Boolean Algebra

Lecture 07

COE211-Digital Logic Design

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Basic Theorem for Boolean Algebra (1)

- **Idempotency:**

(a) $x + x = x$ (b) $x \, . \, x = x$

Proof of (a):

$x + x = (x + x).1$ (identity)

$= (x + x).(x + x')$ (complementarity)

$= x + (x.x')$ (distributivity)

$= x + 0$ (complementarity)

$= x$ (identity)

Basic Theorem for Boolean Algebra (2)

- **Null elements** for + and . operators:

  (a) $x + 1 = 1$ (b) $x \cdot 0 = 0$

- **Involution:** $(x')' = x$

- **Absorption:**

  (a) $x + (x.y) = x$ (b) $x \cdot (x + y) = x$

- **Absorption** (variant):

  (a) $x + (x'.y) = x+y$ (b) $x \cdot (x' + y) = x.y$

Basic Theorem for Boolean Algebra (3)

- **DeMorgan:**
  (a) $(x + y)' = x'.y'$
  (b) $(x.y)' = x' + y'$

- **Consensus:**
  (a) $x.y + x'.z + y.z = x.y + x'.z$
  (b) $(x+y).(x'+z).(y+z) = (x+y).(x'+z)$

How can we prove these theorems?

- Theorems can be proved using the truth table method (Exercise: Prove De-Morgan's theorem using the truth table)
- They can also be proved by **algebraic manipulation** using axioms/postulates or other basic theorems

Proofs (1)

- **The absorption theorem-a** can be proved as follows:

$$x + x.y = x.1 + x.y \quad \text{(identity)}$$
$$= x.(1 + y) \quad \text{(distributivity)}$$
$$= x.(y + 1) \quad \text{(commutativity)}$$
$$= x.1 \quad \text{(Theorem (null element))}$$
$$= x \quad \text{(identity)}$$

- **The absorption theorem-b** can be proved using duality:

$$x.(x+y) = x$$

- Try prove this by algebraic manipulation

Boolean Functions

- **Boolean function** is an expression formed with binary variables, the two binary operators, OR and AND, and the unary operator, NOT, parenthesis and the equal sign
- Its result is also a binary value
- We usually use **. for AND**, **+ for OR**, and **' or ¬ for NOT**
  - Sometimes, we may omit the **.** if there is no ambiguity

Example 1

- Examples:

  F1= xyz'

  F2= x + y'z

  F3=(x'y'z)+(x'yz)+(xy')

  F4=xy'+x'z

x	y	z	F1	F2	F3	F4
0	0	0	0	0	0	0
0	0	1	0	1	1	1
0	1	0	0	0	0	0
0	1	1	0	0	1	1
1	0	0	0	1	1	1
1	0	1	0	1	1	1
1	1	0	1	1	0	0
1	1	1	0	1	0	0

- From the truth table → F3=F4
- Can you also prove by algebraic manipulation that F3=F4?

Complement of Functions (1)

- Given a function, F, the **complement** of this function, F', is obtained by interchanging 1 with 0 in the function's output values
- Example:

F1 = xyz'

F1' = (xyz')' (Complement)

= x' + y' + (z')' (DeMorgan)

= x' + y' + z (Involution)

x	y	z	F1	F1'
0	0	0	0	1
0	0	1	0	1
0	1	0	0	1
0	1	1	0	1
1	0	0	0	1
1	0	1	0	1
1	1	0	1	0
1	1	1	0	1

Complement of Functions (2)

- More general DeMorgan's theorems useful for obtaining complement functions:

$$(A + B + C + \ldots + Z)' = A' \cdot B' \cdot C' \ldots \cdot Z'$$

$$(A \cdot B \cdot C \ldots \cdot Z)' = A' + B' + C' + \ldots + Z'$$

Standard Forms (1)

- Certain types of Boolean expressions lead to gating networks which are desirable from implementation viewpoint
- Two Standard Forms:

  Sum-of-Products and ***Product-of-Sums***

- **Literals**: a variable on its own or in its complemented form. Examples: x, x' , y, y'
- **Product Term**: a single literal or a logical product (AND) of several literals.

