Chapter 2

MATRICES AND DETERMINANTS

2.1 Matrices

Definition : A **matrix A** of order $m \times n$ is a set of real numbers arranged in a rectangular array of m rows and n columns. It is written as

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

Notes :

1. a_{ij} represents the element of the matrix **A** that lies in row i and column j.
2. The matrix **A** can also be written as $\mathbf{A} = (a_{ij})_{m \times n}$.
3. If the number of rows equals the number of columns ($m = n$) then **A** is called a **square** matrix of order n.
4. In a square matrix $\mathbf{A} = (a_{ij})$, the set of elements of the form a_{ii} is called the diagonal of the matrix.

Examples :

1. $\begin{pmatrix} -1 & 4 & 0 \\ 2 & -3 & 7 \end{pmatrix}$ is a matrix of order 2×3.

   $a_{11} = -1$, $a_{12} = 4$, $a_{13} = 0$, $a_{21} = 2$, $a_{22} = -3$ and $a_{23} = 7$.

2. $\begin{pmatrix} 5 & -3 & 2 \\ 0 & 1 & 7 \\ 0 & 8 & 13 \end{pmatrix}$ is a square matrix of order 3.

   The diagonal is the set $\{a_{11}, a_{22}, a_{33}\} = \{5, 1, 13\}$

2.1.1 Special types of matrices :

1. Row vector : A row vector of order n is a matrix of order $1 \times n$, and it is written as $\begin{pmatrix} a_1 & a_2 & \dots & a_n \end{pmatrix}$

Example : $\begin{pmatrix} 2 & 7 & 0 & -1 \end{pmatrix}$ is a row vector of order 4.

2. Column vector : A column vector of order n is a matrix of order $n \times 1$, and it is written as $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$

Example : $\begin{pmatrix} 8 \\ -1 \\ 2 \end{pmatrix}$ is a column vector of order 3.

3. Null matrix : The matrix $(a_{ij})_{m \times n}$ of order $m \times n$ is called a **null matrix** if $a_{ij} = 0$ for all i and j, and it is denoted by $\mathbf{0}$.

$$\mathbf{0} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

Example : $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is a null matrix of order 3×4.

4. Upper triangular matrix : The square matrix $\mathbf{A} = (a_{ij})$ of order n is called an **upper triangular matrix** if $a_{ij} = 0$ for all $i > j$, and it is written as $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{pmatrix}$

Example : $\begin{pmatrix} 8 & 5 & -2 & 1 \\ 0 & 3 & 1 & -6 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ is an upper triangular matrix of order 4.

5. Lower triangular matrix : The square matrix $\mathbf{A} = (a_{ij})$ of order n is called a **lower triangular matrix** if $a_{ij} = 0$ for all $i < j$, and it is written as

$$\mathbf{A} = \begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & 0 & \dots & 0 \\ a_{31} & a_{32} & a_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}$$

Example : $\begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & 0 \\ 3 & -5 & 7 \end{pmatrix}$ is a lower triangular matrix of order 3.

6. Diagonal matrix : The square matrix $\mathbf{A} = (a_{ij})$ of order n is called a **diagonal matrix** if $a_{ij} = 0$ for all $i \neq j$, and it is written as

$$\mathbf{A} = \begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ 0 & 0 & a_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{pmatrix}$$

Example : $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is a diagonal matrix of order 3.

7. Identity matrix : The square matrix $I_n = (a_{ij})$ of order n is called an **identity matrix** if $a_{ij} = 0$ for all $i \neq j$ and $a_{ij} = 1$ for all $i = j$, and it is written as $I_n = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$

Example : $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is an identity matrix of order 3.

