Chapter 4

INTEGRATION

4.1 Indefinite integral

Definition (Antiderivative): A function G is called an antiderivative of the function f on the interval $[a, b]$ if $G'(x) = f(x)$ for all $x \in [a, b]$.

Examples : What is the antiderivative of the following functions

1. $f(x) = 2x$.
2. $f(x) = \cos x$.
3. $f(x) = \sec^2 x$
4. $f(x) = \dfrac{1}{x}$
5. $f(x) = e^x$

Solution :

1. $G(x) = x^2 + c$

   $$G'(x) = \frac{d}{dx}G(x) = \frac{d}{dx}\left(x^2 + c\right) = 2x + 0 = 2x$$
2. $G(x) = \sin x + c$

   $$G'(x) = \frac{d}{dx}G(x) = \frac{d}{dx}(\sin x + c) = \cos x$$
3. $G(x) = \tan x + c$

   $$G'(x) = \frac{d}{dx}G(x) = \frac{d}{dx}(\tan x + c) = \sec^2 x$$
4. $G(x) = \ln|x| + c$

   $$G'(x) = \frac{d}{dx}G(x) = \frac{d}{dx}(\ln|x| + c) = \frac{1}{x}$$
5. $G(x) = e^x + c$

   $$G'(x) = \frac{d}{dx}G(x) = \frac{d}{dx}(e^x + c) = e^x$$

Note: If $G_1(x)$ and $G_2(x)$ are both antiderivatives of the function $f(x)$ then $G_1(x) - G_2(x) =$ constant.

Definition (indefinite integral): If $G(x)$ is the antiderivative of $f(x)$ then $\int f(x)\, dx = G(x) + c$, $\int f(x)\, dx$ is called the indefinite integral of the function $f(x)$.

Basic Rules of integration :

1. $\displaystyle\int 1\ dx = x + c$
2. $\displaystyle\int x^n\ dx = \frac{x^{n+1}}{n+1} + c$, where $n \neq -1$
3. $\displaystyle\int \cos x\ dx = \sin x + c$
4. $\displaystyle\int \sin x\ dx = -\cos x + c$
5. $\displaystyle\int \sec^2 x\ dx = \tan x + c$
6. $\displaystyle\int \csc^2 x\ dx = -\cot x + c$
7. $\displaystyle\int \sec x\ \tan x\ dx = \sec x + c$
8. $\displaystyle\int \csc x\ \cot x = -\csc x + c$
9. $\displaystyle\int \frac{1}{x}\ dx = \ln |x| + c$
10. $\displaystyle\int e^x\ dx = e^x + c$
11. $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\ dx = \sin^{-1} x + c$, where $|x| < 1$
12. $\displaystyle\int \frac{1}{1+x^2}\ dx = \tan^{-1} x + c$
13. $\displaystyle\int \frac{1}{x\sqrt{x^2-1}}\ dx = \sec^{-1} x + c$, where $|x| > 1$

Properties of indefinite integral :

1. $\displaystyle\int k\ f(x)\ dx = k \int f(x)\ dx$, where $k \in \mathbb{R}$
2. $\displaystyle\int (f(x) \pm g(x))\ dx = \int f(x)\ dx \pm \int g(x)\ dx$

Examples : Evaluate the following integrals

1. $\int \left(4x^2 - \frac{5}{x^3}\right)\, dx$

   Solution : $\int \left(4x^2 - \frac{5}{x^3}\right)\, dx = \int 4x^2\, dx - \int \frac{5}{x^3}\, dx$

   $= 4\int x^2\, dx - 5\int x^{-3} dx = 4\,\frac{x^3}{3} - 5\,\frac{x^{-2}}{-2} + c = \frac{4}{3}x^3 + \frac{5}{2x^2} + c$

2. $\int \left(3x^{\frac{1}{3}} + \frac{1}{\sqrt{x}}\right)\, dx$

   Solution : $\int \left(3x^{\frac{1}{3}} + \frac{1}{\sqrt{x}}\right)\, dx = 3\int x^{\frac{1}{3}}\, dx + \int x^{-\frac{1}{2}}\, dx$

