Chapter 7

DIFFERENTIAL EQUATIONS

7.1 Definition of a differential equation

Definition : An equation that involves x , y , y' , y'' , $\cdots$, $y^{(n)}$ for a function $y(x)$ with n^{th} derivative $y^{(n)}$ of y with respect to x is an ordinary differential Equation of order n.

Examples :

1. $y' = x^2 + 5$ is a differential equation of order 1.
2. $y'' + x\,(y')^4 - y = x$ is a differential equation of order 2
3. $\left(y^{(4)}\right)^3 + x^2\ y'' = 2x$ is a differnetial equation of order 4

$y = y(x)$ is called a **solution** of a differential equation if $y = y(x)$ satisfies that differential equation.
Consider the differential equation $y' = 6x + 4$, then $y = 3x^2 + 4x + c$ is the **general solution** of that differential equation.

If an **initial condition** was added to the differential equation to assign a certain vallue for c then $y = y(x)$ is called the **particular solution** of the differential equation .
Consider the differential equation $y' = 6x+4$ with the initial condition $y(0) = 2$,
$y = 3x^2 + 4x + c$ is the general solution of the differential equation ,
$y(0) = 2 \;\Rightarrow\; 3(0)^2 + 4 \times 0 + c = 2 \;\Rightarrow\; c = 2$, hence $y = 3x^2 + 4x + 2$ is the particular solution of the differential equation.

7.2 Separable Differential equations

The separable differential equation has the form $M(x) + N(y)\ y' = 0$.
where $M(x)$ and $N(y)$ are continuous functions.

To solve the separable differential equation :

1. Write it as $M(x)\ dx + N(y)\ dy = 0 \implies N(y)\ dy = -M(x)\ dx$.
2. Integrate the left-hand side with respect to y and the right-hand side with respect to x

$$\int N(y)\ dy = -\int M(x)\ dx$$

Example 1 : Solve the differential equation $y' + y^3 e^x = 0$.
Solution :

$$y' + y^3 e^x = 0 \implies \frac{dy}{dx} = -y^3 e^x\ ,$$

$$\implies -\frac{1}{y^3}\ dy = e^x\ dx \implies -\ y^{-3}\ dy = e^x\ dx$$

$$\implies -\int y^{-3}\ dy = \int e^x\ dx \implies -\frac{y^{-2}}{-2} = e^x + c$$

$$\implies \frac{1}{2y^2} = e^x + c \implies \frac{1}{y^2} = 2\left(e^x + c\right)$$

$$\implies y^2 = \frac{1}{2\left(e^x + c\right)} \implies y = \sqrt{\frac{1}{2\left(e^x + c\right)}}$$

Example 2 : Solve the differential equation $\frac{dy}{dx} = y^2 e^x$, $y(0) = 1$.
Solution :

$$\frac{dy}{dx} = y^2 e^x \implies \frac{1}{y^2}\ dy = e^x\ dx$$

$$\implies y^{-2}\ dy = e^x\ dx \implies \int y^{-2}\ dy = \int e^x\ dx$$

$$\implies \frac{y^{-1}}{-1} = e^x + c \implies y = \frac{-1}{e^x + c}$$

Using the initial condition $y(0) = 1 \implies 1 = \frac{-1}{e^0 + c}$

$$\implies 1 = \frac{-1}{1 + c} \implies 1 + c = -1 \implies c = -2$$

The particular solution is $y = \frac{-1}{e^x - 2}$

Example 3 : Solve the differential equation $dy - \sin x(1+y^2)dx = 0$.
Solution :

$$dy - \sin x(1+y^2)dx = 0 \implies dy = \sin x(1+y^2)dx$$

$$\implies \frac{1}{1+y^2}\ dy = \sin x\ dx \implies \int \frac{1}{1+y^2}\ dy = \int \sin x\ dx$$

$$\implies \tan^{-1} y = -\cos x + c \implies y = \tan(-\cos x + c)$$

Example 4 : Solve the differential equation $e^{-y}\sin x - y'\cos^2 x = 0$.
Solution :

$$e^{-y}\sin x - y'\cos^2 x = 0 \implies -\cos^2 x \frac{dy}{dx} = -e^{-y}\sin x$$

$$\implies \frac{1}{e^{-y}}\ dy = \frac{-\sin x}{-\cos^2 x}\ dx \implies e^y\ dy = \frac{1}{\cos x}\ \frac{\sin x}{\cos x}\ dx$$

$$\implies e^y\ dy = \sec x \tan x\ dx \implies \int e^y\ dy = \int \sec x \tan x\ dx$$

