Differential Equations

MATH 222
L6

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First Order Linear Differential Equations

Any equation containing a derivative is called a differential equation.

A function which satisfies the equation is called a solution to the differential equation.

The order of the differential equation is the order of the highest derivative involved.

The degree of the differential equation is the degree of the power of the highest derivative involved.

When a differential equation is of the form $\frac{dy}{dx} = f(y)$

$\Rightarrow \frac{1}{f(y)} dy = dx$ We can then integrate both sides.

$\int \frac{dy}{f(y)} = \int dx$ This will obtain the general solution.

First Order Linear Differential Equations

Method of solution

When the equation is of the form $\frac{dy}{dx} = f(x)g(y)$, then

$$\frac{1}{g(y)}dy = f(x)dx$$

$$\Rightarrow \int \frac{dy}{g(y)} = \int f(x)dx$$

A first order linear differential equation is an equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad (1)$$

To find a method for solving this equation, lets consider the simpler equation

$$\frac{dy}{dx} + P(x)y = 0$$ Which can be solved by separating the variables.

(Remember: a solution is of the form *y = f(x)*)

First Order Linear Differential Equations

Method of solution

$$\frac{dy}{dx} + P(x)y = 0 \qquad \Rightarrow \frac{dy}{dx} = -P(x)y$$

$$\Rightarrow \int \frac{dy}{y} = -\int P(x)\,dx$$

$$\Rightarrow \ln|y| = -\int P(x)\,dx + c$$

$$\Rightarrow y = e^{-\int P(x)\,dx + c}$$

$$\Rightarrow y = e^{-\int P(x)\,dx} e^{c}$$

$$\Rightarrow y = Ce^{-\int P(x)\,dx}$$

or $\quad ye^{\int P(x)\,dx} = C$

Using the product rule to differentiate the LHS we get:

First Order Linear Differential Equations

Method of solution

$$\frac{d}{dx} ye^{\int P(x)dx} = \frac{dy}{dx} e^{\int P(x)dx} + yP(x)e^{\int P(x)dx}$$

$$= \left(\frac{dy}{dx} + P(x)y \right) e^{\int P(x)dx}$$

Returning to equation 1, $\frac{dy}{dx} + P(x)y = Q(x)$ If we multiply both sides by $e^{\int P(x)dx}$

$$\Rightarrow \left(\frac{dy}{dx} + P(x)y \right) e^{\int P(x)dx} = Q(x)e^{\int P(x)dx}$$

$$\frac{d}{dx} ye^{\int P(x)dx} = Q(x)e^{\int P(x)dx}$$

Now integrate both sides.

$$ye^{\int P(x)dx} = \int Q(x)e^{\int P(x)dx} dx$$

For this to work we need to be able to find $\int P(x)dx$ and $\int Q(x)e^{\int P(x)dx} dx$

First Order Linear Differential Equations

Method of solution

summary

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad (1)$$

$$\frac{dy}{dx} + P(x)y = f(x). \qquad (2)$$

SOLVING A LINEAR FIRST-ORDER EQUATION

(*i*) Put a linear equation of form (1) into the standard form (2).

(*ii*) From the standard form identify $P(x)$ and then find the integrating factor $e^{\int P(x)dx}$.

(*iii*) Multiply the standard form of the equation by the integrating factor. The left-hand side of the resulting equation is automatically the derivative of the integrating factor and y:

$$\frac{d}{dx}\left[e^{\int P(x)dx}y\right] = e^{\int P(x)dx}f(x).$$

(*iv*) Integrate both sides of this last equation.

First Order Linear Differential Equations

Example(1): Solve the differential equation $\frac{dy}{dx}+\frac{2}{x}y=x$

Step 1: Comparing with equation 1, we have $P(x)=\frac{2}{x}$

$$\int P(x)dx=\int \frac{2}{x}dx \;=2\ln|x| \;=\ln|x|^2 \;=\ln x^2$$

Step 2: $e^{\int P(x)dx}=e^{\ln x^2}\;=x^2$ $\left(e^{\int P(x)dx}\text{ is called the integrating factor}\right)$

Step 3: Multiply both sides by the integrating factor.

$$x^2\left(\frac{dy}{dx}+\frac{2}{x}y\right)=x^2x$$

$$\frac{d}{dx}yx^2=x^3$$

First Order Linear Differential Equations

Example(1):

$$\frac{d}{dx}yx^2 = x^3$$

$$yx^2 = \int x^3 dx \qquad yx^2 = \frac{x^4}{4} + c$$

$$y = \frac{1}{4}x^2 + cx^{-2}$$

Note the solution is composed of two parts.

