Laplace transform: Applications

① Obtain the solution of the differential equation:

$$\dot{y} - y = e^{3t} \; ; \; y(0) = 2 \qquad \Big| \; y(t) \xrightarrow{L.T} Y(s)$$

$$\dot{y} \xrightarrow{L.T} sY(s) - y(0)$$

$$y \xrightarrow{L.T} Y(s)$$

$$= \quad e^{3t} \xrightarrow{L.T} \frac{1}{s-3}$$

$$\Rightarrow \; sY(s) - y(0) - Y(s) = \frac{1}{s-3}$$

$$\Rightarrow \; Y(s)(s-1) - 2 = \frac{1}{s-3}$$

$$\Rightarrow \; Y(s) = \frac{1}{(s-3)(s-1)} + \frac{2}{(s-1)} = \frac{2}{s-1} - \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s-3}\right)$$

$$\Rightarrow \; y(t) = \mathcal{L}^{-1}(Y(s)) = 2e^{t} - \frac{1}{2}\left(e^{t} - e^{3t}\right)$$

$$\boxed{y(t) = \frac{3}{2}e^{t} + \frac{1}{2}e^{3t}}$$

② Same question for Equation:

$$\ddot{y} - 3\dot{y} + 2y = e^{3t} \text{ with } y(0) = 1 \,/\, \dot{y}(0) = 0$$

$$\ddot{y} \xrightarrow{LT} s^2Y(s) - sy(0) - \dot{y}(0)$$
$$= s^2Y(s) - s.$$

$$\dot{y} \xrightarrow{LT} sY(s) - y(0) = sY(s) - 1$$

$$e^{3t} \xrightarrow{LT} \frac{1}{s-3}$$

Substitute in Equation, we found:

$$s^2Y(s) - s - 3(sY(s) - 1) + 2Y(s) = \frac{1}{s-3}$$

$$Y(s)\left[s^2 - 3s + 2\right] - s + 3 = \frac{1}{s-3}$$

$$\Rightarrow Y(s) = \frac{1}{(s-3)(s^2-3s+2)} + \frac{s+3}{(s^2-3s+2)}$$

$$Y(s) = \frac{s^2 - 6s + 10}{(s-3)(s-2)(s-1)}$$

can be decomposed as the <u>case 1</u>

$$Y(s) = \frac{r_1}{s-3} + \frac{r_2}{s-2} + \frac{r_3}{s-1}$$

$$r_1 = (s-3)Y(s)\Big|_{s=3} = 1/2$$

$$r_2 = (s-2)Y(s)\Big|_{s=2} = -2$$

$$r_3 = (s-1)Y(s)\Big|_{s=1} = 5/2$$

$$\Rightarrow Y(s) = \frac{1/2}{s-3} - \frac{2}{s-2} + \frac{5/2}{s-1}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}(Y(s)) = \frac{5}{2}e^{t} - 2e^{2t} + 1/2\, e^{3t}$$

3) Solve $\dot{x} + ax = A\sin(\omega t)$; $x(0) = b$

$$\dot{x} \xrightarrow{LT} sX(s) - x(0) = sX(s) - b$$

$$x \longrightarrow X(s)$$

$$\Rightarrow (sX(s) - b) + aX(s) = A\frac{\omega}{s^2+\omega^2}$$

$$\Rightarrow X(s)(s+a) = \frac{A\omega}{s^2+\omega^2} + b$$

$$\Rightarrow X(s) = \frac{A\omega}{(s+a)(s^2+\omega^2)} + \frac{b}{s+a}$$

$$\frac{A\omega}{(s+a)(s^2+\omega^2)} = \frac{\alpha}{s+a} + \frac{\beta s + c}{s^2+\omega^2}$$

$$= \frac{\alpha s^2 + \alpha\omega^2 + (s+a)(\beta s + c)}{(s+a)(s^2+\omega^2)}$$

$$= \frac{\alpha s^2 + \alpha \omega^2 + \beta s^2 + cs + a\beta s + ac}{(s+a)(s^2+\omega^2)}$$

$$= \frac{(\alpha+\beta)s^2 + (c+a\beta)s + \alpha\omega^2 + ac}{(s+a)(s^2+\omega^2)}$$

identification with

$$\frac{A\omega}{(s+a)(s^2+\omega^2)}$$

$$\left\{\begin{array}{l} \alpha+\beta = 0 \\ c + a\beta = 0 \\ \alpha\omega^2 + ac = A\omega \end{array}\right. \Rightarrow \left\{\begin{array}{l} \alpha = -\beta \\ c = -a\beta \\ -\beta\omega^2 - a^2\beta = A\omega \end{array}\right.$$

