Probability

Random experiment التجربة العشوائية

It is an experiment with a predictable outcome

التجربة التي لا يمكن التنبأ بنتائجها عند إجرائها وان كنا نعلم جميع النتائج اللمكنة

مثال تجربة إلقاء قطعة عملة---تجربة إلقاء زهرة النرد (الطاولة)—تجربة قرعة كأس العالم لكرة القدم

Example: to select randomly two students and check if they are smoker or non-smoker we have the following possibilities.

The first student is non – smoker the second is non-smoker too.

The first student is smoker the second is non – smoker.

The first student is non – smoker the second is smoker .

The first student is smoker, the second is smoker too .

If we define smoker by S and Non-smoker by N, We have the following sample points NN, NS , SN, SS.

The set of all possible outcomes of a statistical experiment is called the **sample space** **فضاء العينة** and is represented by the symbol S.

Each outcome in a sample space is called an **element** or a **member** of the sample space, or simply a **sample point**.

Example 1:The sample space S, of possible outcomes when a coin is flipped, may be written

$$S = \{H, T\},$$

where H and T correspond to heads and tails, respectively.

Example 2: Consider the experiment of tossing a die. If we are interested in the number that shows on the top face, the sample space is

$$S1 = \{1, 2, 3, 4, 5, 6\}.$$

Qu1: what is the sample space if we flipped tossing of die two times?????

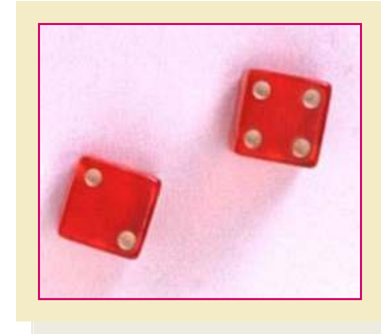
Example 3: If we are interested only in whether the number is even or odd, the sample space is simply

$$S2 = \{\text{even}, \text{odd}\}.$$

Example 4: The sample space *S*, of possible outcomes when a coin is flipped two times, may be written

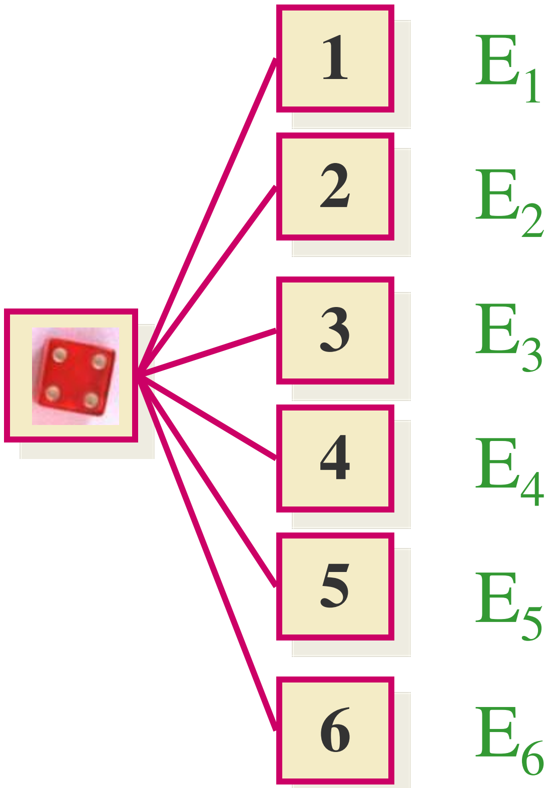
$$S = \{HH, HT, TH, TT\},$$

where *H* and *T* correspond to heads and tails, respectively



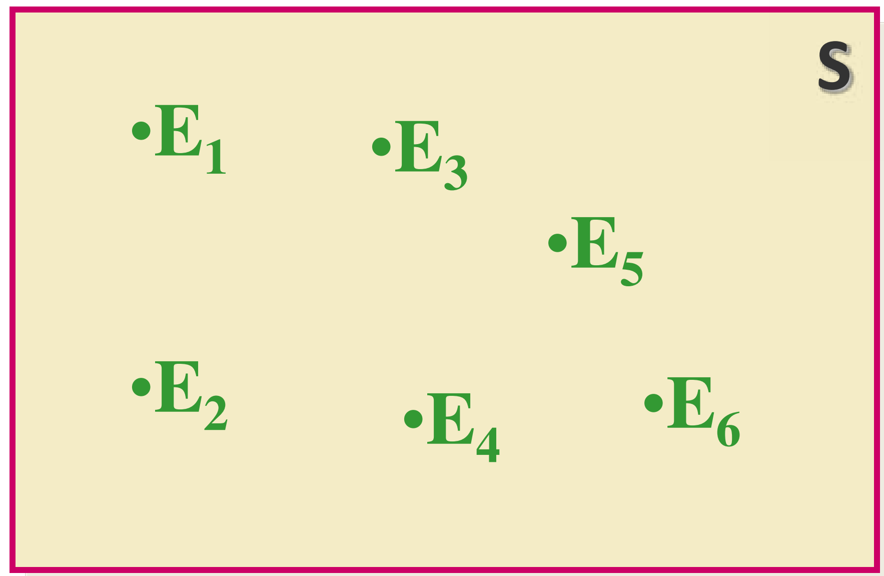
Example

- **The die toss:**
- Simple events:



