Differential Equations

MATH 222

L8

First Semester for Second Year

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Linear second order differential equation

Cauchy Euler's Equation

$$a_n x^n \frac{d^n y}{dx^n} + a_{n_{-1}} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_2 x^2 \frac{d^2 y}{dx^n} + a_1 x \frac{dy}{dx} + a_0 y = f(x)$$

with $a_n, a_{n-1}, \cdots, a_2, a_1$ and a_0 are constants and $a_n \neq 0$

is called Cauchy-Euler equation.

If $x = 2$ $\quad a_2 x^2 \frac{d^2 y}{dx^2} + a_1 x \frac{dy}{dx} + a_0 y = f(x)$

we will discuss method to find solution of homgeneous part of the differential equation that is y_c.

$$a_2 x^2 \frac{d^2 y}{dx^2} + a_1 x \frac{dy}{dx} + a_0 y = 0$$

NOTE:Particular soltion can be obtained by using method of variation of Parameter.

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METHOD

$$\text{Take} \quad y = x^{m} \quad as \quad a \quad solution$$

$$y' = m\ x^{m-1}, y'' = m(m-1)x^{m-2}$$

$$\text{substitute in DE} \quad ax^{2}y'' + bxy' + cy = 0$$

$$am(m-1)x^{2}.x^{m-2} + b.x.mx^{m-1} + cx^{m} = 0$$

$$am(m-1)x^{m}.x^{m-2} + bxmx^{m-1} + cx^{m} = 0$$

$$\left[am^{2} - am + bm + c\right]x^{m} = 0, x^{m} \neq 0$$

$$am^{2} + (b-a)m + c = 0 \quad \text{is the } characterisitc \text{ equation}$$

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THREE CASES

There are three cases to consider, depending on the roots of the auxiliary equation.

Case I: Distinct Real Roots

Case II: Repeated Real Roots

Case III: Conjugate Complex Roots

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CASE I — DISTINCT REAL ROOTS

Let m_1 and m_2 denote the real roots of the auxiliary equation where $m_1 \neq m_2$. Then

$$y_1 = x^{m_1} \quad \text{and} \quad y_2 = x^{m_2}$$

form a fundamental solution set, and the general solution is

$$y = c_1 x^{m_1} + c_2 x^{m_2}.$$

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CASE II — REPEATED REAL ROOTS

Let $m_1 = m_2$ denote the real root of the auxiliary equation. Then we obtain one solution—namely,

$$y_1 = x^{m_1}$$

we obtain the solution

$$y_2 = x^{m_1} \ln x.$$

Then the general solution is

$$y = c_1 x^{m_1} + c_2 x^{m_1} \ln x.$$

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CASE III — CONJUGATE COMPLEX ROOTS

Let $m_1 = \alpha + \beta i$ and $m_2 = \alpha - \beta i$, then the solution is

$$y = C_1 x^{\alpha+\beta i} + C_2 x^{\alpha-\beta i}.$$

Using Euler's formula, we can obtain the fundamental set of solutions

$$y_1 = x^{\alpha}\cos(\beta \ln x) \quad \text{and} \quad y_2 = x^{\alpha}\sin(\beta \ln x).$$

Then the general solution is

$$\begin{aligned} y &= c_1 x^{\alpha}\cos(\beta \ln x) + c_2 x^{\alpha}\sin(\beta \ln x) \\ &= x^{\alpha}\left[c_1\cos(\beta \ln x) + c_2\sin(\beta \ln x)\right]. \end{aligned}$$

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Example: **Solve Cauchy Euler's differential equation**

$$x^2 y'' + xy' - y = 0$$

Solution: $Let \quad y = x^m, \quad characteristic$ equation is

$$(\mathrm{m})(m-1) + m - 1 = 0$$

$$m^2 - m + m - 1 = 0$$

$$m^2 - 1 = 0$$

$$(m-1)(m+1) = 0$$

$\Rightarrow$ Roots are 1,-1

$$\mathrm{y_c} = c_1 x + c_2 x^{-1}$$

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Example: **Solve Cauchy Euler's differential equation**

$$(x+2)^2 y'' - 3(x+2)y' + 4y = 0$$

Solution:

