Differential Equations

MATH 222

L4

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Homogeneous Equations

Equation $\frac{dy}{dx} = f(x,y)$ is homogeneous from degree n if

$$f(tx,ty) = t^n f(x,y)$$

Example

$$f(x,y) = x^2 - 3xy + 5y^2$$

Solution:

$$f(tx,ty) = (tx)^2 - 3(tx)(ty) + 5(ty)^2$$

$$= t^2\left(x^2 - 3xy + 5y^2\right) = t^2 f(x,y)$$

This equation homogeneous from degree two

Homogeneous Equations

Example

$$f(x,y)=\frac{x}{y}+\frac{y^2}{x^2}+1$$

Solution:

$$f(tx,ty)=\frac{tx}{ty}+\frac{(ty)^2}{(tx)^2}+1$$

$$=\frac{x}{y}+\frac{y^2}{x^2}+1$$

$$=t^0 f(x,y)$$

This equation homogeneous from degree zero

Homogeneous Equations

Example

$$f(x,y) = \ln x - \ln y + 1$$

$$= \ln \frac{x}{y} + 1$$

Solution:

$$f(tx,ty) = \ln \frac{tx}{ty} + 1$$

$$= \ln \frac{x}{y} + 1$$

$$= t^0 f(x;y)$$

This equation homogeneous from degree zero

Homogeneous Equations

Example

$$f(x,y)=xy+1$$

$$f(tx,ty)=(tx)(ty)+1$$

Solution:

$$=t^2xy+1\neq t^2(xy+1)=t^2xy+t^2$$

This equation nonhomogeneous

$$f(x,y)=xy^2+y$$ **nonhomogeneous**

$$f(x,y)=e^{y/x}$$ **homogeneous from degree zero**

Homogeneous Equations

Example. Show that differential equation $xydy = \left(x^2 + y^2\right)dx$ is homogenous differential equation.

Solution:

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy}$$

Differential equation is homogeneous

$$f(x, y) = \frac{x^2 + y^2}{xy}$$

$$f(tx, ty) = \frac{t^2x^2 + t^2y^2}{t^2xy} = \frac{t^2\left(x^2 + y^2\right)}{t^2xy} = \frac{x^2 + y^2}{xy} = f(x, y)$$

$\Rightarrow$ Differential equation is homogeneous

Homogeneous Equations

METHOD for solving Homogenous differential equations

Substitute $y = ux$

$$\frac{dy}{dx} = u + x\frac{du}{dx}$$

$$dy = udx + xdu$$

OR

Substitute $x = vy$

$$\frac{dx}{dy} = v + y\frac{dv}{dy}$$

$$dx = vdy + ydv$$

Homogeneous Equations

Note. Selection of substitution Differential Equation depends on number of terms of coefficients $M(x, y)$ *and* $N(x, y)$

1. If $(1+2+3)dx+(1)dy=0$, then take $y=ux$
2. If $1dx+(1+2+3)dy=0$, then take $x=vy$
3. If $(1+2)dx+(1+2)dy=0$, then take $x = vy$ *or* $y = ux$

Homogeneous Equations

(1 / 3)

Using substitution the homogeneous differential equation is reduce to separable variable form.

Example: Solve the homogenous differential equation

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy}$$

Solution:

Rewriting in the form : $M(x, y)dx + N(x, y)dy = 0$

$$\left(x^2 + y^2\right)dx - xydy = 0$$

substitute $y = ux$ and $dy = udx + xdu$

Homogeneous Equations

(2 / 3)

$$\left(x^2+x^2u^2\right)dx-ux^2\left(udx+xdu\right)=0$$

$$x^2dx+x^2u^2dx-u^2x^2dx-ux^3du=0$$

$$x^2dx-ux^3du=0$$

$$x^2dx=ux^3du$$

$$\frac{x^2dx}{x^3}=udu$$

Homogeneous Equations

(3 / 3)

$$\frac{dx}{x} = udu$$ **is variable separable form**

$$\int \frac{dx}{x} = \int udx$$

$$\ln|x| = \frac{u^2}{2} + c$$

$$\ln|x| = \frac{1}{2}\left(\frac{y^2}{x^2}\right) + c$$ **is general solution.**