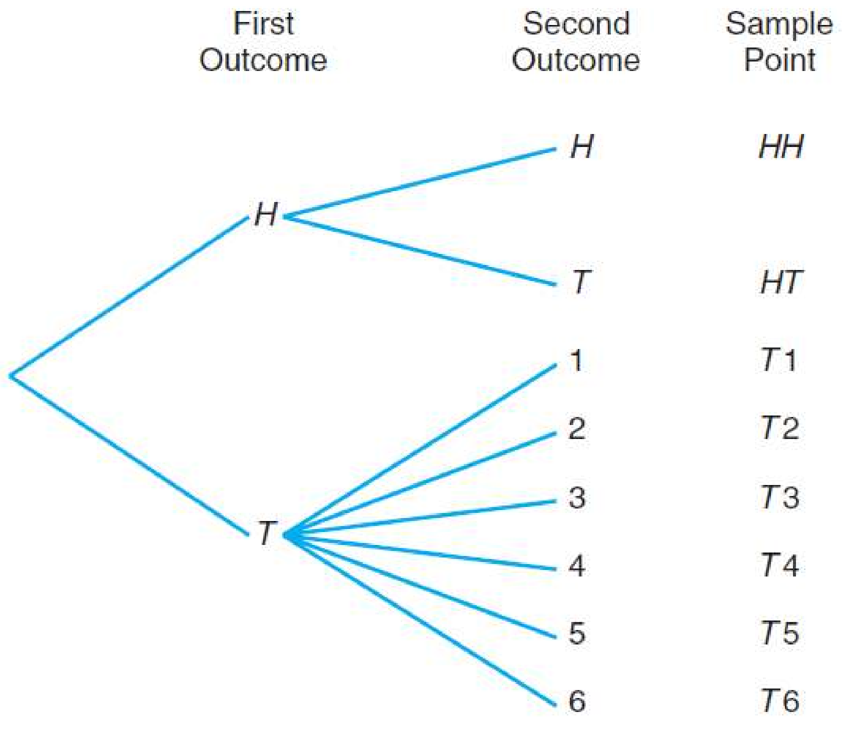
Sample space:

$$S = \{E_1, E_2, E_3, E_4, E_5, E_6\}$$



Example 5: An experiment consists of flipping a coin and then flipping it a second time if a head الصورةoccurs. If a tail occurs كتابةon the first flip, then a die is tossed once.

$$S = \{HH, HT, T1, T2, T3, T4, T5, T6\}.$$



tree diagram

An **event** الحدث is a subset of a sample spaceفضاء العينة.

هناك نوعين من الحدث

1- الحدث المستحيل **وهو عدم ظهور الحدث ويكون نتيجة**

Example:1:The sample space *S*, of possible outcomes when tossing a die and the Event more than 6, the Sample space S will be φ

2- الحدث المؤكد وهو ظهور كل بيانات العينة

Example 1::The sample space *S*, of possible outcomes when tossing a die and the Event less than 7, the Sample space S will be *all elements S={1,2,3,4,5,6}*

الحدث المكمل

The **complement** of an event A with respect to S is the subset of all elements of S that are not in A.
We denote the complement of A by the symbol $\acute{A}$

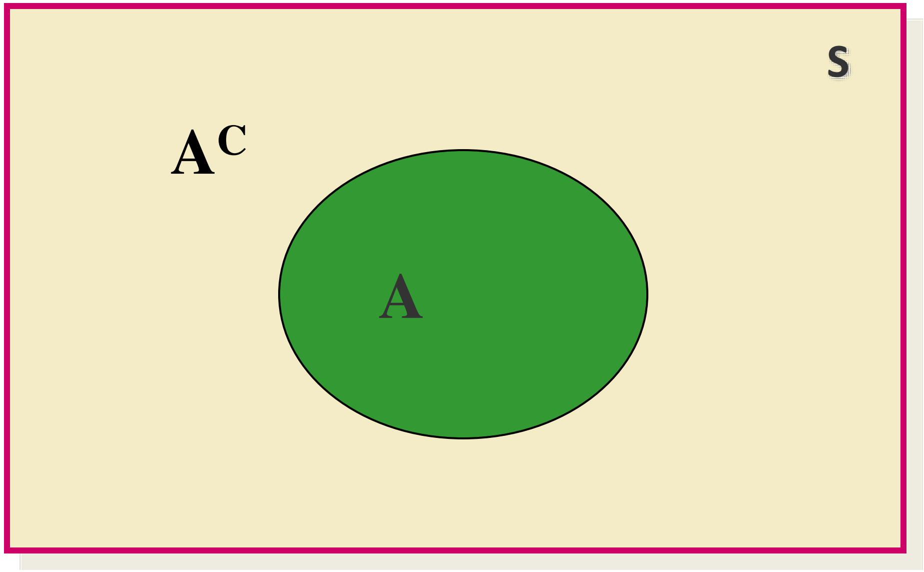
Example 1

Consider the sample space S = {book, cell phone, mp3, paper, stationery, laptop}.
Let A = {book, stationery, laptop, paper}.
Then the complement of A is $\acute{A}$ = {cell phone, mp3}.

The Event E' is Called the complement of the Event E if E'= S-E
i.e. P(E')= P(S)- P(E) then P(E')= 1- P(E)

Event Relations

The **complement** of an event A consists of all outcomes of the experiment that do not result in event A. We write A^C.



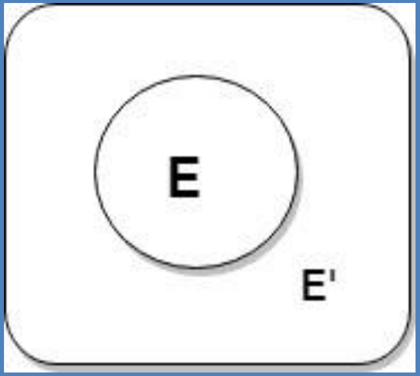
Example 2: $S = \{1, 2, 3, 4, 5, 6\}$

E is the event getting and odd number

$$E = \{1, 3, 5\}$$

Then E complement is $E' = \{2, 4, 6\}$, Note that E and E' has no elements in common .

The following Venn – Diagram shows the complement of event E .



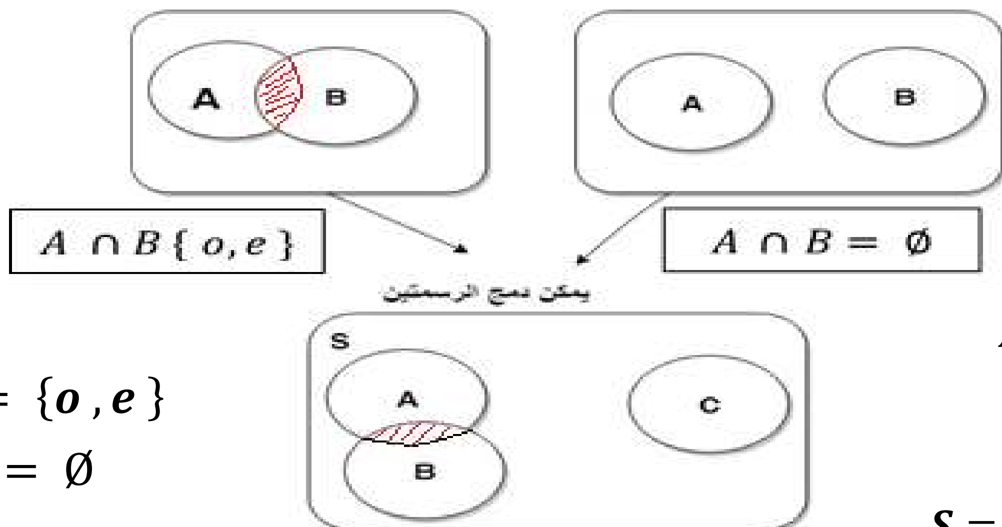
$$S = \{1,2,3,4,5,6\}$$
$$E = \{1,3,5,\}$$
$$E' = \{2,4,6,\}$$

The **intersection** of two events *A* and *B*, denoted by the symbol $A \cap B$, is the event containing all elements that are common to *A* and *B*.

