Chapter 6

PARTIAL DERIVATIVES

6.1 Functions of several variables

6.1.1 Functions of two variables :

Definition: A function of two variables is a rule that assigns an ordered pair (x, y) (in the domain of the function) to a real number w.

$$\begin{aligned} f : \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x, y) &\longrightarrow w \end{aligned}$$

Example :

$f(x, y) = \dfrac{y}{x^2 + y^2}$ is a function of two variables x and y

$f(3, 1) = \dfrac{1}{3^2 + 1^2} = \dfrac{1}{10}.$

Note that $f(x, y)$ takes $(3, 1) \in \mathbb{R}^2$ to $\dfrac{1}{10} \in \mathbb{R}$

6.1.2 Functions of three variables :

Definition: A function of three variables is a rule that assigns an ordered triple (x, y, z) (in the domain of the function) to a real number w .

$$\begin{aligned} f : \mathbb{R}^3 &\longrightarrow \mathbb{R} \\ (x, y, z) &\longrightarrow w \end{aligned}$$

Example :

$f(x, y, z) = \dfrac{z}{x + y^2 + 3}$ is a function of three variables x , y and z

$f(1, -2, 4) = \dfrac{4}{1 + (-2)^2 + 3} = \dfrac{4}{8} = \dfrac{1}{2}.$

Note that $f(x, y, z)$ takes $(1, -2, 4) \in \mathbb{R}^3$ to $\dfrac{1}{2} \in \mathbb{R}$

6.2 Partial derivatives

6.2.1 Partial derivatives of a function of two variables :
If $w = f(x, y)$ is a function of two variables, then :

1. The partial derivative of f with respect to x is denoted by $\frac{\partial f}{\partial x}$, $\frac{\partial w}{\partial x}$, f_x or w_x , and it is calculated by applying the rules of differentiation to x and regarding y as a constant .

2. The partial derivative of f with respect to y is denoted by $\frac{\partial f}{\partial y}$, $\frac{\partial w}{\partial y}$, f_y or w_y , and it is calculated by applying the rules of differentiation to y and regarding x as a constant .

Example 1: Calculate f_x and f_y of the functiuon $f(x, y) = x^2y^3 + xy\ln(x + y)$
Solution:

1. $f_x = \frac{\partial}{\partial x}\left(x^2y^3 + xy\ln(x + y)\right)$

   $$f_x = (2x)y^3 + \left[(1)y\ln(x + y) + xy\frac{1}{x + y}\right] = 2xy^3 + y\ln(x + y) + \frac{xy}{x + y}$$

2. $f_y = \frac{\partial}{\partial y}\left(x^2y^3 + xy\ln(x + y)\right)$

   $$f_y = x^2(3y^2) + \left[x(1)\ln(x + y) + xy\frac{1}{x + y}\right] = 3x^2y^2 + x\ln(x + y) + \frac{xy}{x + y}$$

Example 2: Calculate f_x and f_y of the functiuon $f(x, y) = \frac{x + y^2}{x + y}$
Solution:

1. $$f_x = \frac{\partial f}{\partial x} = \frac{(1 + 0)(x + y) - (x + y^2)(1 + 0)}{(x + y)^2} = \frac{x + y - (x + y^2)}{(x + y^2)}$$

   $$f_x = \frac{x + y - x - y^2}{(x + y)^2} = \frac{y - y^2}{(x + y)^2}$$

2. $$f_y = \frac{\partial f}{\partial y} = \frac{(0 + 2y)(x + y) - (x + y^2)(0 + 1)}{(x + y)^2} = \frac{2y(x + y) - (x + y^2)}{(x + y)^2}$$

   $$f_y = \frac{2xy + 2y^2 - x - y^2}{(x + y)^2} = \frac{2xy - x + y^2}{(x + y)^2}$$

6.2.2 Partial derivatives of a function of three variables :
If $w = f(x, y, z)$ is a function of three variables, then :

1. The partial derivative of f with respect to x is denoted by $\frac{\partial f}{\partial x}$, $\frac{\partial w}{\partial x}$, f_x or w_x , and it is calculated by applying the rules of differentiation to x and regarding y and z as constants .