2.1.2 Elementary matrix operations :
1. Addition and subtraction of matrices :
Addition or subtraction of two matrices is defined if the two matricest have the same order.
If $\mathbf{A} = (a_{ij})_{m\times n}$ and $\mathbf{B} = (b_{ij})_{m\times n}$ any two matrices of order $m \times n$ then

1. $\mathbf{A}+\mathbf{B} = (a_{ij}+b_{ij})_{m\times n}$.

$$\mathbf{A}+\mathbf{B} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{pmatrix}$$

2. $\mathbf{A}-\mathbf{B} = (a_{ij}-b_{ij})_{m\times n}$.

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix} a_{11}-b_{11} & a_{12}-b_{12} & \dots & a_{1n}-b_{1n} \\ a_{21}-b_{21} & a_{22}-b_{22} & \dots & a_{2n}-b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}-b_{m1} & a_{m2}-b_{m2} & \dots & a_{mn}-b_{mn} \end{pmatrix}$$

Example : If $\mathbf{A} = \begin{pmatrix} 2 & -3 & 0 \\ 1 & -4 & 6 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 5 & 2 & 1 \\ -3 & 7 & -2 \end{pmatrix}$ then

$$\mathbf{A}+\mathbf{B} = \begin{pmatrix} 2+5 & -3+2 & 0+1 \\ 1+(-3) & -4+7 & 6+(-2) \end{pmatrix} = \begin{pmatrix} 7 & -1 & 1 \\ -2 & 3 & 4 \end{pmatrix}$$

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix} 2-5 & -3-2 & 0-1 \\ 1-(-3) & -4-7 & 6-(-2) \end{pmatrix} = \begin{pmatrix} -3 & -5 & -1 \\ 4 & -11 & 8 \end{pmatrix}$$

Notes:

1. The addition of matrices is commutative : if $\mathbf{A}$ and $\mathbf{B}$ any two matrices of the same order then $\mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A}$.

2. The null matrix is the identity element of addition : if $\mathbf{A}$ is any matrix then $\mathbf{A}+\mathbf{0} = \mathbf{A}$.

2. Multiplying a matrix by a scalar :
If $\mathbf{A} = (a_{ij})$ is a matrix of order $m \times n$ and $c \in \mathbb{R}$ then $c\mathbf{A} = (ca_{ij})$.

$$c\mathbf{A} = \begin{pmatrix} ca_{11} & ca_{12} & \dots & ca_{1n} \\ ca_{21} & ca_{22} & \dots & ca_{2n} \\ \vdots & \vdots & & \vdots \\ ca_{m1} & ca_{m2} & \dots & ca_{mn} \end{pmatrix}$$

Example : If $\mathbf{A} = \begin{pmatrix} 3 & -1 & 4 \\ 2 & -2 & 0 \end{pmatrix}$ then $3\mathbf{A} = \begin{pmatrix} 9 & -3 & 12 \\ 6 & -6 & 0 \end{pmatrix}$

3. Multiplying a row vector by a column vector :
If $\mathbf{A} = \begin{pmatrix} a_1 & a_2 & \dots & a_n \end{pmatrix}$ is a row vector of order n and
$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ is a column vector of order n then

$$\mathbf{AB} = \begin{pmatrix} a_1 & a_2 & \dots & a_n \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1b_1 + a_2b_2 + \dots + a_nb_n$$

Example : If $\mathbf{A} = \begin{pmatrix} -1 & 2 & 0 & 5 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 \\ -2 \\ 1 \\ -1 \end{pmatrix}$ then

$$\mathbf{AB} = \begin{pmatrix} -1 & 2 & 0 & 5 \end{pmatrix} \begin{pmatrix} 4 \\ -2 \\ 1 \\ -1 \end{pmatrix} = -4 - 4 + 0 - 5 = -13$$

4. Multiplication of matrices :

1. If $\mathbf{A}$ and $\mathbf{B}$ any two matrices then $\mathbf{AB}$ is defined if the number of columns of $\mathbf{A}$ equals the number of rows of $\mathbf{B}$.
2. If $\mathbf{A} = (a_{ij})_{m\times n}$ and $\mathbf{B} = (b_{ij})_{n\times p}$ then $\mathbf{AB} = (c_{ij})_{m\times p}$.