Example 2 : Suppose $A = \{a, e, i, o, u\}$, $B = \{t, o, e\}$, $C = \{r, s, t\}$

$$find\ A \cap B, A \cap C$$

Ans: $A \cap B = \{o, e\}$, $A \cap C = \{\varphi\}$



$A \cap B = \{o, e\}$

$A \cap C = \emptyset$

$A = \{a, e, i, o, u\}$

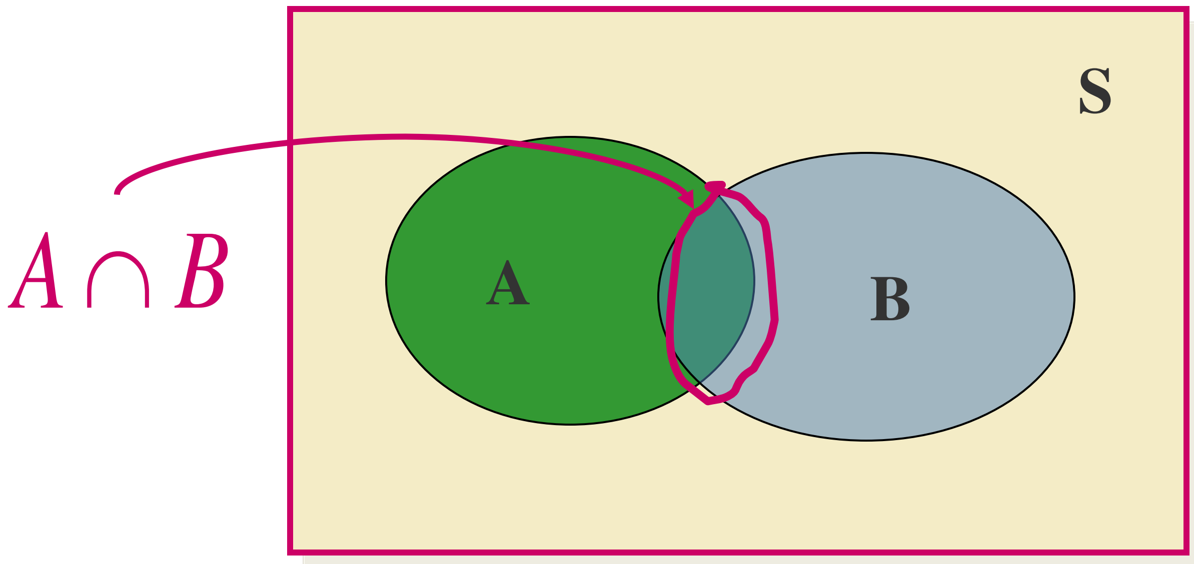
$B = \{t, o, e\}$

$C = \{r, s, t\}$

$S = \{set\ of\ all\ alphabets\}$

Event Relations

The **intersection** of two events, **A** and **B**, is the event that both A **and** B occur when the experiment is performed. We write **A ∩ B**.



- If two events A and B are **mutually exclusive**, then P(**A ∩ B) = 0**.

Basic Concepts

- Two events are **mutually exclusive** if, when one event occurs, the other cannot, and vice versa.

Two events A and B are **mutually exclusive** **الاحداث المتنافية**, or **disjoint**, if $A \cap B = \varphi$, that is, if A and B have no elements in common.

- **Experiment: Toss a die**
  - A: observe an odd number
  - B: observe a number greater than 2
  - C: observe a 6
  - D: observe a 3

Not Mutually Exclusive

Mutually Exclusive

B and C?
B and D?

The **union** of the two events *A* and *B*, denoted by the symbol $A \cup B$, is the event containing all the elements that belong to *A* or *B* or both.

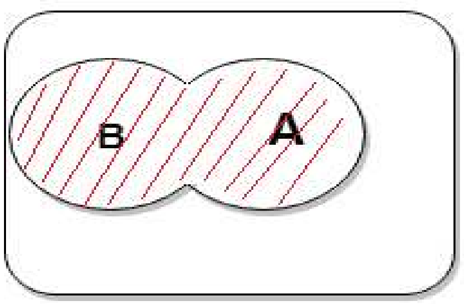
The union of two events A, B denoted as $A \cup B$ is the set of all elements in A, B without repetition.

Example 3: Suppose $A = \{a, r, s, t, u\}, R = \{L, m, n\}$ and $B = \{s, t, w, z, k\}, C = \{s, t, u\}$

Find $A \cup B, A \cup C$

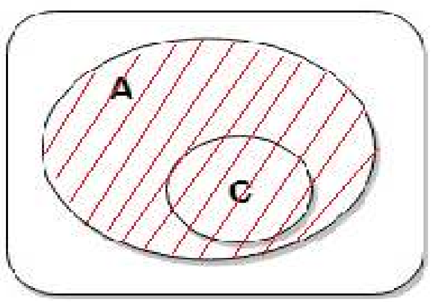
Ans:

$$A \cup B = \{a, r, r\ s, t, u, w, z, k\}$$



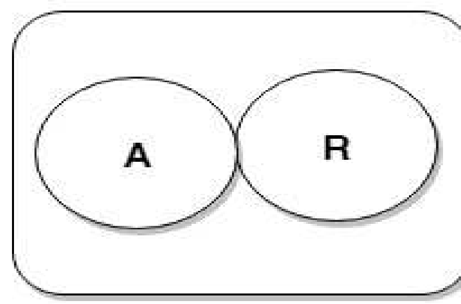
$A \cup B$

يوجد قاسم مشترك



$A \cup C$

A تقع داخل C



A ∪R

لايوجد قاسم مشترك

The Probability of an Event

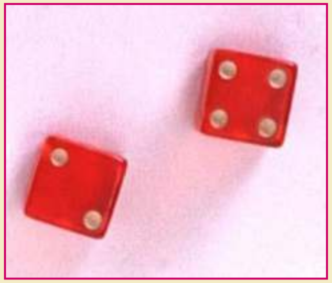
$$P(A) = m/n = \frac{\text{The number of the event occur}}{\text{The Total number of the sample Space}}$$

