Differential Equations

MATH 222

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Bernoulli's Equation

A Bernoulli equation is a differential equation of the form:

$$\frac{dy}{dx} + p(x)\,y = Q(x)\,y^n \qquad (1) \qquad\qquad n \in \Re \setminus \{0,1\}$$

Note that for n=0 and n=1, equation (1) is linear

This is solved by:

Divide both sides by *y^n* and using following substitute :

$$\begin{cases} w = y^{1-n} \\ \dfrac{dw}{dx} = (1-n)\,y^{-n}\dfrac{dy}{dx} \end{cases} \qquad (2)$$

We have :

$$\frac{1}{1-n}\frac{dw}{dx} + p(x)w = Q(x)$$ **This is linear differential equation:**

Bernoulli's Equation

Example(1): **Solving a Bernoulli DE**

$$x\frac{dy}{dx}+y=x^2y^2$$

We first rewrite the equation as

$$\frac{dy}{dx}+\frac{y}{x}=x\ \ y^2$$ **by dividing by x**

We then substitute

$$w=y^{-1}\ \ or\ \ y=w^{-1}$$

$$\frac{dy}{dx}=\frac{dy}{dw}\frac{dw}{dx}=-w^{-2}\frac{dw}{dx}\qquad \textit{Chain Rule}$$

into the given equation

$$-w^{-2}\frac{dw}{dx}+\frac{w^{-1}}{x}=x\ \ w^{-2}\qquad\Rightarrow\qquad \frac{dw}{dx}-\frac{w}{x}=-x$$

Bernoulli's Equation

$$\frac{dw}{dx}-\frac{w}{x}=-x$$

The integrating factor for this linear equation is

$$p(x)=-\frac{1}{x} \quad \Rightarrow \quad \mu(x)=e^{-\int \frac{1}{x}dx}=e^{-lin\,x}=e^{lin\,x^{-1}}=x^{-1}$$

Then

$$x^{-1}\frac{dw}{dx}-w=-1$$

Integrating

$$\frac{d}{dx}(x^{-1}w)=-1 \quad \textbf{gives} \quad x^{-1}w=-x+c \quad \Rightarrow \quad w=-x^{2}+cx$$

Since $y=w^{-1}$ **we have** $y=\frac{1}{w}$

so a solution of the given equation is

$$y=\frac{1}{-x^{2}+cx}$$

Bernoulli's Equation

Example(2): **Solving a Bernoulli DE**

$$x\frac{dy}{dx} + y = \frac{1}{y^2}$$

We first rewrite the equation as

$$xy^2 \frac{dy}{dx} + y^3 = 1$$

We then substitute

$$w = y^3 \quad \Rightarrow \quad \frac{dw}{dx} = 3y^2 \frac{dy}{dx}$$

into the given equation

$$\frac{x}{3}\frac{dw}{dx} + w = 1 \quad \Rightarrow \quad \frac{dw}{dx} + \frac{3}{x}w = \frac{3}{x}$$

Bernoulli's Equation

$$\frac{dw}{dx}+\frac{3}{x}w=\frac{3}{x}$$

The integrating factor for this linear equation is

$$p(x)=\frac{3}{x} \quad \Rightarrow \quad \mu(x)=e^{\int \frac{3}{x}dx}=x^3$$

Then

$$x^3\frac{dw}{dx}+3x^2w=3x^2$$

Integrating

$$\frac{d}{dx}(wx^3)=3x^2 \quad \textbf{gives} \quad w\,x^3=x^3+c$$

Since

$$y^3=w \quad \textbf{we have} \quad y^3x^3=x^3+c$$

so a solution of the given equation is $y^3x^3=x^3+c$

Bernoulli's Equation

Example(3): **Solving a Bernoulli DE**

$$t\frac{dy}{dt} + y = t^2 y^2$$

We first rewrite the equation as

$$\frac{dy}{dt} + \frac{y}{t} = t\, y^2$$ **by dividing by t**

We then substitute

$$w = y^{-1} \ \ or \ \ y = w^{-1}$$

$$\frac{dy}{dt} = \frac{dy}{dw}\frac{dw}{dt} = -w^{-2}\frac{dw}{dt} \qquad \textit{Chain Rule}$$

into the given equation

$$-w^{-2}\frac{dw}{dt} + \frac{w^{-1}}{t} = t\, w^{-2} \qquad \Rightarrow \qquad \frac{dw}{dt} - \frac{w}{t} = -t$$

Bernoulli's Equation

$$\frac{dw}{dt} - \frac{w}{t} = -t$$

The integrating factor for this linear equation is

$$p(t) = -\frac{1}{t} \quad \Rightarrow \quad \mu(t) = e^{-\int \frac{1}{x} dt} = e^{-\ln t} = e^{\ln t^{-1}} = t^{-1}$$

Then

$$t^{-1}\frac{dw}{dt} - w = -1$$

Integrating

$$\frac{d}{dt}(t^{-1}w) = -1$$ **gives** $$t^{-1}w = -t + c \quad \Rightarrow \quad w = -t^2 + ct$$

Since $y = w^{-1}$ **we have** $y = \frac{1}{w}$

so a solution of the given equation is

$$y = \frac{1}{-t^2 + ct}$$

EXERCISES

Solve Bernoulli differential equation

$$(1)\quad xy\,\frac{dy}{dx}+y^2=\frac{1}{y}$$

Hint: you can use example 2

$$(2)\quad t^2\frac{dy}{dt}+y^2=t\,y$$