The particular solution: $y = \frac{1}{4}x^2$ and the complementary function: $y = cx^{-2}$

The particular solution satisfies the equation: $\frac{dy}{dx} + \frac{2}{x}y = Q(x)$

The complementary function satisfies the equation: $\frac{dy}{dx} + \frac{2}{x}y = 0$

First Order Linear Differential Equations

Example(2): Solve the differential equation $\frac{dy}{dx} - 2xy = 3x$

Step 1: Comparing with equation 1, we have $P(x) = -2x$ and $Q(x) = 3x$

$$\int P(x)dx = \int -2x dx \ = -x^2$$

Step 2: Integrating Factor $e^{\int P(x)dx} = e^{-x^2}$

Step 3: Multiply both sides by the integrating factor.

$$e^{-x^2}\left(\frac{dy}{dx} - 2xy\right) = e^{-x^2} 3x$$

$$\frac{d}{dx} ye^{-x^2} = 3xe^{-x^2}$$

First Order Linear Differential Equations

Example(2):

$$\frac{d}{dx} ye^{-x^2} = 3xe^{-x^2}$$

$$ye^{-x^2} = \int 3xe^{-x^2} dx$$ To solve this integration we need to use substitution.

$$\text{Let } t = x^2 \Rightarrow dt = 2xdx$$

$$\int 3xe^{-x^2} dx = \frac{3}{2}\int e^{-t} dt = -\frac{3}{2}e^{-t} = -\frac{3}{2}e^{-x^2}$$

$$ye^{-x^2} = -\frac{3}{2}e^{-x^2} + c$$

$$y = -\frac{3}{2} + ce^{x^2}$$

First Order Linear Differential Equations

Example(3): Solve the differential equation $x^2 \dfrac{dy}{dx} - x^3 + xy = 0$

rewriting in standard form: $\dfrac{dy}{dx} + \dfrac{1}{x} y = x$

Step 1: Comparing with equation 1, we have $P(x) = \dfrac{1}{x}$ and $Q(x) = x$

$$\int P(x)dx = \int \frac{dx}{x} \quad = \ln|x|$$

Step 2: Integrating Factor $e^{\int P(x)dx} = e^{\ln|x|} = |x|$

Step 3: Multiply both sides by the integrating factor.

$\dfrac{d}{dx} yx = x^2$ (note the shortcut I have taken here)

$$yx = \int x^2 dx \qquad y = \frac{1}{3}x^2 + cx^{-1}$$

First Order Linear Differential Equations

Example(4): Solve the differential equation $\cos x\frac{dy}{dx}+y\sin x=\cos^2 x$

rewriting in standard form: $\frac{dy}{dx}+y\tan x=\cos x$

Step 1: Comparing with equation 1, we have $P(x)=\tan x$ and $Q(x)=\cos x$

$$\int P(x)dx=\int \tan x\,dx=\ln|\sec x|$$

Step 2: Integrating Factor $e^{\int P(x)dx}=e^{\ln|\sec x|}=|\sec x|$

Step 3: Multiply both sides by the integrating factor.

$\frac{d}{dx}y\sec x=1$ (note the shortcut I have taken here)

$$y\sec x=\int 1dx$$

$$y=x\cos x+c\cos x$$

First Order Linear Differential Equations

Example(5):

Solve the differential equation $x\frac{dy}{dx} - y = x^2$ given $x = 1$ when $y = 0$

rewriting in standard form: $\frac{dy}{dx} - \frac{1}{x}y = x$

Step 1: Comparing with equation 1, we have $P(x) = -\frac{1}{x}$

$$\int P(x)dx = -\int \frac{1}{x}dx = -\ln|x|$$

Step 2: Integrating Factor $e^{\int P(x)dx} = e^{-\ln|x|} = \frac{1}{|x|}$

Step 3: Multiply both sides by the integrating factor.

$$\frac{d}{dx}\frac{y}{x} = 1 \qquad \frac{y}{x} = \int 1dx \qquad y = x^2 + cx$$

First Order Linear Differential Equations

Example(5):

$$y = x^2 + cx \quad \text{given } x = 1 \text{ when } y = 0$$

$$0 = 1 + c$$

$$c = -1$$

Hence the particular solution is $\underline{\underline{y = x^2 - x}}$

EXERCISES

(1) Find the general solution of

$$(x^2 - 9)\frac{dy}{dx} + xy = 0$$

(2) Solve

$$\frac{dy}{dx} + y = x \qquad ; \qquad y(0) = 4$$