   c_{ij} is calculated by multiplying the i^{th} row of $\mathbf{A}$ by the j^{th} column of $\mathbf{B}$.

   $$c_{ij} = \begin{pmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{pmatrix} \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}$$

Example 1 :

1. $\begin{pmatrix} -1 & 3 & 4 \\ -2 & 0 & 5 \end{pmatrix}_{2\times 3} \begin{pmatrix} 1 & 3 \\ -1 & -2 \\ 4 & 0 \end{pmatrix}_{3\times 2}$

   $$= \begin{pmatrix} (-1\times 1)+(3\times -1)+(4\times 4) & (-1\times 3)+(3\times -2)+(4\times 0) \\ (-2\times 1)+(0\times -1)+(5\times 4) & (-2\times 3)+(0\times -2)+(5\times 0) \end{pmatrix}_{2\times 2}$$

   $$= \begin{pmatrix} -1-3+16 & -3-6+0 \\ -2+0+20 & -6+0+0 \end{pmatrix}_{2\times 2} = \begin{pmatrix} 12 & -9 \\ 18 & -6 \end{pmatrix}_{2\times 2}$$

2. $\begin{pmatrix} 3 & -1 \\ -2 & 5 \end{pmatrix}_{2\times 2} \begin{pmatrix} 0 & -3 & 4 \\ -2 & 0 & 1 \end{pmatrix}_{2\times 3}$

$$= \begin{pmatrix} (3\times 0)+(-1\times -2) & (3\times -3)+(-1\times 0) & (3\times 4)+(-1\times 1) \\ (-2\times 0)+(5\times -2) & (-2\times -3)+(5\times 0) & (-2\times 4)+(5\times 1) \end{pmatrix}_{2\times 3}$$

$$\begin{pmatrix} 0+2 & -9+0 & 12-1 \\ 0-10 & 6+0 & -8+5 \end{pmatrix}_{2\times 3} = \begin{pmatrix} 2 & -9 & 11 \\ -10 & 6 & -3 \end{pmatrix}_{2\times 3}$$

Example 2: Let $\mathbf{A} = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & 6 \\ 2 & 0 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ 2 & 3 \\ 0 & 4 \end{pmatrix}$

Compute (if possible) : $2\mathbf{BA}$ and $\mathbf{AB}$
Solution : $\mathbf{A}$ is of order 3×3 and $\mathbf{B}$ is of order 3×2

$2\mathbf{BA}$ is not possible because the number of columns of $\mathbf{B}$ is not equal to the number of rows of $\mathbf{A}$.

$$\mathbf{AB} = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & 6 \\ 2 & 0 & 1 \end{pmatrix}_{3\times 3} \begin{pmatrix} 1 & -1 \\ 2 & 3 \\ 0 & 4 \end{pmatrix}_{3\times 2} = \begin{pmatrix} (1-4+0) & (-1-6+12) \\ (4+10+0) & (-4+15+24) \\ (2+0+0) & (-2+0+4) \end{pmatrix}_{3\times 2}$$

$$\mathbf{AB} = \begin{pmatrix} -3 & 5 \\ 14 & 35 \\ 2 & 2 \end{pmatrix}_{3\times 2}$$

Notes :

1. The identity matrix is the identity element in matrix multiplication :

   If A is a matrix of order $m \times n$ and $\mathbf{I}_n$ is the identity matrix of order n then $\mathbf{A}\ \mathbf{I}_n = \mathbf{I}_n\mathbf{A} = \mathbf{A}$.

2. Matrix multiplication is not commutative :

   If $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 3 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

   $$\mathbf{AB} = \begin{pmatrix} -1 & 0 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 8 & 5 \end{pmatrix}$$

   $$\mathbf{BA} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$$

   $\mathbf{AB} \neq \mathbf{BA}$.