احتمال حدوث الحدث هوعدد مرات حدوث الحدث علي عدد المرات الكلية لفراغ العينة

The Probability Function Rules:

1) $0 <= P(A) <= 1$
2) $P(\varphi) = 0$
3) $P(S) = 1$

The Probability of an Event



- P(A) must be between 0 and 1.
  - If event A can never occur, P(A) = 0. If event A always occurs when the experiment is performed, P(A) =1.
- The sum of the probabilities for all simple events in S equals 1.

- The **probability of an event A** is found by adding the probabilities of all the simple events contained in A.

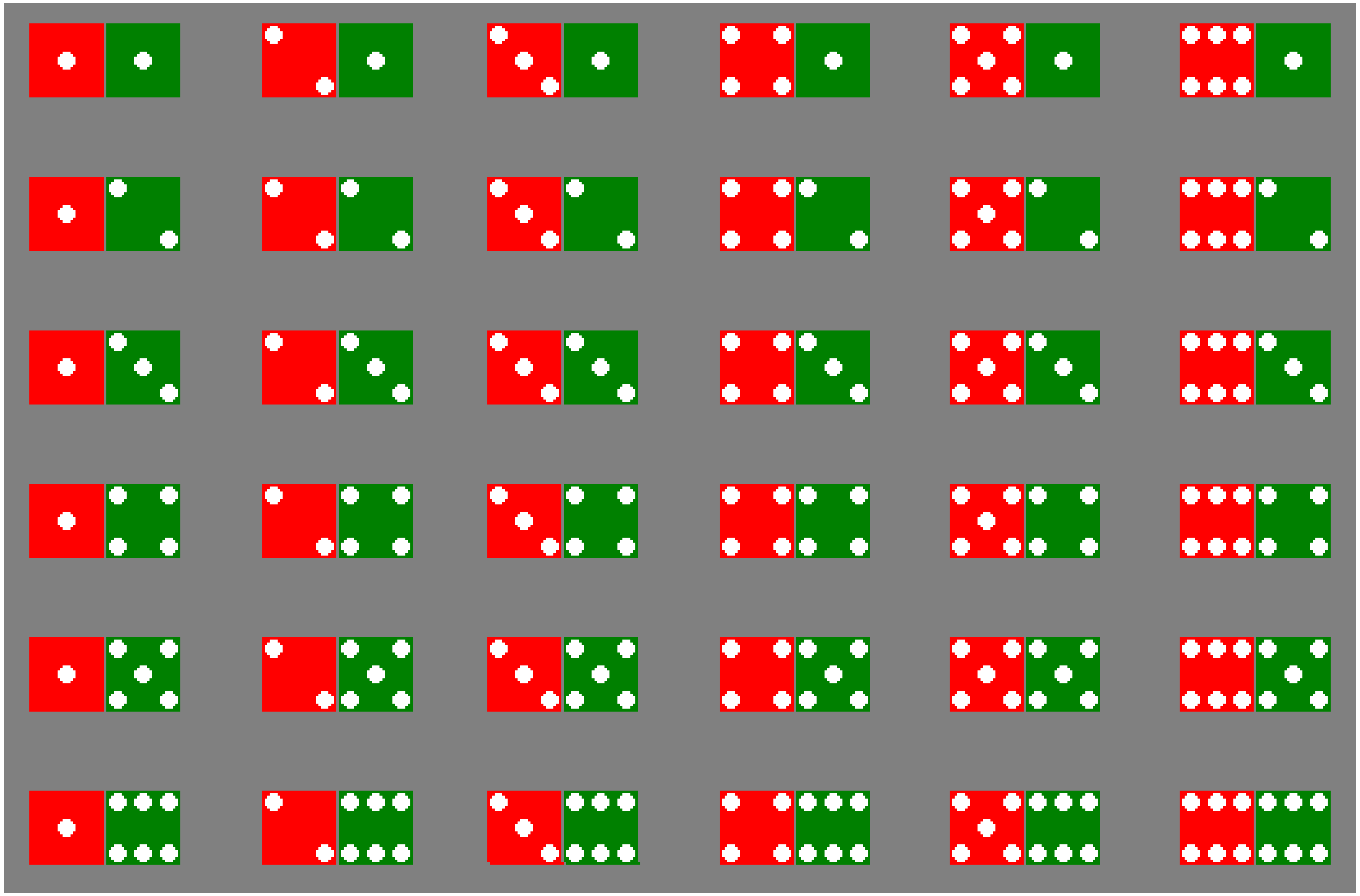
Using Simple Events

- The probability of an event A is equal to the sum of the probabilities of the simple events contained in A
- If the simple events in an experiment are **equally likely,** you can calculate

$$P(A) = \frac{n_A}{N} = \frac{\text{number of simple events in A}}{\text{total number of simple events}}$$

Example 3

The sample space of throwing a pair of dice is



Example 3

Event	Simple events	Probability
Dice add to 3	(1,2),(2,1)	2/36
Dice add to 6	(1,5),(2,4),(3,3),(4,2),(5,1)	5/36
Red die show 1	(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)	6/36
Green die show 1	(1,1),(2,1),(3,1),(4,1),(5,1),(6,1)	6/36

If an experiment can result in any one of N different equally likely outcomes, and if exactly n of these outcomes correspond to event A, then the probability of event A is

$$P(A) = \frac{n}{N}.$$

A statistics class for engineers consists of 25 industrial, 10 mechanical, 10 electrical, and 8 civil engineering students. If a person is randomly selected by the instructor to answer a question, find the probability that the student chosen is (a) an industrial engineering major and (b) a civil engineering or an electrical engineering major.

Denote by I, M, E, and C the students majoring in industrial, mechanical, electrical, and civil engineering, respectively. The total number of students in the class is 53, all of whom are equally likely to be selected.