Homogeneous Equations

Example:. Solve the Differential Equation by using appropriate substitution

$$\left(y^2 + xy + x^2\right)dx - x^2 dy = 0 \qquad (1/2)$$

Solution: Differential equation is homogeneous as degree of each term is same, hence we can use either *y = ux or x = vy* as substitution

$$\text{Let} \quad y = ux$$

$$dy = udx + xdu$$

Substituting *y* and *dy* in the given equation

$$\left(u^2x^2 + ux^2 + x^2\right)dx - x^2\left(udx + xdu\right) = \circ$$

$$u^2x^2dx + ux^2dx + x^2dx - x^2udx - x^3du = \circ$$

$$u^2x^2dx + x^2dx - x^3du = \circ$$

$$x^2\left(u^2 + 1\right)dx = x^3du$$

Homogeneous Equations

(2 / 2)

Separating variable u and x

$$\frac{dx}{x} = \frac{du}{1+u^2}$$ is Separable form

Integrating both the sides

$$\int \frac{dx}{x} = \int \frac{du}{1+u^2}$$

$$\ln|x| = \tan^{-1} u + c$$

$$\ln|x| = \tan^{-1}\left(\frac{y}{x}\right) + c.$$

is general solution of the differential equation

Homogeneous Equations

Example: Show that differential equation

$$3xy\frac{dy}{dx} = 4x^2 + 9y^2 \qquad \text{is homogeneous} \qquad (1/2)$$

Solution:

$$3xydy - \left(4x^2 + 9y^2\right)dx = 0$$

$$y = ux, \qquad dy = udx + xdv$$

$$3x.ux\left(udx + xdu\right) - \left(4x^2 + 9u^2x^2\right)dx = 0$$

$$3x^2u^2dx + 3ux^3du - 4x^2dx - 9u^2x^2dx = 0$$

$$3ux^3du = 4x^2dx + 6u^2x^2dx = x^2\left(4 + 6u^2\right)dx$$

Homogeneous Equations

$$\frac{3udu}{4+6u^2}=\frac{dx}{x}$$ is Separable form (2 / 2)

Integrating both the sides

$$\int\frac{3udu}{4+6u^2}=\int\frac{dx}{x}$$

Let $z=4+6u^2$

$dz=12udu$

$$\frac{1}{4}\int\frac{dz}{z}=\int\frac{dx}{x}$$

$$\frac{1}{4}\ln|z|=\ln|x|+c$$

$$\frac{1}{4}\ln\left(4+6\frac{y^2}{x^2}\right)=\ln|x|+c.$$

is general solution of the differential equation

Homogeneous Equations (1 / 3)

Example:5. Solve the differential equation

$$\frac{dy}{dx}+\frac{3xy+y^2}{x^2+xy}=0$$

Solution: Differential equation is homogeneous

$$(x^2+xy)dy+(3xy+y^2)dx=0$$

[Note: We may take $y = ux$ or $x = vy$ as substitution]

Let $x = vy$ $dx = vdy + ydv$

$$(v^2y^2+vy^2)dy+(3vy^2+y^2)(vdy+ydv)=0$$

$$v^2y^2+vy^2dy+3v^2y^2dy+3vy^3dv+y^2vdy+y^3dv=0$$

$$2vy^2dy+4v^2y^2dy=-y^3(3v+1)dv$$

$$(2v+4v^2)y^2dy=-y^3(3v+1)dv$$

$$-\frac{dy}{y}=\frac{3v+1}{2v(2v+1)}$$ is variable separable form

$$-\int\frac{dy}{y}=\int\frac{3v+1}{2v(2v+1)}dv$$

EXERCISES

Show that differential equation is homogenous differential equation.

(1) $dy+\left(x^2+xy+y^2\right)dx=0$ (2) $\dfrac{dy}{dx}=\dfrac{y}{x}+\cosh\dfrac{y}{x}$

Using substitution the homogeneous differential equation is reduce to separable variable form.

(1) $2x^2y\ dx-\left(3x^3+y^3\right)dy=0$

(2) $\left(y^2+xy+x^2\right)dx-x^2dy=0$