   $= 3\left(\frac{x^{\frac{4}{3}}}{\frac{4}{3}}\right) + \left(\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right) + c = \frac{9}{4}x^{\frac{4}{3}} + 2x^{\frac{1}{2}} + c$

3. $\int (2\cos x - 3\sec^2 x)\, dx$

   Solution : $\int (2\cos x - 3\sec^2 x)\, dx = 2\int \cos x\, dx - 3\int \sec^2 x\, dx$

   $= 2\sin x - 3\tan x + c$

4. $\int (7\sec x\ \tan x + 5\csc^2 x)\, dx$

   Solution : $\int (7\sec x\ \tan x + 5\csc^2 x)\, dx = 7\int \sec x \tan x\, dx + 5\int \csc^2 x\, dx$

   $= 7\sec x + 5(-\cot x) + c = 7\sec x - 5\cot x + c$

5. $\int \left(\frac{2}{x} - \frac{3}{x^2}\right)\, dx$

   Solution : $\int \left(\frac{2}{x} - \frac{3}{x^2}\right)\, dx = 2\int \frac{1}{x}\, dx - 3\int x^{-2}\, dx$

   $= 2\ln|x| - 3\left(\frac{x^{-1}}{-1}\right) + c = 2\ln|x| + \frac{3}{x} + c$

6. $\int \left(9e^x - \frac{3}{1+x^2}\right)\, dx$

Solution : $\int \left(9e^x - \frac{3}{1+x^2}\right)\, dx = 9\int e^x\, dx - 3\int \frac{1}{1+x^2}\, dx$

$= 9e^x - 3\tan^{-1} x + c$

7. $\int \left(\frac{4}{\sqrt{1-x^2}} + \frac{1}{\sqrt[3]{x}}\right)\, dx$

Solution : $\int \left(\frac{4}{\sqrt{1-x^2}} + \frac{1}{\sqrt[3]{x}}\right)\, dx = 4\int \frac{1}{\sqrt{1-x^2}}\, dx + \int x^{-\frac{1}{3}}\, dx$

$= 4\sin^{-1} x + \left(\frac{x^{\frac{2}{3}}}{\frac{2}{3}}\right) + c = 4\sin^{-1} x + \frac{3}{2}x^{\frac{2}{3}} + c$

The definite integral :
If f is a continuous function on the interval $[a, b]$ and G is the antiderivative of f on $[a, b]$ then the definite integral of f on $[a, b]$ is

$$\int_a^b f(x)\, dx = [G(x)]_a^b = G(b) - G(a)$$

Examples : Evaluate the following integrals :

1. $\int_1^3 (3x^2 + 5)dx$

Solution : $\int_1^3 (3x^2 + 5)dx = \left[x^3 + 5x\right]_1^3$

$= \left(3^3 + 5 \times 3\right) - \left(1^3 + 5 \times 1\right) = (27 + 15) - (1 + 5) = 36$

2. $\int_0^1 (2x + e^x)\, dx$

Solution : $\int_0^1 (2x + e^x)\, dx = \left[x^2 + e^x\right]_0^1$

$= \left(1^2 + e^1\right) - \left(0^2 + e^0\right) = 1 + e - 1 = e$

4.2 Integration by substitution

The main idea of integration by substitution is to use a suitable substitution to transform the given integral to an easier integral that can be solved by one of the basic rules of integration.

Example : Evaluate the integral $\int x(x^2+3)^6\ dx$

Solution : Use the substitution $u = x^2 + 3$

Then $du = 2x\ dx \ \Rightarrow\ \frac{1}{2}\ du = x\ dx$

$\int x(x^2+3)^6\ dx = \int u^6 \frac{1}{2}\ du = \frac{1}{2} \int u^6\ du$

$= \frac{1}{2}\ \frac{u^7}{7} + c = \frac{(x^2+3)^7}{14} + c$

By the chain rule $\frac{d}{dx}\left[f(x)\right]^{n+1} = (n+1)\ \left[f(x)\right]^n\ f'(x)$, where $n \neq -1$

Hence $\int \left[f(x)\right]^n\ f'(x)\ dx = \frac{\left[f(x)\right]^{n+1}}{n+1} + c$, where $n \neq -1$