Let $y = (x+2)^m, y' = m(x+2)^{m-1}, y'' = m(m-1)(x+2)^{m-2}$

$$\left[m(m-1) - 3m + 4\right](x+2)^m = 0, (x+2)^m \neq 0 \Rightarrow$$

$$m^2 - 4m + 4 = 0, \text{or } (m-2)^2 = 0 \quad \text{or Roots are } 2,2$$

$$y = c_1(x+2)^2 + c_2(x+2)^2 \ln(x+2)$$

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ALTERNATE METHOD OF SOLUTION

Any **Cauchy-Euler equation can be reduced to an equation with constant coefficients by means of the substitution**

$$x = e^{t}, x > 0$$

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ALTERNATE METHOD OF SOLUTION

$$x = e^t, \quad x > 0$$

$$t = \ln x$$

$$\frac{dy}{dx} = \frac{dy}{dt} . \frac{dt}{dx} = \frac{1}{x} \frac{dy}{dt} \qquad \rightarrow 1$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dx}..\left(\frac{dy}{dt}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d^2y}{dt^2}..\frac{dt}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2} = \frac{1}{x^2}\left[\frac{d^2y}{dt^2} - \frac{dy}{dt}\right] \rightarrow 2$$

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Example: Solve the Cauchy- Euler equation

$$x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=x^3, x>0$$

by transforming it into a differential equation with constant coefficients.

Solution:

$$x=e^t$$

$$x\frac{dy}{dx}=\frac{dy}{dt}, x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dt^2}-\frac{dy}{dt}$$

$$\frac{d^2y}{dt^2}-\frac{dy}{dt}-2\frac{dy}{dt}+2y=e^{3t}$$

$$y''-3y'+2y=e^{3t}$$

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$$y'' - 3y' + 2y = e^{3t}$$

Homogeneous solution y_c,

$$y = e^{mt}$$

$$y'' - 3y' + 2y = 0,$$

$$m^2 - 3m + 2 = 0$$

$$(m-1)(m-2) = 0 \quad \Rightarrow \quad \text{Roots are } 1, 2$$

$$\boxed{y_c = c_1 e^{t} + c_2 e^{2t}}$$

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$$y_P = Ae^{3t},$$

$$y'_P = 3Ae^{3t},$$

$$y''_P = 9Ae^{3t}$$

$$9Ae^{3t} - 9Ae^{3t} + 2Ae^{3t} = e^{3t}$$

$$+2Ae^{3t} = e^{3t} \Rightarrow +2A = 1$$

$$A = +\frac{1}{2} \Rightarrow \qquad y_P = +\frac{1}{2}e^{3t}$$

$$y = y_c + y_P = c_1e^{t} + c_2e^{2t} + \frac{1}{2}e^{3t},$$

substitute $e^{t} = x$

$$y = c_1x + c_2x^{2} + \frac{1}{2}x^{3}$$

Cauchy Euler's Equation

Example: Solve the Cauchy- Euler equation

$$x^3 y''' - 3x^2 y'' + 6xy' - 6y = x^4 \ln x \qquad x > \circ$$

Solution: **Substitute** $y = x^m$ to find y_c

$$y' = mx^{m-1}, y''' = m(m-1)(m-2)x^{m-3}$$

characteristic eq. is

$$m(m-1)(m-2) - 3m(m-1) + 6m - 6 = 0$$

Cauchy Euler's Equation

$$(m-1)\left[m^2-2m-3m+6\right]=0$$

$$(m-1)\left(m^2-5m+6\right)=0$$

$$(m-1)(m-2)(m-3)=0$$

$$\Rightarrow \quad \text{Roots are } 1,2,3$$

$$y_c = c_1 x + c_2 x^2 + c_3 x^3$$

EXERCISES

Solve the Cauchy- Euler equation

$$(1)\quad x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-4y=0$$

$$(2)\quad 4x^2\frac{d^2y}{dx^2}+8x\frac{dy}{dx}+y=0$$