2. The partial derivative of f with respect to y is denoted by $\frac{\partial f}{\partial y}$, $\frac{\partial w}{\partial y}$, f_y or w_y , and it is calculated by applying the rules of differentiation to y and regarding x and z as constants .

3. The partial derivative of f with respect to z is denoted by $\frac{\partial f}{\partial z}$, $\frac{\partial w}{\partial z}$, f_z or w_z , and it is calculated by applying the rules of differentiation to z and regarding x and y as constants .

Example : If $f(x, y, z) = 2z^3x - 4(x^2 + y^2)z$, then calculate f_x , f_y and f_z at $(0, 1, 2)$.
Solution :

1. $f_x = \frac{\partial}{\partial x}\left(2z^3x - 4(x^2 + y^2)z\right) = 2z^3 - 4(2x)z = 2z^3 - 8xz$

   $f_x(0, 1, 2) = 2\ (2^3) - 8(0)(2) = 16$

2. $f_y = \frac{\partial}{\partial y}\left(2z^3x - 4(x^2 + y^2)z\right) = 0 - 4(0 + 2y)z = -8yz$

   $f_y(0, 1, 2) = -8(1)(2) = -16$

3. $f_z = \frac{\partial}{\partial z}\left(2z^3x - 4(x^2 + y^2)z\right) = 6z^2x - 4(x^2 + y^2)$

   $f_z(0, 1, 2) = 6(2^2)(0) - 4(0^2 + 1^2) = -4$

6.2.3 Second partial derivatives :
If $w = f(x, y)$ is a function of two variables , then :

1. $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(f_x) = f_{xx}$.

2. $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(f_y) = f_{yy}$.

3. $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(f_y) = f_{yx}$.

4. $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(f_x) = f_{xy}$.

Note : Second partial derivatives of a function of three variables are defined in a same manner.

Theorem : Let $f(x, y)$ be a function of two variables. If f , f_x , f_y , f_{xy} and f_{yx} are continuous, then $f_{xy} = f_{yx}$.

Note : If $f(x, y, z)$ is a function of three variables and f has continuous second partial derivatives, then $f_{xy} = f_{yx}$, $f_{xz} = f_{zx}$ and $f_{yz} = f_{zy}$.

Example 1: Let $f(x, y) = x^3y + xy^2 \sin(x + y)$, calculate $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$

Solution :

$$f_x = 3x^2y + y^2 \sin(x + y) + xy^2 \cos(x + y)$$

$$f_y = x^3 + 2xy \sin(x + y) + xy^2 \cos(x + y)$$

$$f_{xy} = 3x^2 + 2y \sin(x + y) + y^2 \cos(x + y) + 2xy \cos(x + y) - xy^2 \sin(x + y)$$

$$f_{yx} = 3x^2 + 2y \sin(x + y) + 2xy \cos(x + y) + y^2 \cos(x + y) - xy^2 \sin(x + y)$$

Note : $f_{xy} = f_{yx}$ according to the theorem .

Example 2: Let $f(x, y, z) = x^3y^2z + xy \sin(y + z)$, calculate $\dfrac{\partial^2 f}{\partial y \partial x}$ and $\dfrac{\partial^2 f}{\partial x \partial z}$

Solution :

$$f_x = 3x^2y^2z + y \sin(y + z)$$

$$f_z = x^3y^2 + xy \cos(y + z)$$

$$\frac{\partial^2 f}{\partial y \partial x} = f_{xy} = 6x^2yz + \sin(y + z) + y \cos(y + z)$$

$$\frac{\partial^2 f}{\partial x \partial z} = f_{zx} = 3x^2y^2 + y \cos(y + z)$$

Example 3: Let $f(x, y, z) = 2z^3 - 3(x^2 + y^2)z$, Show that $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2} = 0$

Solution :

$$f_x = 0 - 3z(2x) = -6xz$$

$$f_y = 0 - 3z(2y) = -6yz$$

$$f_z = 6z^2 - 3(x^2 + y^2)$$

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} = -6z$$

$$\frac{\partial^2 f}{\partial y^2} = f_{yy} = -6z$$

$$\frac{\partial^2 f}{\partial z^2} = f_{zz} = 12z$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = -6z - 6z + 12z = 0$$

6.3 Chain Rules

Theorem (Chain Rules):