(a) Since 25 of the 53 students are majoring in industrial engineering, the probability of event I, selecting an industrial engineering major at random, is

$$P(I) = \frac{25}{53}.$$

(b) Since 18 of the 53 students are civil or electrical engineering majors, it follows that

$$P(C \cup E) = \frac{18}{53}.$$

Example**:** what is the sample space of selecting 3 items from a manufacturing line. given that each item can be classified as either Defective D or Non – Defective N.

What the event of:

Getting at least علي الاقل two defective items.

Getting at most علي الاكثر one defective item.

Getting 3 defectives.

Answer: 3 items with 2 possibilities each, we get the following sample Space:

$$S = \{DDD, DDN, DND, DNN, NDD, NDN, NND, NNN\}$$

We call the first event A, the second event B, the third event C.

$$A = \{DDD, DDN, DND, NDD\}$$

$$B = \{DNN, NDN, NND, NNN\}$$

$$C = \{DDD\}$$

$$A, B, C \text{ are } \subseteq S$$

P(A)= 4/8 P(B)= 4/8 P(C)= 1/8

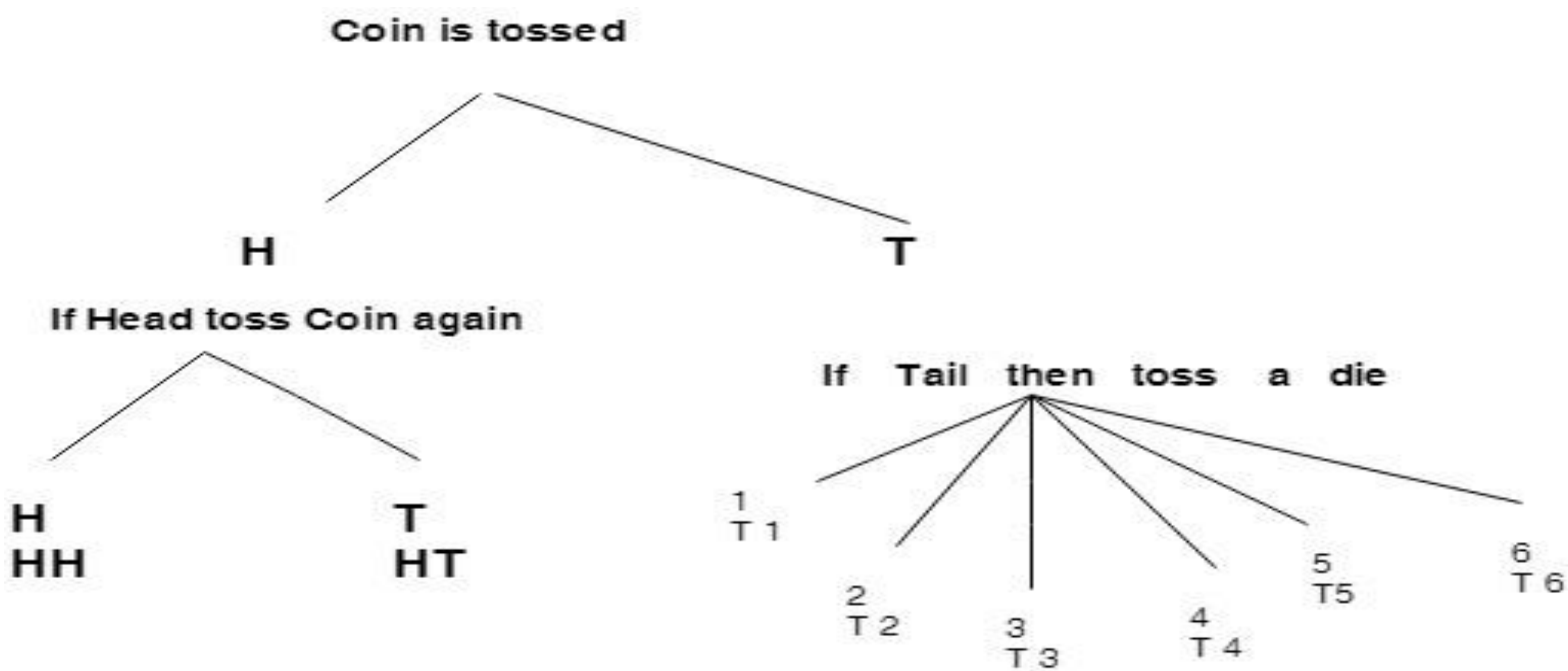
Example: An exponent consists of tossing a coin and then throw Adie if tail occurs in the coin otherwise the coin is tossed another time find the sample space and the event of getting at least علي الاقل one tail .

Ans : The sample space $S = \{HH, HT, T1, T2, T3, T4, T5, T6\}$

The event of getting at least one tail

$$Et = \{HT, T1, T2, T3, T4, T5, T6\}$$

P(Et)= 7/8



$Et = \emptyset$ If the event is not possible such as getting – 1 is case of tossing die

Example 1

Toss a fair coin twice. What is the probability of observing at least one head?

1st Coin	2nd Coin	E_i	$P(E_i)$
H	H	HH	1/4
	T	HT	1/4
T	H	TH	1/4
	T	TT	1/4

P(at least 1 head)

$= P(E_1) + P(E_2) + P(E_3)$

$= 1/4 + 1/4 + 1/4 = 3/4$

If A and B are two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

If A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B).$$

Ali is going to graduate from an industrial engineering department in a university by the end of the semester. After being interviewed at two companies he likes, he assesses that his probability of getting an offer from company *A* is 0.8, and his probability of getting an offer from company *B* is 0.6. If he believes that the probability that he will get offers from both companies is 0.5, what is the probability that he will get at least one offer from these two companies?

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.8 + 0.6 - 0.5 = 0.9.$$

The **probability** of an event A is the sum of the weights of all sample points in A. Therefore,

$$0 \leq P(A) \leq 1, \quad P(\phi) = 0, \quad \text{and} \quad P(S) = 1.$$

Furthermore, if $A_1, A_2, A_3, \ldots$ is a sequence of mutually exclusive events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = P(A_1) + P(A_2) + P(A_3) + \cdots.$$

A coin is tossed twice. What is the probability that at least 1 head occurs?