$$\Rightarrow \beta = -\frac{A\omega}{\omega^2+a^2};$$

$$\alpha = \frac{A\omega}{\omega^2+a^2}; \quad c = \frac{Aa\omega}{\omega^2+a^2}$$

$$\Rightarrow \frac{A\omega}{(s+a)(s^2+\omega^2)} = \frac{A\omega}{a^2+\omega^2}\left(\frac{1}{s+a} - \frac{s-a}{s^2+\omega^2}\right)$$

so, $$X(s) = \frac{A\omega}{a^2+\omega^2}\left(\frac{1}{s+a} - \frac{s-a}{s^2+\omega^2}\right) + \frac{b}{s+a}$$

$$\Rightarrow X(s) = \left(b + \frac{Aw}{a^2+w^2}\right)\frac{1}{s+a} + \left(\frac{Aa}{a^2+w^2}\right)\frac{w}{s^2+w^2} - \left(\frac{Aw}{a^2+w^2}\right)\left(\frac{s}{s^2+w^2}\right)$$

$$\Rightarrow x(t) = \mathcal{L}^{-1}(X(s))$$

$$\boxed{x(t) = \left(b + \frac{Aw}{a^2+w^2}\right)e^{-at} + \frac{Aa}{a^2+w^2}\sin wt - \frac{Aw}{a^2+w^2}\cos wt}$$

(4) Solve $\ddot{y} - 10\dot{y} + 9y = 5t$ $\begin{cases} y(0) = -1 \\ \dot{y}(0) = 2 \end{cases}$

$$\ddot{y} \xrightarrow{L.T} S^2Y(S) - Sy(0) - \dot{y}(0) = S^2Y(S) + S - 2$$

$$\dot{y} \xrightarrow{L.T} SY(S) - y(0) = SY(S) + 1$$

$$5t \rightarrow \frac{5}{S^2}$$

$$\Rightarrow \left(S^2Y(S) + S - 2\right) - 10\left(SY(S) + 1\right) + 9Y(S) = \frac{5}{S^2}$$

$$\Rightarrow Y(S)\left[S^2 - 10S + 9\right) + S - 2 - 10 = \frac{5}{S^2}$$

$$\Rightarrow Y(S) = \frac{5 + 12S^2 - S^3}{S^2(S-9)(S-1)}$$

$$= \frac{A}{S} + \frac{B}{S^2} + \frac{C}{S-9} + \frac{D}{S-1}$$

$$B = S^2Y(S)\Big|_{S=0} = \frac{5}{9}$$

$$C = (S-9)Y(S)\Big|_{S=9} = \frac{31}{81}$$

$$D = (S-1)Y(S)\Big|_{S=1} = -2$$

$$\Rightarrow A = \frac{50}{81}$$

$$\Rightarrow Y(s) = \frac{(50/81)}{s} + \frac{(5/9)}{s^2} + \frac{\left(\frac{31}{81}\right)}{s-9} - \frac{2}{s-1}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}(Y(s)) = \frac{50}{81} + \frac{5}{9}t + \frac{31}{81}e^{9t} - 2e^{t}$$

(5) $\ddot{y} - 6\dot{y} + 15y = 2\sin(3t) : y(0) = -1$
$\dot{y}(0) = -4$

after developp $\Rightarrow$

$$Y(s) = \frac{-s^3 + 2s^2 - 9s + 24}{(s^2+9)(s^2-6s+15)}$$

$$= \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2-6s+15}$$

After identification

$$\begin{cases} A + C = -1 \\ -6A + B + D = 2 \\ 15A - 6B + 9C = -9 \\ 15B + 9D = 24 \end{cases}$$

the solution of this system leads:

$A = 1/10$; $B = 1/10$; $C = -\frac{11}{10}$; $D = 5/2$

$$Y(s) = \frac{1}{10}\left(\frac{s+1}{s^2+9} + \frac{-11s + 25}{s^2-6s+15}\right)$$

..... show that

$$y(t) = \mathcal{L}^{-1}(Y(s)) = \frac{1}{10}\left(\cos 3t + \frac{1}{3}\sin 3t - 11e^{3t}\cos\sqrt{6}t - \frac{8}{\sqrt{6}}e^{3t}\sin\sqrt{6}\,t\right)$$