3. $\mathbf{AB} = \mathbf{0}$ does not imply that $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$.

   For example, $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq \mathbf{0}$ and $\mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \neq \mathbf{0}$

   But $\mathbf{AB} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0}$

2.1.3 Transpose of a matrix :
If $\mathbf{A} = (a_{ij})_{m\times n}$ then the transpose of $\mathbf{A}$ is $\mathbf{A}^t = (a_{ji})_{n\times m}$.

Example : If $\mathbf{A} = \begin{pmatrix} 4 & 0 & -2 \\ -3 & 5 & 1 \end{pmatrix}$ then $\mathbf{A}^t = \begin{pmatrix} 4 & -3 \\ 0 & 5 \\ -2 & 1 \end{pmatrix}$

Note : The transpose of a lower triangular matrix is an upper triangular matrix , and the transpose of an upper triangular matrix is a lower triangular matrix .

Theorem :
If $\mathbf{A}$ and $\mathbf{B}$ any two matrices and $\lambda \in \mathbb{R}$ then

1. $\left(\mathbf{A}^t\right)^t = \mathbf{A}$.
2. $(\mathbf{A}+\mathbf{B})^t = \mathbf{A}^t + \mathbf{B}^t$.
3. $(\lambda \mathbf{A})^t = \lambda\ \mathbf{A}^t$.
4. $(\mathbf{AB})^t = \mathbf{B}^t\mathbf{A}^t$.

2.1.4 Properties of operations on matrices :

1. If $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ any three matrices of the same order then

   $\mathbf{A}+\mathbf{B}+\mathbf{C} = (\mathbf{A}+\mathbf{B})+\mathbf{C} = \mathbf{A}+(\mathbf{B}+\mathbf{C}) = (\mathbf{A}+\mathbf{C})+\mathbf{B}$

2. If $\mathbf{A}$, $\mathbf{B}$ any two matrices of order $m \times n$ and $\mathbf{C}$ a matrix of order $n \times p$ then $(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC}+\mathbf{BC}$

3. If $\mathbf{A}$, $\mathbf{B}$ any two matrices of order $m \times n$ and $\mathbf{C}$ a matrix of order $p \times m$ then $\mathbf{C}(\mathbf{A}+\mathbf{B}) = \mathbf{CA}+\mathbf{CB}$

4. If $\mathbf{A}$ a matrix of order $m \times n$, $\mathbf{B}$ a matrix of order $n \times p$ and $\mathbf{C}$ a matrix of order $p \times q$ then $\mathbf{ABC} = (\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$

2.2 Determinants

If $\mathbf{A}$ is a square matrix then the determinant of $\mathbf{A}$ is denoted by $\det(\mathbf{A})$ or $|\mathbf{A}|$.

2.2.1 The determinant of a 2×2 matrix :

If $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ then $|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

Example :

If $\mathbf{A} = \begin{pmatrix} 5 & -1 \\ 2 & 3 \end{pmatrix}$ then $|\mathbf{A}| = (5 \times 3) - (2 \times -1) = 15 + 2 = 17$

2.2.2 The determinant of a 3×3 matrix :

Let $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be a square matrix of order 3.

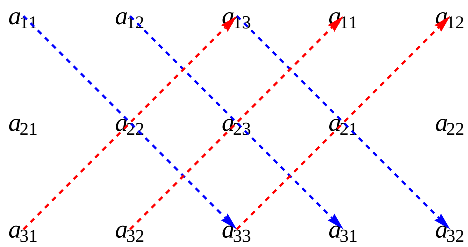
1). The determinant of $\mathbf{A}$ is defined as :

$$|\mathbf{A}| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$|\mathbf{A}| = a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) - a_{12}\left(a_{21}a_{33} - a_{23}a_{31}\right) + a_{13}\left(a_{21}a_{32} - a_{22}a_{31}\right)$$