Solution: The sample space for this experiment is

$$S = \{HH, HT, TH, TT\}.$$

If the coin is balanced, each of these outcomes is equally likely to occur. Therefore, we assign a probability of ω to each sample point. Then $4\omega = 1$, or $\omega = 1/4$. If A represents the event of at least 1 head occurring, then

$$A = \{HH, HT, TH\} \text{ and } P(A) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}.$$

What is the probability of getting a total of 7 or 11 when a pair of fair dice is tossed?
Let A be the event that 7 occurs and B the event that 11 comes up. Now, a total of 7 occurs for 6 of the 36 sample points, and a total of 11 occurs for only 2 of the sample points. Since all sample points are equally likely, we have $P(A) = 1/6$ and $P(B) = 1/18$. The events A and B are mutually exclusive, since a total of 7 and 11 cannot both occur on the same toss. Therefore,

$$P(A \cup B) = P(A) + P(B) = \frac{1}{6} + \frac{1}{18} = \frac{2}{9}.$$

If the probabilities are, respectively, 0.09, 0.15, 0.21, and 0.23 that a person purchasing a new automobile will choose the color green, white, red, or blue, what is the probability that a given buyer will purchase a new automobile that comes in one of those colors?
Let G, W, R, and B be the events that a buyer selects, respectively, a green, white, red, or blue automobile. Since these four events are mutually exclusive, the probability is

$$\begin{aligned} P(G \cup W \cup R \cup B) &= P(G) + P(W) + P(R) + P(B) \\ &= 0.09 + 0.15 + 0.21 + 0.23 = 0.68. \end{aligned}$$

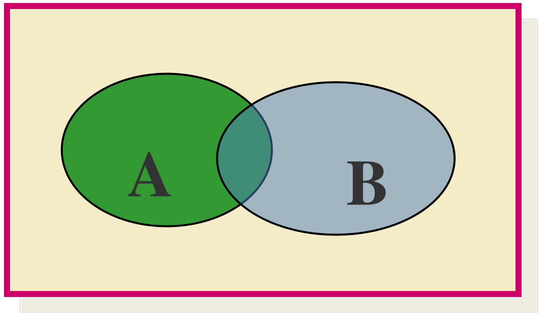
Calculating Probabilities for Unions and Complements

There are special rules that will allow you to calculate probabilities for composite events.

The Additive Rule for Unions:

For any two events, **A** and **B**, the probability of their union, **P(A ∪ B)**, is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



Example: Additive Rule

Example: Suppose that there were 120 students in the classroom, and that they could be classified as follows:

A: brown hair

P(A) = 50/120

B: female

P(B) = 60/120

	Brown	Not Brown
Male	20	40
Female	30	30

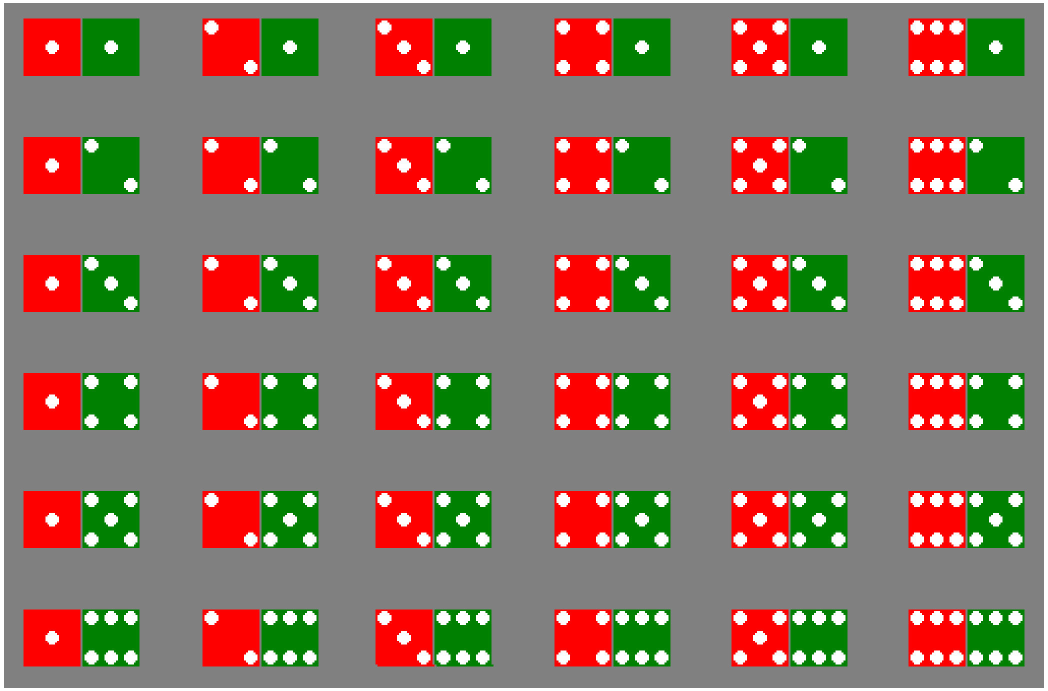
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$= 50/120 + 60/120 - 30/120$
$= 80/120 = 2/3$

Check: $P(A \cup B)$
$= (20 + 30 + 30)/120$

Example: Two Dice

A: red die show 1

B: green die show 1



$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= 6/36 + 6/36 - 1/36$$
$$= 11/36$$

A Special Case

When two events A and B are **mutually exclusive**, $P(A \cap B) = 0$ and $P(A \cup B) = P(A) + P(B)$.

A: male with brown hair

$P(A) = 20/120$

B: female with brown hair

$P(B) = 30/120$

	Brown	Not Brown
Male	20	40
Female	30	30

A and B are mutually exclusive, so that →

$P(A \cup B) = P(A) + P(B)$
$= 20/120 + 30/120$
$= 50/120$

Theorem

If A and A' are complementary events, then

$$P(A) + P(A') = 1.$$

Proof: Since $A \cup A' = S$ and the sets A and A' are disjoint,

$$1 = P(S) = P(A \cup A') = P(A) + P(A').$$

If the probabilities that an automobile mechanic will service 3, 4, 5, 6, 7, or 8 or more cars on any given workday are, respectively, 0.12, 0.19, 0.28, 0.24, 0.10, and 0.07, what is the probability that he will service at least 5 cars on his next day at work?

