Differential Equations

MATH 222

L5

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College of Science and Computer Engineering

Taibah University

Yanbu

Saudi Arabian .

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Dr. Essam Othman Abdel-Rahman

Exact Equations

DEFINITION 2.4.1

Exact Equation

$M(x, y)\,dx + N(x, y)\,dy$ is an *exact differential* in a region R of the xy-plane, if it corresponds to the *differential* of some function $f(x, y)$. A first-order DE of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is said to be an *exact equation*, if the left side is an *exact differential*.

For example,

$x^2y^3dx + x^3y^2dy = 0$ **is an exact equation, because its left-hand side is an exact differential:**

$$d\left(\tfrac{1}{3}x^3y^3\right) = x^2y^3dx + x^3y^2dy$$

Exact Equations

THEOREM 2.1 Criterion for an Extra Differential

Let $M(x, y)$ and $N(x, y)$ be continuous and have continuous first partial derivatives in a region R defined by $a < x < b$, $c < y < d$. Then a necessary and sufficient condition that $M(x, y)\,dx + N(x, y)\,dy$ be an ***exact differential*** is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad (1)$$

Exact Equations

Method of Solution

- Since $\partial f/\partial x = M(x, y)$, we have

$$f(x, y) = \int M(x, y)dx + g(y) \qquad (2)$$

- Differentiating (5) with respect to y and assume $\partial f/\partial y = N(x, y)$ Then

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x, y)dx + g'(y) = N(x, y)$$

- and

$$g'(y) = N(x, y) - \frac{\partial}{\partial y}\int M(x, y)dx \qquad (3)$$

Which holds if (1) is satisfied.

Integrate (3) with respect to y to get g(y), and substitute the result into (2) to obtain the implicit solution f(x, y) = c.

Exact Equations

Example 1

- **Consider the following differential equation.**

$$2xy\,dx + (x^2 - 1)dy = 0$$

- **Then**

$$M(x,y) = 2xy,\; N(x,y) = x^2 - 1$$

and hence

$$M_y(x,y) = 2x = N_x(x,y) \;\Rightarrow\; \text{ODE is exact}$$

- **From Theorem1,**

$$\psi_x(x,y) = 2xy,\;\; \psi_y(x,y) = x^2 - 1$$

- **Thus**

$$\psi(x,y) = \int \psi_x(x,y)dx = \int 2xy\,dx = x^2 y + C(y)$$

Exact Equations

- **We have**

$$\psi_x(x, y) = 2xy, \;\; \psi_y(x, y) = x^2 - 1$$

and

$$\psi(x, y) = \int \psi_x(x, y)dx = \int 2xy\, dx = x^2 y + C(y)$$

- **It follows that**

$$\psi_y(x, y) = x^2 - 1 = x^2 + C'(y) \;\Rightarrow\; C'(y) = -1 \;\Rightarrow\; C(y) = -y \; + k$$

- **Thus**

$$\psi(x, y) = x^2 y - y + k$$

- **By Theorem 1, the solution is given implicitly by**

$$x^2 y - y = c \qquad where \quad c = -k$$

Exact Equations

Example 2

- **Consider the following differential equation.**

$$\frac{dy}{dx} = -\frac{x+4y}{4x-y} \iff (x+4y)dx + (4x-y)dy = 0$$

- **Then**

$$M(x,y) = x+4y,\ N(x,y) = 4x-y$$

and hence

$$M_y(x,y) = 4 = N_x(x,y) \Rightarrow \text{ODE is exact}$$

- **From Theorem1,**

$$\psi_x(x,y) = x+4y,\ \psi_y(x,y) = 4x-y$$

- **Thus**

$$\psi(x,y) = \int \psi_x(x,y)dx = \int (x+4y)dx = \frac{1}{2}x^2 + 4xy + C(y)$$

Exact Equations

(2 of 2)