2). Sarrus Method for calculating the deteminant of a 3×3 matrix :

Write the first two columns to the right of the matrix to get a 3×5 matrix



$$|\mathbf{A}| = (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) - (a_{31}a_{22}a_{13} + a_{32}a_{23}a_{11} + a_{33}a_{21}a_{12})$$

Example : If $\mathbf{A} = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 2 & 4 \\ -5 & 0 & 1 \end{pmatrix}$

1) Using the definition of the determinant of a 3×3 matrix

$$|\mathbf{A}| = 2 \begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} - 3 \begin{vmatrix} 1 & 4 \\ -5 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 1 & 2 \\ -5 & 0 \end{vmatrix}$$

$$|\mathbf{A}| = 2\,(2 \times 1 - 4 \times 0) - 3\,(1 \times 1 - 4 \times -5) - 1\,(1 \times 0 - 2 \times -5)$$

$$|\mathbf{A}| = 2(2 - 0) - 3(1 + 20) - 1(0 + 10) = 4 - 63 - 10 = -69$$

2) Using Sarrus Method

$$\begin{matrix} 2 & 3 & -1 & 2 & 3 \\ 1 & 2 & 4 & 1 & 2 \\ -5 & 0 & 1 & -5 & 0 \end{matrix}$$

$|\mathbf{A}| = (2 \times 2 \times 1 + 3 \times 4 \times -5 + (-1) \times 1 \times 0) - (-5 \times 2 \times -1 + 0 \times 4 \times 2 + 1 \times 1 \times 3)$

$|\mathbf{A}| = (4 - 60 + 0) - (10 + 0 + 3) = -56 - 13 = -69$.

2.2.3 The determinant of a 4×4 matrix :

Let $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$ be a 4×4 matrix , then

$|\mathbf{A}| = a_{11} |\mathbf{A}_1| - a_{12} |\mathbf{A}_2| + a_{13} |\mathbf{A}_3| - a_{14} |\mathbf{A}_4|$

where

$\mathbf{A}_1 = \begin{pmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{pmatrix}$, $\mathbf{A}_2 = \begin{pmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{pmatrix}$

$\mathbf{A}_3 = \begin{pmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{pmatrix}$, $\mathbf{A}_4 = \begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}$

Example : If $\mathbf{A} = \begin{pmatrix} 3 & 1 & -2 & 1 \\ 0 & 4 & -1 & 5 \\ 2 & 1 & -3 & 0 \\ 1 & -2 & -1 & 3 \end{pmatrix}$

$|\mathbf{A}| = (3) |\mathbf{A}_1| - (1) |\mathbf{A}_2| + (-2) |\mathbf{A}_3| - (1) |\mathbf{A}_4|$

where

$\mathbf{A}_1 = \begin{pmatrix} 4 & -1 & 5 \\ 1 & -3 & 0 \\ -2 & -1 & 3 \end{pmatrix}$, $\mathbf{A}_2 = \begin{pmatrix} 0 & -1 & 5 \\ 2 & -3 & 0 \\ 1 & -1 & 3 \end{pmatrix}$

$\mathbf{A}_3 = \begin{pmatrix} 0 & 4 & 5 \\ 2 & 1 & 0 \\ 1 & -2 & 3 \end{pmatrix}$, $\mathbf{A}_4 = \begin{pmatrix} 0 & 4 & -1 \\ 2 & 1 & -3 \\ 1 & -2 & -1 \end{pmatrix}$

- To calculate $|\mathbf{A}_1|$

$$\begin{matrix} 4 & -1 & 5 & 4 & -1 \\ 1 & -3 & 0 & 1 & -3 \\ -2 & -1 & 3 & -2 & -1 \end{matrix}$$