Let E be the event that at least 5 cars are serviced. Now, $P(E) = 1 - P(E')$, where E' is the event that fewer than 5 cars are serviced. Since

$$P(E') = 0.12 + 0.19 = 0.31,$$

$$P(E) = 1 - 0.31 = 0.69.$$

Counting Techniques

Counting techniques are used to estimate the possible number of sample points in a random experiment. Some experiments produce thousands of cases which can't be listed one by one .

we talk about the following counting techniques.

a) Multiplication rule .
b) Permutations . تباديل
c) Combinations . توافيق

a– Multiplication rule

If an operation can be performed in *n*1 ways, and if for each of these ways a second operation can be performed in *n*2 ways, then the two operations can be performed together in *n*1*n*2 ways.

Example 1: Tossing two dice produces 36 ways. Because n1 = 6 ways and n2 = 6 ways , and hence the two dice produce $6\ x\ 6 = 36$ **points or ways .**

If an operation can be performed in *n*1 ways, and if for each of these a second operation can be performed in *n*2 ways, and for each of the first two a third operation can be performed in *n*3 ways, and so forth, then the sequence of *k* operations can be performed in *n*1*n*2 · · · *nk* ways.

Example 2: An ATM machine is hacked by using a card with 4 digits secret number, how many trials the hacker can do to guess the secure number if he knows the following .

The numbers have repeated digits.

The numbers has no repeated digits.

The numbers starts with 19 and the last digits are not similar

Ans:

The number has repeated digits the four digits are guessed among the 10 digits (0 ,1,2,3,4,5,6,7,8,9). Guesses = 10x10x10x10 = 10000 guesses .

No repeated digits, So if the first digit can be tried in 10 ways, the second digit on be tried in 10 – 1 or 9 ways and so on. Guesses = 10 x9x8x7 = 5040

Starts with 19 and no the last digits are not similar but including 1 and 9. guesses = 1 X 1 X 10 X 9 = 90 guesses

Example 3

Ali is going to assemble a computer by himself. He has the choice of **chips** from **two** brands, a **hard drive** from **four**, **memory** from **three**, and an **accessory** bundle from **five** local stores. How many different ways can Ali order the parts?

Solution : Since $n1 = 2$, $n2 = 4$, $n3 = 3$, and $n4 = 5$, there are

$nl \times n2 \times n3 \times n4 = 2 \times 4 \times 3 \times 5 = 120$

different ways to order the parts

permutation التباديل

A **permutation** التباديل is an arrangement of all or part of a set of objects. الترتيب مهم- وتحديد الاشياء مهم

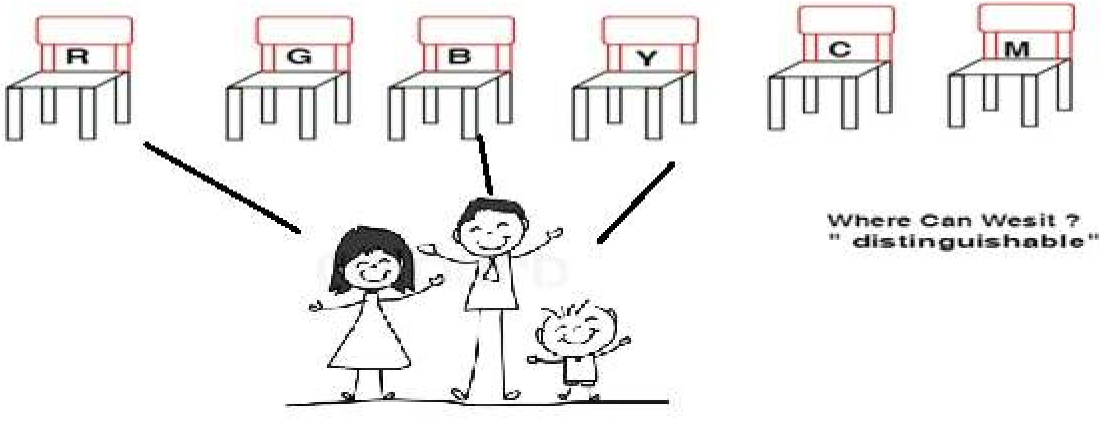
The possible arrangements of all or part of a set of objects. It is applied when objects are "Ordinal". In other words, when order matters.

The number of permutations of n distinct objects taken r at a time is

$$_nP_r = \frac{n!}{(n-r)!}.$$

permutation التباديل

How many possible ways of assigning <u>mother, father and son</u> on 6 possible colored chairs .



Solution

$$6P3 = \frac{6!}{(6-3)} = \frac{6x5x4x\not{3!}}{\not{3!}}$$

$$= 6\,x\,5\,x4 = 20\,x\,6 = 120\,ways\,.$$

permutation التباديل

In one year, **three** awards (**research, teaching, and service**) will be given to a class of **25** graduate students in a statistics department. If each student can receive at most one award, how many possible selections are there?

***Solution* :** Since the awards are distinguishable, it is a permutation problem. The total number of sample points is

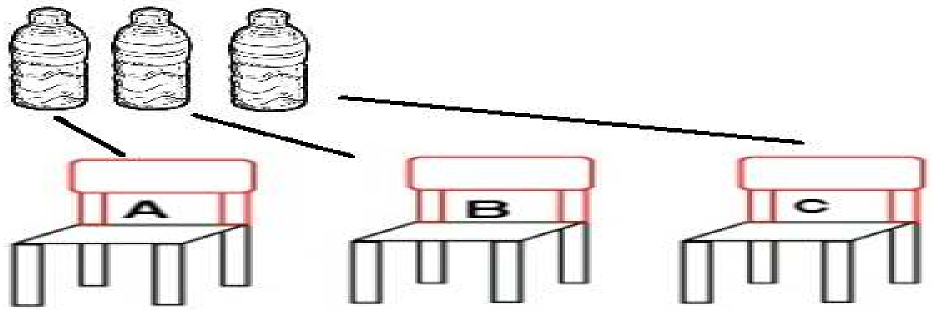
$$_{25}P_3 = \frac{25!}{(25-3)!} = \frac{25!}{22!} = (25)(24)(23) = 13,800.$$