- **We have**

$$\psi_x(x,y) = x + 4y, \ \ \psi_y(x,y) = 4x - y$$

and

$$\psi(x,y) = \int \psi_x(x,y)dx = \int (x + 4y)dx = \frac{1}{2}x^2 + 4xy + C(y)$$

- **It follows that**

$$\psi_y(x,y) = 4x - y = 4x + C'(y) \ \Rightarrow \ C'(y) = -y \ \Rightarrow \ C(y) = -\frac{1}{2}y^2 + k$$

- **Thus**

$$\psi(x,y) = \frac{1}{2}x^2 + 4xy - \frac{1}{2}y^2 + k$$

- **By Theorem 1, the solution is given implicitly by**

$$x^2 + 8xy - y^2 = c \qquad where \quad c = -2k$$

Non-Exact Equations

- **Consider the following differential equation.**

$$(3xy + y^2)dx + (2xy + x^3)dy = 0$$

- **Then**

$$M(x, y) = 3xy + y^2,\ N(x, y) = 2xy + x^3$$

and hence

$$M_y(x, y) = 3x + 2y \neq 2y + 3x^2 = N_x(x, y) \ \Rightarrow \ \text{ODE is not exact}$$

Integrating Factors

- It is sometimes possible to convert a differential equation that is not exact into an exact equation by multiplying the equation by a suitable integrating factor $\mu(x, y)$:

$$M(x,y)\,dx + N(x,y)dy = 0$$

$$\mu(x,y)M(x,y)\,dx + \mu(x,y)N(x,y)\,dy = 0$$

- For this equation to be exact, we need

$$(\mu M)_y = (\mu N)_x \;\Leftrightarrow\; M\mu_y - N\mu_x + (M_y - N_x)\mu = 0$$

- This partial differential equation may be difficult to solve. If μ is a function of x alone, then $\mu_y = 0$ and hence we solve

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu, \quad \Leftrightarrow \quad \mu = e^{\int \frac{M_y - N_x}{N} dx}$$

provided right side is a function of x only. Similarly if μ is a function of y alone.

Non-Exact Equations

- **Consider the following non-exact differential equation.**

$$(3xy + y^2)\,dx + (x^2 + xy)\,dy = 0$$

- **Seeking an integrating factor, we solve the linear equation**

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu \quad \Leftrightarrow \quad \frac{d\mu}{dx} = \frac{\mu}{x} \quad \Rightarrow \quad \mu(x) = x$$

- **Multiplying our differential equation by μ, we obtain the exact equation**

$$(3x^2 y + xy^2)\,dx + (x^3 + x^2 y)\,dy = 0,$$

Non-Exact Equations

$$(3x^2 y + xy^2)\,dx + (x^3 + x^2 y)\,dy = 0,$$

- **We have**

$$\psi_x(x,y) = (3x^2 y + xy^2),\;\; \psi_y(x,y) = (x^3 + x^2 y)$$

and

$$\psi(x,y) = \int \psi_x(x,y)dx = \int (3x^2 y + xy^2)\,dx = x^3 y + \tfrac{1}{2}x^2 y^2 + C(y)$$

- **It follows that**

$$\psi_y(x,y) = x^3 + x^2 y + C'(y) \;\Rightarrow\; C'(y) = 0 \;\Rightarrow\; C(y) = k$$

- **Thus**

$$\psi(x,y) = x^3 y + \tfrac{1}{2}x^2 y^2 + k$$

- **By Theorem 1, the solution is given implicitly by**

$$x^3 y + \frac{1}{2}x^2 y^2 = c \qquad \textit{where}\;\; c = -k$$

EXERCISES

Solve exact differential equation

$$(e^{2y} - y\cos(xy))\,dx + (2x\,e^{2y} - x\cos(xy) + 2y)dy = 0$$

Find the value of k so that the given differential equation is exact

$$(y^3 + k\,x\,y^4 - 2x)\,dx + (3x\,y^2 + 20\,x^2\,y^3)dy = 0$$