$|\mathbf{A}_1| = (-36 + 0 - 5) - (30 + 0 - 3) = -36 - 5 - 30 + 3 = -68$

- To calculate $|\mathbf{A}_2|$

$$\begin{matrix} 0 & -1 & 5 & 0 & -1 \\ 2 & -3 & 0 & 2 & -3 \\ 1 & -1 & 3 & 1 & -1 \end{matrix}$$

$|\mathbf{A}_2| = (0 + 0 - 10) - (-15 + 0 - 6) = -10 + 21 = 11$

- To calculate $|\mathbf{A}_3|$

$$\begin{matrix} 0 & 4 & 5 & 0 & 4 \\ 2 & 1 & 0 & 2 & 1 \\ 1 & -2 & 3 & 1 & -2 \end{matrix}$$

$|\mathbf{A}_3| = (0 + 0 - 20) - (5 + 0 + 24) = -20 - 29 = -49$

- To calculate $|\mathbf{A}_4|$

$$\begin{matrix} 0 & 4 & -1 & 0 & 4 \\ 2 & 1 & -3 & 2 & 1 \\ 1 & -2 & -1 & 1 & -2 \end{matrix}$$

$|\mathbf{A}_4| = (0 - 12 + 4) - (-1 + 0 - 8) = -8 + 9 = 1$

$|\mathbf{A}| = (3)\,|\mathbf{A}_1| - (1)\,|\mathbf{A}_2| + (-2)\,|\mathbf{A}_3| - (1)\,|\mathbf{A}_4|$
$|\mathbf{A}| = (3 \times -68) - (1 \times 11) + (-2 \times -49) - (1 \times 1)$
$|\mathbf{A}| = -204 - 11 + 98 - 1 = -216 + 98 = -118$.

2.2.4 Properties of determinants :

1. If $\mathbf{A}$ is a square matrix that contains a zero row (or a zero column) then $|\mathbf{A}| = 0$.

   Examples :

   $$\begin{vmatrix} 3 & -1 & 1 \\ 0 & 0 & 0 \\ 2 & -2 & 4 \end{vmatrix} = 0 \text{ (the second row } R_2 \text{ is a zero row)}$$

   $$\begin{vmatrix} 3 & -1 & 0 \\ -1 & 5 & 0 \\ 2 & -2 & 0 \end{vmatrix} = 0 \text{ (the third column } C_3 \text{ is a zero column)}$$

2. If $\mathbf{A}$ is a square matrix that contains two equal rows (or two equal columns) then $|\mathbf{A}| = 0$.

   Examples :

   $$\begin{vmatrix} 4 & -5 & 4 \\ 0 & 2 & 0 \\ -3 & 1 & -3 \end{vmatrix} = 0 \text{ (because } C_1 = C_3\text{).}$$

   $$\begin{vmatrix} 1 & -1 & 2 \\ 3 & 2 & -2 \\ 3 & 2 & -2 \end{vmatrix} = 0 \text{ (because } R_2 = R_3\text{)}$$

3. If $\mathbf{A}$ is a square matrix that contains a row which is a multiple of another row (or a column which is a multiple of another column) then $|\mathbf{A}| = 0$.

   Examples :

   $$\begin{vmatrix} 2 & 1 & -3 \\ 0 & 5 & 1 \\ 4 & 2 & -6 \end{vmatrix} = 0 \text{ (because } R_3 = 2R_1\text{)}.$$

   $$\begin{vmatrix} -2 & 1 & 3 \\ 0 & 0 & 1 \\ 2 & -1 & 1 \end{vmatrix} = 0 \text{ (because } C_1 = -2C_2\text{)}.$$

4. If $\mathbf{A}$ is a diagonal matrix or an upper triangular matrix or a lower triangular matrix the $|\mathbf{A}|$ is the the product of the elements of the main diagonal.