So, the above integral can be solved as follows

$\int x(x^2+3)^6\ dx = \frac{1}{2} \int (x^2+3)^6\ (2x)\ dx = \frac{1}{2}\ \frac{(x^2+3)^7}{7} + c$

Basic rules of integrations and their general forms :

1. $\int x^n\ dx = \frac{x^{n+1}}{n+1} + c$, where $n \neq -1$

   $\int \left[f(x)\right]^n f'(x)\ dx = \frac{\left[f(x)\right]^{n+1}}{n+1} + c$, where $n \neq -1$

2. $\int \frac{1}{x}\ dx = \ln|x| + c$

   $\int \frac{f'(x)}{f(x)}\ dx = \ln|f(x)| + c$

3. $\int e^x\ dx = e^x + c$

   $\int e^{f(x)} f'(x)\ dx = e^{f(x)} + c$

4. $\int \cos x\ dx = \sin x + c$

   $\int \cos\left(f(x)\right) f'(x)\ dx = \sin\left(f(x)\right) + c$

5. $$\int \sin x \; dx = -\cos x + c$$

$$\int \sin\left(f(x)\right) f'(x) \; dx = -\cos\left(f(x)\right) + c$$

6. $$\int \sec^2 x \; dx = \tan x + c$$

$$\int \sec^2\left(f(x)\right) f'(x) \; dx = \tan\left(f(x)\right) + c$$

7. $$\int \csc^2 x \; dx = -\cot x + c$$

$$\int \csc^2\left(f(x)\right) f'(x) \; dx = -\cot\left(f(x)\right) + c$$

8. $$\int \sec x \tan x \; dx = \sec x + c$$

$$\int \sec\left(f(x)\right) \tan\left(f(x)\right) f'(x) \; dx = \sec\left(f(x)\right) + c$$

9. $$\int \csc x \cot x \; dx = -\csc x + c$$

$$\int \csc\left(f(x)\right) \cot\left(f(x)\right) f'(x) \; dx = -\csc\left(f(x)\right) + c$$

10. $$\int \frac{1}{\sqrt{a^2 - x^2}} \; dx = \sin^{-1}\left(\frac{x}{a}\right) + c \text{ , where } a > 0 \text{ and } |x| < a$$

$$\int \frac{f'(x)}{\sqrt{a^2 - \left[f(x)\right]^2}} \; dx = \sin^{-1}\left(\frac{f(x)}{a}\right) + c \text{ , where } a > 0 \text{ and } |f(x)| < a$$

11. $$\int \frac{1}{a^2 + x^2} \; dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c \text{ , where } a > 0$$

$$\int \frac{f'(x)}{a^2 + \left[f(x)\right]^2} \; dx = \frac{1}{a} \tan^{-1}\left(\frac{f(x)}{a}\right) + c \text{ , where } a > 0$$

12. $$\int \frac{1}{x\sqrt{x^2 - a^2}} \; dx = \frac{1}{a} \sec^{-1}\left(\frac{x}{a}\right) + c \text{ , where } a > 0 \text{ and } |x| > a$$

$$\int \frac{f'(x)}{f(x)\sqrt{\left[f(x)\right]^2 - a^2}} \; dx = \frac{1}{a} \sec^{-1}\left(\frac{f(x)}{a}\right) + c \text{ , where } |f(x)| > a$$

Examples : Evaluate the following integrals

1. $\displaystyle\int (x^2+2x)(x^3+3x^2+5)^{10}\ dx$

   Solution :

   $$\int (x^2+2x)(x^3+3x^2+5)^{10}\ dx = \frac{1}{3}\int (x^3+3x^2+5)^{10}\left[3(x^2+2x)\right]\ dx$$

   $$= \frac{1}{3}\int (x^3+3x^2+5)^{10}(3x^2+6x)\ dx = \frac{1}{3}\ \frac{(x^3+3x^2+5)^{11}}{11} + c$$

