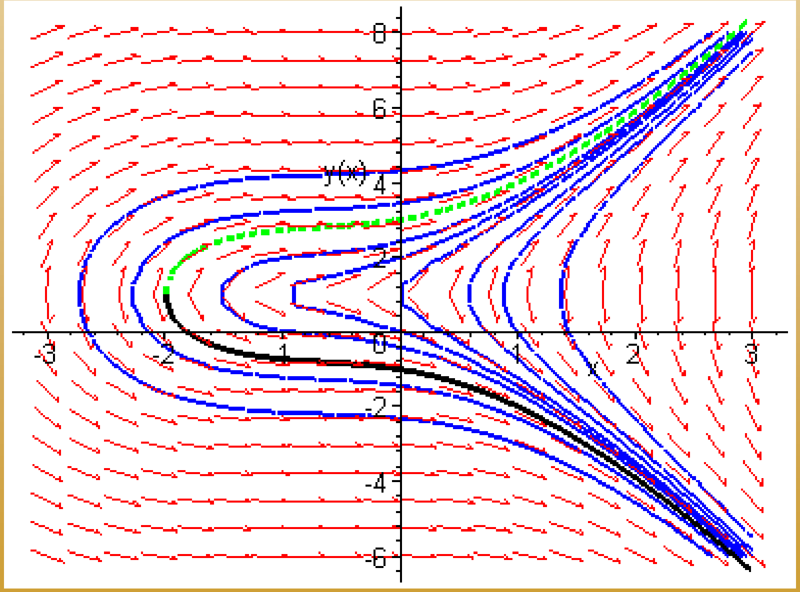
$$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}$$



Initial Condition y(0) = 3

- Note that if initial condition is $y(0) = 3$, then we choose the positive sign, instead of negative sign, on square root term:

$$y = 1 + \sqrt{x^3 + 2x^2 + 2x + 4}$$



Implicit Solution of Initial Value Problem

- Consider the following initial value problem:

$$y' = \frac{y\cos x}{1+3y^3}, \quad y(0)=1$$

- Separating variables and using calculus, we obtain

$$\frac{1+3y^3}{y}dy = \cos x dx$$

$$\int\left(\frac{1}{y}+3y^2\right)dy = \int \cos x dx$$

$$\ln|y| + y^3 = \sin x + C$$

- Using the initial condition, it follows that

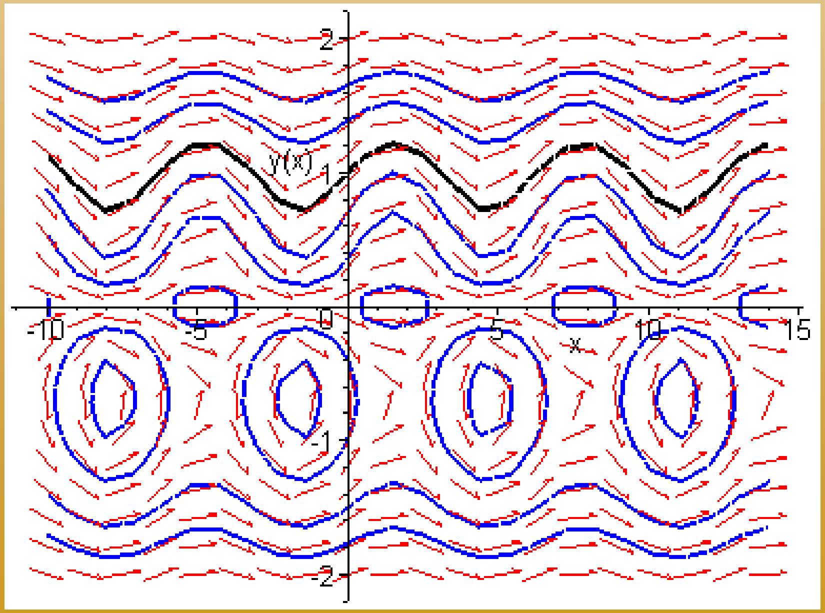
$$\ln y + y^3 = \sin x + 1$$

Graph of Solutions

- Thus

$$y' = \frac{y\cos x}{1+3y^3}, \quad y(0)=1 \Rightarrow \ln y + y^3 = \sin x + 1$$

- The graph of this solution (black), along with the graphs of the direction field and several integral curves (blue) for this differential equation, is given below.



EXERCISES

In Problems 1–6 solve the given differential equation by separation of variables.

1. $\dfrac{dy}{dx} = \sin 5x$

2. $\dfrac{dy}{dx} = (x + 1)^2$

3. $dx + e^{3x}dy = 0$

4. $dy - (y - 1)^2dx = 0$

5. $x\dfrac{dy}{dx} = 4y$

6. $\dfrac{dy}{dx} + 2xy^2 = 0$