Combinations التوافيق

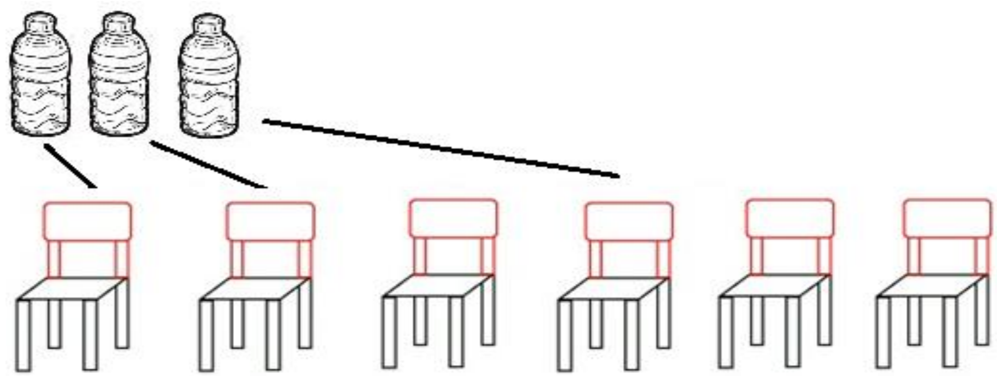
In many problems we are interested to find how many ways of selecting objects out of n objects without regard to order. الترتيب غير مهم- تحديد الاشياء غير مهم

$$^{n}C_{r} = \frac{n!}{(n-r)!\,r!}$$

Example 1 if we have 3 indistinguishable water bottles to be put on the three chairs we would have only single permutation .



If we changed the position of one bottle to another chair, that will not affect the permutation it will remain single combination .



If there are six chairs and 3 bottles, the resetting permutation would be.

$$\frac{6\,!}{3!\,(6-3)!} = \frac{6 \times 5 \times 4 \times 3!}{3! \times 3\,!} = 20\ ways$$

Example 2: we have 10 operation rooms, 4 patients. How many ways can we assign the 4 patients to the rooms?

Ans: $n = 10 \qquad r = 4$

$$\binom{10}{4} = \frac{10!}{4!\,(10-4)!} = \frac{10\,!}{4! \times 6!} = 210\ Ways$$

Example 3: **A club has 25 members.**

- How many ways are there to choose four members of the club to serve on an executive committee?
  - **Order not important الترتيب غير مهم**
  - C(25,4) = 25!/21!4! = 25*24*23*22/4*3*2*1 =25*23*22 = 12,650
- How many ways are there to choose a president, vice president, secretary, and treasurer of the club?
  - **Order is important الترتيب مهم**
  - P(25,4) = 25!/21! = 303,600

A club has 25 members.

- How many ways are there to choose four members of the club to serve on an executive committee?
  - Order not important
  - C(25,4) = 25!/21!4! = 25*24*23*22/4*3*2*1 =25*23*22 = 12,650
- How many ways are there to choose a president, vice president, secretary, and treasurer of the club?
  - Order is important
  - P(25,4) = 25!/21! = 303,600

Conditional Probability الاحتمال الشرطي

The probability of B with the Information that A already occurred

The conditional probability of B, given A, denoted by $P(B|A)$, is defined by

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \quad \text{provided} \quad P(A) > 0.$$

Categorization of the Adults in a Small Town

	Employed	Unemployed	Total
Male	460	40	500
Female	140	260	400
Total	600	300	900

M: a man is chosen,

E: the one chosen is employed.

Find $P(M|E)$

$$P(M|E) = \frac{P(E \cap M)}{P(E)},$$

$$P(E) = \frac{600}{900} = \frac{2}{3} \quad \text{and} \quad P(E \cap M) = \frac{460}{900} = \frac{23}{45}.$$

$$P(M|E) = \frac{23/45}{2/3} = \frac{23}{30},$$

The probability that a regularly scheduled flight departs on time is $P(D) = 0.83$; the probability that it arrives on time is $P(A) = 0.82$; and the probability that it departs and arrives on time is $P(D \cap A) = 0.78$. Find the probability that a plane (a) arrives on time, given that it departed on time, and (b) departed on time, given that it has arrived on time.

(a) The probability that a plane arrives on time, given that it departed on time, is

$$P(A|D) = \frac{P(D \cap A)}{P(D)} = \frac{0.78}{0.83} = 0.94.$$

(b) The probability that a plane departed on time, given that it has arrived on time, is

$$P(D|A) = \frac{P(D \cap A)}{P(A)} = \frac{0.78}{0.82} = 0.95.$$

Theorem of Total Probability

If the events $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space S such that $P(B_i) \neq 0$ for $i = 1, 2, \ldots, k$, then for any event A of S,

$$P(A) = \sum_{i=1}^{k} P(B_i \cap A) = \sum_{i=1}^{k} P(B_i)P(A|B_i).$$

In a certain assembly plant, three machines, B_1, B_2, and B_3, make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective. Now, suppose that a finished product is randomly selected. What is the probability that it is defective?

Consider the following events:

A: the product is defective,

B_1: the product is made by machine B_1,

B_2: the product is made by machine B_2,

B_3: the product is made by machine B_3.

Applying the rule of elimination, we can write

$$P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3).$$

$$P(B_1)P(A|B_1) = (0.3)(0.02) = 0.006,$$
$$P(B_2)P(A|B_2) = (0.45)(0.03) = 0.0135,$$
$$P(B_3)P(A|B_3) = (0.25)(0.02) = 0.005,$$

$$P(A) = 0.006 + 0.0135 + 0.005 = 0.0245.$$

(Bayes' Rule) If the events $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space S such that $P(B_i) \neq 0$ for $i = 1, 2, \ldots, k$, then for any event A in S such that $P(A) \neq 0$,

$$P(B_r|A) = \frac{P(B_r \cap A)}{\sum_{i=1}^{k} P(B_i \cap A)} = \frac{P(B_r)P(A|B_r)}{\sum_{i=1}^{k} P(B_i)P(A|B_i)} \quad \text{for } r = 1, 2, \ldots, k.$$

A committee of 18 members was formed to represent the four grades in one of the faculties as follows:

3 members of the first grade one of them is female

5 members of the second grade two of them are female

6 members of the third grade three of them are female

4 members of the fourth grade two of them are female

The piece is manufactured in a factory and directed to test its validity by two workers. If the probability of dealing by the first worker is 0 .6 and second worker is 0.4. If it valid and chosen by first worker is 0.94. and by second worker is 0.98. What is the probability that the piece chosen by first worker.