  Examples: x, xyz', A'B, AB

Standard Forms (2)

- Every boolean expression can either be expressed as sum-of-products or product-of-sums expression.
- Examples:
  - SOP: $x'y + xy' + xyz$
  - POS: $(x + y')(x' + y)(x' + z')$
  - both: $x' + y + z$ or xyz'

Minterm

- Consider two binary variables: x, y
- Each variable may appear as itself or in complemented form as literals (*i.e.*, x, x' & y, y’)
- For two variables, there are four possible combinations with the AND operator, namely:

  x'y', x'y, xy', xy

- These product terms are called the ***minterms***
- A **minterm** of *n* variables is the product of *n* literals from the different variables
- In general, *n* variables can give **2^{n} minterms**

Maxterm

- In a similar fashion, a **maxterm** of *n* variables is the sum of *n* literals from the different variables.

  Examples: x'+y', x'+y, x+y',x+y

- In general, *n* variables can give **2^n maxterms**

Minterm and Maxterm

- The minterms and maxterms of 2 variables are denoted by m0 to m3 and M0 to M3 respectively:

Minterms				Maxterms	
x	y	term	notation	term	notation
0	0	x'y'	m0	x+y	M0
0	1	x'y	m1	x+y'	M1
1	0	xy'	m2	x'+y	M2
1	1	xy	m3	x'+y'	M3

- **Each minterm is the complement of the corresponding maxterm:**
  - Example: $m2 = xy'$

    $m2' = (xy')' = x' + (y')' = x'+y = M2$

Canonical Form: Sum-of-Minterms

- What is a canonical/normal form?
  - A unique form for representing something
- Minterms are product terms
  - We can express Boolean functions using Sum-of-Minterms form

Example 2 (1)

a) Obtain the truth table:

x	y	z	F1	F2	F3
0	0	0	0	0	0
0	0	1	0	1	1
0	1	0	0	0	0
0	1	1	0	0	1
1	0	0	0	1	1
1	0	1	0	1	1
1	1	0	1	1	0
1	1	1	0	1	0

Example 2 (2)

b) Obtain Sum-of-Minterms by gathering/summing the minterms of the function (**where result is a 1**)

F1 = xyz' = Σ(m6)

F2 = x'y'z+xy'z'+xy'z+xyz'+xyz = Σ(m1,m4,m5,m6,m7)

F3 = x'y'z+x'yz+xy'z'+xy'z

= Σ(m1,m3,m4,m5)

x	y	z	F1	F2	F3
0	0	0	0	0	0
0	0	1	0	1	1
0	1	0	0	0	0
0	1	1	0	0	1
1	0	0	0	1	1
1	0	1	0	1	1
1	1	0	1	1	0
1	1	1	0	1	0

Canonical Form: Product-of-Maxterms

- Maxterms are sum terms
- For Boolean functions, the maxterms of a function are the terms for which **the result is 0**.
- Boolean functions can be expressed as Products-of-Maxterms.

Example 3

F2 = Π(M0,M2,M3) = (x+y+z)(x+y'+z)(x+y'+z')

F3 = Π(M0,M2,M6,M7)

= (x+y+z)(x+y'+z)(x'+y'+z)(x'+y'+z')

x	y	z	F1	F2	F3
0	0	0	0	0	0
0	0	1	0	1	1
0	1	0	0	0	0
0	1	1	0	0	1
1	0	0	0	1	1
1	0	1	0	1	1
1	1	0	1	1	0
1	1	1	0	1	0

Why is this so?

- Take F2 as an example.

$$F2 = \Sigma(m1,m4,m5,m6,m7)$$

- The complement function of F2 is:

$$F2' = \Sigma(m0,m2,m3)$$

$$= m0 + m2 + m3$$

(Complement functions' minterms are the opposite of their original functions, i.e. when original function = 0)

x	y	z	F2	F2'
0	0	0	0	1
0	0	1	1	0
0	1	0	0	1
0	1	1	0	1
1	0	0	1	0
1	0	1	1	0
1	1	0	1	0
1	1	1	1	0

Relation between Boolean expressions and sum-of-minterms and product-of-maxterms

- From previous slide:
  - $F2' = m0 + m2 + m3$

Therefore:

$$F2 = (m0 + m2 + m3)'$$

$$= m0' . m2' . m3' \qquad \text{DeMorgan}$$

$$= M0 . M2 . M3 \qquad mx' = Mx$$

$$= \Pi(M0,M2,M3)$$

- Every Boolean function can be expressed as either Sum-of-Minterms or Product-of-Maxterms