2. $\displaystyle\int \frac{x+1}{(x^2+2x+6)^5}\ dx$

   Solution :

   $$\int \frac{x+1}{(x^2+2x+6)^5}\ dx = \int (x^2+2x+6)^{-5}(x+1)\ dx$$

   $$= \frac{1}{2}\int (x^2+2x+6)^{-5}(2x+2)\ dx = \frac{1}{2}\ \frac{(x^2+2x+6)^{-4}}{-4} + c$$

3. $\displaystyle\int \frac{x^3+x}{\sqrt{x^4+2x^2+5}}\ dx$

   Solution :

   $$\int \frac{x^3+x}{\sqrt{x^4+2x^2+5}}\ dx = \int (x^4+2x^2+5)^{-\frac{1}{2}}(x^3+x)\ dx$$

   $$= \frac{1}{4}\int (x^4+2x^2+5)^{-\frac{1}{2}}(4x^3+4x)\ dx = \frac{1}{4}\ \frac{(x^4+2x^2+5)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

4. $\displaystyle\int \frac{x^2+1}{x^3+3x+8}\ dx$

   Solution :

   $$\int \frac{x^2+1}{x^3+3x+8}\ dx = \frac{1}{3}\int \frac{3(x^2+1)}{x^3+3x+8}\ dx$$

   $$= \frac{1}{3}\int \frac{3x^2+3}{x^3+3x+8}\ dx = \frac{1}{3}\ln|x^3+3x+8| + c$$

5. $\displaystyle\int \frac{\sin x}{1+\cos x}\ dx$

   Solution :

   $$\int \frac{\sin x}{1+\cos x}\ dx = -\int \frac{-\sin x}{1+\cos x}\ dx = -\ln|1+\cos x| + c$$

6. $\displaystyle\int \frac{e^{5x}}{e^{5x}-2}\ dx$

Solution :

$$\int \frac{e^{5x}}{e^{5x}-2}\ dx = \frac{1}{5}\int \frac{5e^{5x}}{e^{5x}-2}\ dx = \frac{1}{5}\ln|e^{5x}-2| + c$$

7. $\displaystyle\int (3x^2+1)\sin(x^3+x+1)\ dx$

Solution :

$$\int (3x^2+1)\sin(x^3+x+1)\ dx = \int \sin(x^3+x+1)\ (3x^2+1)\ dx$$

$$= -\cos(x^3+x+1) + c$$

8. $\displaystyle\int \frac{\sec^2\sqrt{x}}{\sqrt{x}}\ dx$

Solution :

$$\int \frac{\sec^2\sqrt{x}}{\sqrt{x}}\ dx = \int \sec^2\sqrt{x}\ \frac{1}{\sqrt{x}}\ dx$$

$$= 2\int \sec^2\sqrt{x}\ \frac{1}{2\sqrt{x}}\ dx = 2\tan\sqrt{x} + c$$

9. $\displaystyle\int x\ \csc(x^2+2)\cot(x^2+2)\ dx$

Solution :

$$\int x\ \csc(x^2+2)\cot(x^2+2)\ dx = \int \csc(x^2+2)\cot(x^2+2)\ x\ dx$$

$$\frac{1}{2}\int \csc(x^2+2)\cot(x^2+2)\ (2x)\ dx = -\frac{1}{2}\csc(x^2+2) + c$$

10. $\displaystyle\int e^{7\sin x}\cos x\ dx$

Solution :

$$\int e^{7\sin x}\cos x\ dx = \frac{1}{7}\int e^{7\sin x}(7\cos x)\ dx = \frac{1}{7}e^{7\sin x} + c$$

11. $\displaystyle\int \frac{e^{\frac{3}{x}}}{x^2}\ dx$

Solution :

$$\int \frac{e^{\frac{3}{x}}}{x^2}\ dx = \int e^{\frac{3}{x}}\ \frac{1}{x^2}\ dx$$

$$= -\frac{1}{3}\int e^{\frac{3}{x}}\ \frac{-3}{x^2}\ dx = -\frac{1}{3}e^{\frac{3}{x}} + c$$