$$\implies e^y = \sec x + c \implies y = \ln|\sec x + c|$$

Example 5 : Solve the differential equation $y' = 1 - y + x^2 - yx^2$.
Solution :

$$y' = 1 - y + x^2 - yx^2 \implies \frac{dy}{dx} = 1 - y + x^2(1-y)$$

$$\implies \frac{dy}{dx} = (1-y)(1+x^2) \implies \frac{1}{1-y}\ dy = (1+x^2)\ dx$$

$$\implies \int \frac{1}{1-y}\ dy = \int (1+x^2)\ dx \implies -\int \frac{-1}{1-y}\ dy = \int (1+x^2)\ dx$$

$$\implies -\ln|1-y| = x + \frac{x^3}{3} + c \implies \ln|1-y| = -x - \frac{x^3}{3} - c$$

$$\implies 1 - y = e^{-x-\frac{x^3}{3}-c} \implies y = 1 - e^{-x-\frac{x^3}{3}-c}$$

7.3 First-order linear differential equations

The first-order linear differential equation has the form $y' + P(x)\ y = Q(x)$, where $P(x)$ and $Q(x)$ are continuous functions of x

To solve the first-order linear differential equation :

1. Compute the integrating factor $u(x) = e^{\int P(x)\ dx}$
2. The general solution of the first-order linear differential equation is

   $$y(x) = \frac{1}{u(x)} \int u(x)\ Q(x)\ dx$$

Example 1 : Solve the differential equation $x\dfrac{dy}{dx} + y = x^2 + 1$.
Solution :

$$x\frac{dy}{dx} + y = x^2 + 1 \quad \Longrightarrow \quad y' + \left(\frac{1}{x}\right) y = \frac{x^2+1}{x}$$

$$\Longrightarrow \quad y' + \left(\frac{1}{x}\right) y = x + \frac{1}{x}$$

$P(x) = \dfrac{1}{x}$ and $Q(x) = x + \dfrac{1}{x}$

The integrating factor is $u(x) = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

The general solution is $y = \dfrac{1}{x} \displaystyle\int x \left(x + \frac{1}{x}\right)\ dx$

$$y = \frac{1}{x} \int (x^2+1)\ dx = \frac{1}{x}\left(\frac{x^3}{3} + x + c\right) = \frac{x^2}{3} + 1 + \frac{c}{x}$$

Example 2 : Solve the differential equation $y' - \dfrac{2}{x}y = x^2 e^x$, $y(1) = e$.
Solution :

$P(x) = -\dfrac{2}{x}$ and $Q(x) = x^2 e^x$

The integrating factor is

$$u(x) = e^{\int -\frac{2}{x} dx} = e^{-2\int \frac{1}{x} dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2}$$

The general solution is $y = \dfrac{1}{x^{-2}} \displaystyle\int x^{-2} x^2 e^x dx$

$$y = x^2 \int e^x dx = x^2(e^x + c) = x^2 e^x + cx^2$$

Using the initial condition $y(1) = e$

$y(1) = e \implies e = (1)^2 e^1 + c\ (1)^2 \implies e = e + c \implies c = 0$

The particular solution is $y = x^2 e^x$

Example 3 : Solve the differential equation $y' + y = \cos(e^x)$
Solution :

$P(x) = 1$ and $Q(x) = \cos(e^x)$

The integrating factor is $u(x) = e^{\int 1\ dx} = e^x$

The general solution is $y = \dfrac{1}{e^x} \displaystyle\int e^x \cos(e^x)\ dx$

$$y = e^{-x} \int \cos(e^x) e^x\ dx = e^{-x} \left(\sin(e^x) + c\right) = e^{-x} \sin(e^x) + ce^{-x}$$

Example 4 : Solve the differential equation $xy' - 3y = x^2$
Solution :

$xy' - 3y = x^2 \implies y' - \dfrac{3}{x} y = x$

$P(x) = -\dfrac{3}{x}$ and $Q(x) = x$

The integrating factor is

$u(x) = e^{\int -\frac{3}{x} dx} = e^{-3 \int \frac{1}{x} dx} = e^{-3 \ln x} = e^{\ln x^{-3}} = x^{-3}$

The general solution is $y = \dfrac{1}{x^{-3}} \displaystyle\int x^{-3} x\ dx$

$$y = x^3 \int x^{-2} dx = x^3 \left(\frac{x^{-1}}{-1} + c \right)$$

$$y = x^3 \left(-\frac{1}{x} + c \right) = -x^2 + cx^3$$