   Examples :

   $$\begin{vmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{vmatrix} = 2 \times -1 \times 5 = -10 \text{ (Diagonal matrix)}$$

   $$\begin{vmatrix} 1 & 3 & -7 \\ 0 & 5 & 4 \\ 0 & 0 & -3 \end{vmatrix} = 1 \times 5 \times -3 = 15 \text{ (Upper triangular matrix)}$$

   $$\begin{vmatrix} 3 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 7 & 2 \end{vmatrix} = 3 \times 1 \times 2 = 6 \text{ (Lower triangular matrix)}$$

5. The determinant of the null matrix is 0 and the determinant of the identity matrix is 1.

6. If $\mathbf{A}$ is a square matrix and $\mathbf{B}$ is the matrix formed by multiplying one of the rows (or columns) of $\mathbf{A}$ by a non-zero constant λ then $|\mathbf{B}| = \lambda|\mathbf{A}|$.

7. If $\mathbf{A}$ is a square matrix and $\mathbf{B}$ is the matrix formed by interchanging two rows (or two columns) of $\mathbf{A}$ then $|\mathbf{B}| = -|\mathbf{A}|$.

   Example :

   $$\begin{vmatrix} 3 & 0 & 4 \\ 6 & -1 & 2 \\ 0 & 0 & 5 \end{vmatrix} \xrightarrow{R_1 \longleftrightarrow R_2} -1 \times \begin{vmatrix} 6 & -1 & 2 \\ 3 & 0 & 4 \\ 0 & 0 & 5 \end{vmatrix}$$

   $$\xrightarrow{C_1 \longleftrightarrow C_2} -1 \times -1 \times \begin{vmatrix} -1 & 6 & 2 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{vmatrix} = -1 \times -1 \times -1 \times 3 \times 5 = -15$$

8. If **A** is a square matrix and **B** is the matrix formed by mutliplying a row by a non-zero constant and adding the result to another row (or mutliplying a column by a non-zero constant and adding the result to another column) then $|\mathbf{B}| = |\mathbf{A}|$.

   Example :

$$\begin{vmatrix} 5 & 2 & 3 \\ 15 & 8 & 1 \\ 10 & 6 & 2 \end{vmatrix} \xrightarrow{-3R_1+R_2} \begin{vmatrix} 5 & 2 & 3 \\ 0 & 2 & -8 \\ 10 & 6 & 2 \end{vmatrix} \xrightarrow{-2R_1+R_3} \begin{vmatrix} 5 & 2 & 3 \\ 0 & 2 & -8 \\ 0 & 2 & -4 \end{vmatrix}$$

$$\xrightarrow{-R_2+R_3} \begin{vmatrix} 5 & 2 & 3 \\ 0 & 2 & -8 \\ 0 & 0 & 4 \end{vmatrix} = 5 \times 2 \times 4 = 40$$

Examples : Use properties of determinants to calculate the derminants of the following matrices

1. $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 0 \\ 1 & 2 & 3 & 5 \\ 3 & 0 & 0 & 0 \end{vmatrix} = 0$ (because $C_3 = \dfrac{3}{2}C_2$)

2. $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & -4 \\ 1 & 2 & 3 & 5 \\ 3 & 0 & 1 & 0 \end{vmatrix} \xrightarrow{-R_1+R_2} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & -8 \\ 1 & 2 & 3 & 5 \\ 3 & 0 & 1 & 0 \end{vmatrix}$

   $\xrightarrow{-R_1+R_3} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & -8 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & 1 & 0 \end{vmatrix} = 0$ (because $R_2 = -8R_3$)

3. $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 4 & 3 & 2 & 1 \\ 8 & 7 & 6 & 5 \end{vmatrix} \xrightarrow{-R_1+R_2} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 3 & 2 & 1 \\ 8 & 7 & 6 & 5 \end{vmatrix}$

   $\xrightarrow{-R_3+R_4} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 3 & 2 & 1 \\ 4 & 4 & 4 & 4 \end{vmatrix} = 0$ (because $R_2 = R_4$)