12. $\displaystyle\int \frac{x}{\sqrt{9-x^4}}\ dx$

Solution :

$$\int \frac{x}{\sqrt{9-x^4}}\ dx = \int \frac{x}{\sqrt{3^2-(x^2)^2}}\ dx$$

$$= \frac{1}{2}\int \frac{2x}{\sqrt{3^2-(x^2)^2}}\ dx = \frac{1}{2}\sin^{-1}\left(\frac{x^2}{3}\right) + c$$

13. $\displaystyle\int \frac{1}{x^2-6x+10}\ dx$

Solution

$$\int \frac{1}{x^2-6x+10}\ dx = \int \frac{1}{(x^2-6x+9)+(10-9)}\ dx$$

$$= \int \frac{1}{(x-3)^2+1}\ dx = \tan^{-1}(x-3) + c$$

14. $\displaystyle\int \frac{3}{x^2+2x+5}\ dx$

Solution

$$\int \frac{3}{x^2+2x+5}\ dx = \int \frac{3}{(x^2+2x+1)+(5-1)}\ dx$$

$$= 3\int \frac{1}{(x+1)^2+2^2}\ dx = 3\ \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + c$$

15. $\displaystyle\int \frac{1}{x\ln|x|}\ dx$

Solution

$$\int \frac{1}{x\ln|x|}\ dx = \int \frac{\frac{1}{x}}{\ln|x|}\ dx = \ln|\ln|x|| + c$$

16. $\displaystyle\int \frac{2x-1}{x^2+1}\ dx$

Solution :

$$\int \frac{2x-1}{x^2+1}\ dx = \int \frac{2x}{x^2+1}\ dx - \int \frac{1}{x^2+1}\ dx$$

$$= \ln(x^2+1) - \tan^{-1}x + c$$

4.3 Integration by parts

It is used to solve an integral of a product of two functions using the formula

$$\int u\ dv = u\ v - \int v\ du$$

Examples : Evaluate the following integrals

1. $\int xe^x\ dx$

   Solution : Using integration by parts

   $$\begin{array}{ll} u = x & dv = e^x\ dx \\ du = dx & v = e^x \end{array}$$

   $$\int xe^x\ dx = xe^x - \int e^x\ dx = xe^x - e^x + c$$

2. $\int x^2 \sin x\ dx$

   Solution : Using integration by parts

   $$\begin{array}{ll} u = x^2 & dv = \sin x\ dx \\ du = 2xdx & v = -\cos x \end{array}$$

   $$\int x^2 \sin x\ dx = -x^2 \cos x - \int 2x(-\cos x)\ dx$$

   $$= -x^2 \cos x + 2 \int x \cos x\ dx$$

   Using integration by parts again

   $$\begin{array}{ll} u = x & dv = \cos x\ dx \\ du = dx & v = \sin x \end{array}$$

   $$\int x^2 \sin x\ dx = -x^2 \cos x + 2 \left(x \sin x - \int \sin x\ dx \right)$$

   $$= -x^2 \cos x + 2 (x \sin x - (-\cos x)) + c$$

   $$= -x^2 \cos x + 2x \sin x + 2 \cos x + c$$

3. $\int x \ln|x|\ dx$

   Solution : Using integration by parts

   $$\begin{array}{ll} u = \ln|x| & dv = x\ dx \\ du = \dfrac{1}{x}\ dx & v = \dfrac{x^2}{2} \end{array}$$

$$\int x \ln|x| \ dx = \frac{x^2}{2} \ln|x| - \int \frac{1}{x} \ \frac{x^2}{2} \ dx$$

$$= \frac{x^2}{2} \ln|x| - \frac{1}{2} \int x \ dx = \frac{x^2}{2} \ln|x| - \frac{1}{2} \ \frac{x^2}{2} + c = \frac{x^2}{2} \ln|x| - \frac{x^2}{4} + c$$

4. $\int \ln|x| \ dx$

Solution : Using integration by parts

$$u = \ln|x| \qquad dv = dx$$
$$du = \frac{1}{x} \ dx \qquad v = x$$

$$\int \ln|x| \ dx = x \ln|x| - \int x \ \frac{1}{x} \ dx = x \ln|x| - \int 1 \ dx$$

$$= x \ln|x| - x + c$$

5. $\int \tan^{-1} x \ dx$

Solution : Using integration by parts

$$u = \tan^{-1} x \qquad dv = dx$$
$$du = \frac{1}{1+x^2} \ dx \qquad v = x$$

$$\int \tan^{-1} x \ dx = x \tan^{-1} x - \int x \ \frac{1}{1+x^2} \ dx$$

$$= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} \ dx = x \tan^{-1} x - \frac{1}{2} \ln\left(1+x^2\right) + c$$

6. $\int \sin^{-1} x \ dx$

Solution : Using integration by parts

$$u = \sin^{-1} x \qquad dv = dx$$
$$du = \frac{1}{\sqrt{1-x^2}} \ dx \qquad v = x$$

$$\int \sin^{-1} x \ dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} \ dx$$

$$= x \sin^{-1} x + \frac{1}{2} \int (1-x^2)^{-\frac{1}{2}} (-2x) \ dx = x \sin^{-1} x + \frac{1}{2} \ \frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= x \sin^{-1} x + \sqrt{1-x^2} + c$$

7. $\int e^x \sin x \; dx$

Solution : Using integration by parts

$$\begin{array}{ll} u = \sin x & dv = e^x \; dx \\ du = \cos x \; dx & v = e^x \end{array}$$

$$\int e^x \sin x \; dx = e^x \sin x - \int e^x \cos x \; dx$$

Using integration by parts again

$$\begin{array}{ll} u = \cos x & dv = e^x \; dx \\ du = -\sin x \; dx & v = e^x \end{array}$$

$$\int e^x \sin x \; dx = e^x \sin x - \left(e^x \cos x - \int e^x (-\sin x) \; dx \right)$$

$$\int e^x \sin x \; dx = e^x \sin x - \left(e^x \cos x + \int e^x \sin x \; dx \right)$$

$$\int e^x \sin x \; dx = e^x \sin x - e^x \cos x - \int e^x \sin x \; dx$$

$$2 \int e^x \sin x \; dx = e^x \sin x - e^x \cos x + c$$

$$\int e^x \sin x \; dx = \frac{1}{2} \left(e^x \sin x - e^x \cos x + c \right)$$

4.4 Integral of rational functions (The metod of partial fractions)

Method of partial fractions is used to solve integrals of the form $\int \frac{P(x)}{Q(x)}\,dx$ where $P(x)$, $Q(x)$ are polynomials and degree $P(x) <$ degree $Q(x)$.
If degree $P(x) \geq$ degree $Q(x)$ use long division of polynomials .

Definition (linear factor) :
A linear factor is a polynomial of degree 1.
It has the form $ax + b$ where $a, b \in \mathbb{R}$ and $a \neq 0$.

Examples :
x , $3x$, $2x - 7$ are examples of linear factors .

Definition (irreducible quadratic) :
An irreducible quadratic is a polynomial of degree 2.
It has the form $ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$, $a \neq 0$ and $b^2 - 4ac < 0$.

Examples :

1. $x^2 + 9$ and $x^2 + x + 1$ are examples of irreducible quadratics.
2. $x^2 = x\ x$ and $x^2 - 1 = (x - 1)(x + 1)$ are reducible quadratics .

How to write $\frac{P(x)}{Q(x)}$ as partial fractions decomposition ?
Write $Q(x)$ as a product of linear factors and irreducible quadratics (if possible).
If $Q(x) = (a_1x + a_2)^m\ (b_1x^2 + b_2x + b_3)^n$ where $m, n \in \mathbb{N}$ then

$$\begin{aligned}\frac{P(x)}{Q(x)} &= \frac{A_1}{a_1x + a_2} + \frac{A_2}{(a_1x + a_2)^2} + \cdots + \frac{A_m}{(a_1x + a_2)^m}\\ &+ \frac{B_1x + C_1}{b_1x^2 + b_2x + b_3} + \frac{B_2x + C_2}{(b_1x^2 + b_2x + b_3)^2} + \cdots + \frac{B_nx + C_n}{(b_1x^2 + b_2x + b_3)^n}\end{aligned}$$

Where $A_1, A_2, \cdots, A_m, B_1, B_2, \cdots, B_n, C_1, C_2, \cdots, C_n \in \mathbb{R}$.

Examples : Write the partial fractions decomposition of the follwoing

1. $\frac{2x + 6}{x^2 - 2x - 3}$

   Solution :

   $$\frac{2x + 6}{x^2 - 2x - 3} = \frac{2x + 6}{(x - 3)(x + 1)} = \frac{A_1}{x - 3} + \frac{A_2}{x + 1}$$

2. $\dfrac{x+5}{x^2+4x+4}$

   Solution :

   $$\frac{x+5}{x^2+4x+4} = \frac{x+5}{(x+2)^2} = \frac{A_1}{x+2} + \frac{A_2}{(x+2)^2}$$

3. $\dfrac{x^2+1}{x^4+4x^2}$

   Solution :

   $$\frac{x^2+1}{x^4+4x^2} = \frac{x^2+1}{x^2(x^2+4)} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{B_1x+C_1}{x^2+4}$$

4. $\dfrac{2x+7}{(x+1)(x^2+9)^2}$

   Solution :

   $$\frac{2x+7}{(x+1)(x^2+9)^2} = \frac{A_1}{x+1} + \frac{B_1x+C_1}{x^2+9} + \frac{B_2x+C_2}{(x^2+9)^2}$$

5. $\dfrac{x}{(x-1)(x^2-1)}$

   Solution :

   $$\frac{x}{(x-1)(x^2-1)} = \frac{x}{(x+1)(x-1)^2} = \frac{A_1}{x+1} + \frac{A_2}{x-1} + \frac{A_3}{(x-1)^2}$$

6. $\dfrac{x^3+x}{x^2-1}$

   Solution : Using long division of polynomials

   $$\frac{x^3+x}{x^2-1} = \frac{(x^3-x)+2x}{x^2-1} = \frac{x(x^2-1)+2x}{x^2-1} = x + \frac{2x}{x^2-1}$$

   $$\frac{x^3+x}{x^2-1} = x + \frac{2x}{(x-1)(x+1)} = x + \frac{A_1}{x-1} + \frac{A_2}{x+1}$$

Examples : Evaluate the following integrals

1. $\displaystyle\int \frac{x+3}{(x-3)(x-2)}\, dx$

   Solution : Using the method of partial fractions

   $$\frac{x+3}{(x-3)(x-2)} = \frac{A_1}{x-3} + \frac{A_2}{x-2}$$

   $$\frac{x+3}{(x-3)(x-2)} = \frac{A_1(x-2)}{(x-3)(x-2)} + \frac{A_2(x-3)}{(x-2)(x-3)}$$

$$\frac{x+3}{(x-3)(x-2)} = \frac{A_1(x-2)+A_2(x-3)}{(x-3)(x-2)}$$

$$x+3 = A_1(x-2)+A_2(x-3) = A_1x - 2A_1 + A_2x - 3A_2$$

$$x+3 = (A_1+A_2)\,x + (-2A_1 - 3A_2)$$

By comparing the coefficients of the polynomials

$$\begin{cases} A_1 + A_2 = 1 & \longrightarrow (1) \\ -2A_1 - 3A_2 = 3 & \longrightarrow (2) \end{cases}$$

Muliplying equation (1) by 2 and adding it to equation (2) :

$$-A_2 = 5 \implies A_2 = -5$$

From Equation (1) : $A_1 - 5 = 1 \implies A_1 = 1+5 = 6$

$$\frac{x+3}{(x-3)(x-2)} = \frac{6}{x-3} + \frac{-5}{x-2}$$

$$\int \frac{x+3}{(x-3)(x-2)}\,dx = \int \left(\frac{6}{x-3} - \frac{5}{x-2}\right)\,dx$$

$$= 6\int \frac{1}{x-3}\,dx - 5\int \frac{1}{x-2}\,dx = 6\ln|x-3| - 5\ln|x-2| + c$$

2. $\displaystyle\int \frac{x+1}{x^2-1}\,dx$

Solution :

$$\int \frac{x+1}{x^2-1}\,dx = \int \frac{x+1}{(x-1)(x+1)}\,dx$$

$$= \int \frac{1}{x-1}\,dx = \ln|x-1| + c$$

3. $\displaystyle\int \frac{x-1}{(x+1)(x+2)^2}\,dx$

Solution : Using the method of partial fractions

$$\frac{x-1}{(x+1)(x+2)^2} = \frac{A_1}{x+1} + \frac{A_2}{x+2} + \frac{A_3}{(x+2)^2}$$

$$\frac{x-1}{(x+1)(x+2)^2} = \frac{A_1(x+2)^2}{(x+1)(x+2)^2} + \frac{A_2(x+1)(x+2)}{(x+1)(x+2)^2} + \frac{A_3(x+1)}{(x+1)(x+2)^2}$$

$$x-1 = A_1(x+2)^2 + A_2(x+1)(x+2) + A_3(x+1)$$

$$x-1 = A_1(x^2+4x+4) + A_2(x^2+3x+2) + A_3(x+1)$$

$$x-1 = A_1x^2 + 4A_1x + 4A_1 + A_2x^2 + 3A_2x + 2A_2 + A_3x + A_3$$

$$x-1 = (A_1+A_2)\,x^2 + (4A_1+3A_2+A_3)\,x + (4A_1+2A_2+A_3)$$

By comparing the coefficients of the polynomials

$$\begin{cases} A_1 + A_2 = 0 \longrightarrow (1) \\ 4A_1 + 3A_2 + A_3 = 1 \longrightarrow (2) \\ 4A_1 + 2A_2 + A_3 = -1 \longrightarrow (3) \end{cases}$$

Subtracting equation (3) from equation (2) : $A_2 = 2$

From equation (1) : $A_1 + 2 = 0 \Rightarrow A_1 = -2$

From equation (2) :

$(4 \times -2) + (3 \times 2) + A_3 = 1 \Rightarrow -8 + 6 + A_3 = 1 \Rightarrow A_3 = 3$

$$\frac{x-1}{(x+1)(x+2)^2} = \frac{-2}{x+1} + \frac{2}{x+2} + \frac{3}{(x+2)^2}$$

$$\int \frac{x-1}{(x+1)(x+2)^2}\, dx = \int \left(\frac{-2}{x+1} + \frac{2}{x+2} + \frac{3}{(x+2)^2} \right) dx$$

$$= -2\int \frac{1}{x+1}\, dx + 2\int \frac{1}{x+2}\, dx + 3\int (x+2)^{-2}\, dx$$

$$= -2\ln|x+1| + 2\ln|x+2| + 3\,\frac{(x+2)^{-1}}{-1} + c$$

$$= -2\ln|x+1| + 2\ln|x+2| - \frac{3}{x+2} + c$$

4. $\displaystyle\int \frac{2x^2+3x+2}{x^3+x}\, dx$

Solution : Using the method of partial functions

$$\frac{2x^2+3x+2}{x^3+x} = \frac{2x^2+3x+2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$\frac{2x^2+3x+2}{x^3+x} = \frac{A(x^2+1)}{x(x^2+1)} + \frac{x(Bx+C)}{x(x^2+1)}$$

$$2x^2+3x+2 = A(x^2+1) + x(Bx+C) = Ax^2 + A + Bx^2 + Cx$$

$$2x^2+3x+2 = (A+B)x^2 + Cx + A$$

By comparing the coefficients of the polynomials

$$\begin{cases} A + B = 2 \longrightarrow (1) \\ C = 3 \longrightarrow (2) \\ A = 2 \longrightarrow (3) \end{cases}$$

From equation (1) : $2 + B = 2 \Rightarrow B = 0$

$$\frac{2x^2+3x+2}{x^3+x} = \frac{2}{x} + \frac{3}{x^2+1}$$

$$\int \frac{2x^2+3x+2}{x^3+x}\ dx = \int \left(\frac{2}{x} + \frac{3}{x^2+1}\right)\ dx$$
$$= 2\int \frac{1}{x}\ dx + 3\int \frac{1}{x^2+1}\ dx$$
$$= 2\ln|x| + 3\tan^{-1} x + c$$