1. If $w = f(x, y)$ and $x = g(t)$, $y = h(t)$, such that f , g and h are differentiable then

$$\frac{df}{dt} = \frac{dw}{dt} = \frac{\partial w}{\partial x}\ \frac{dx}{dt} + \frac{\partial w}{\partial y}\ \frac{dy}{dt}$$

2. If $w = f(x, y)$ and $x = g(t, s)$, $y = h(t, s)$, such that f , g and h are differentiable then

$$\frac{\partial f}{\partial t} = \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\ \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\ \frac{\partial y}{\partial t}$$

$$\frac{\partial f}{\partial s} = \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\ \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\ \frac{\partial y}{\partial s}$$

3. If $w = f(x, y, z)$ and $x = g(t, s)$, $y = h(t, s)$, $z = k(t, s)$ such that f , g , h and k are differentiable then

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\ \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\ \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\ \frac{\partial z}{\partial t}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\ \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\ \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\ \frac{\partial z}{\partial s}$$

Example 1 : Let $f(x, y) = xy + y^2$, $x = s^2t$, and $y = s + t$, calculate $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$.

Solution :

1. $$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\ \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\ \frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial x} = y\ ,\ \frac{\partial x}{\partial s} = 2st$$

$$\frac{\partial f}{\partial y} = x + 2y\ ,\ \frac{\partial y}{\partial s} = 1$$

$$\frac{\partial f}{\partial s} = y\ (2st) + (x + 2y)(1) = (s + t)2st + \left[s^2t + 2(s + t)\right]$$

$$= 2s^2t + 2st^2 + s^2t + 2s + 2t = 3s^2t + 2st^2 + 2s + 2t$$

2. $$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\ \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\ \frac{\partial y}{\partial t}$$

$$\frac{\partial f}{\partial x} = y\ ,\ \frac{\partial x}{\partial t} = s^2$$

$$\frac{\partial f}{\partial y} = x + 2y\ ,\ \frac{\partial y}{\partial t} = 1$$

$$\frac{\partial f}{\partial t} = ys^2 + (x+2y)(1) = (s+t)s^2 + s^2t + 2(s+t)$$
$$= s^3 + s^2t + s^2t + 2s + 2t = s^3 + 2s^2t + 2s + 2t$$

Example 2 : Let $f(x, y, z) = x + \sin(xy) + \cos(xz)$, $x = ts$, $y = s + t$ and $z = \frac{s}{t}$, calculate $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$.
Solution :

1. $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\ \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\ \frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\ \frac{\partial z}{\partial s}$

   $\frac{\partial f}{\partial x} = 1 + y\cos(xy) - z\sin(xz)$, $\frac{\partial x}{\partial s} = t$

   $\frac{\partial f}{\partial y} = x\cos(xy)$, $\frac{\partial y}{\partial s} = 1$

   $\frac{\partial f}{\partial z} = -x\sin(xz)$, $\frac{\partial z}{\partial s} = \frac{1}{t}$

   $\frac{\partial f}{\partial s} = t\ \ [1 + y\cos(xy) - z\sin(xz)] + x\cos(xy) + \left(\frac{1}{t}\right)(-x\sin(xz))$

   $\frac{\partial f}{\partial s} = t + ty\cos(xy) - tz\sin(xz) + x\cos(xy) - \frac{x\sin(xz)}{t}$

2. $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\ \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\ \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\ \frac{\partial z}{\partial t}$

   $\frac{\partial f}{\partial x} = 1 + y\cos(xy) - z\sin(xz)$, $\frac{\partial x}{\partial t} = s$

   $\frac{\partial f}{\partial y} = x\cos(xy)$, $\frac{\partial y}{\partial t} = 1$

   $\frac{\partial f}{\partial z} = -x\sin(xz)$, $\frac{\partial z}{\partial t} = \frac{-s}{t^2}$

   $\frac{\partial f}{\partial t} = s\ \ [1 + y\cos(xy) - z\sin(xz)] + x\cos(xy) + \left(\frac{-s}{t^2}\right)(-x\sin(xz))$

   $\frac{\partial f}{\partial t} = s + sy\cos(xy) - sz\sin(xz) + x\cos(xy) + \frac{sx\sin(xz)}{t^2}$

6.4 Implicit differentiation

1. Suppose that the equation $F(x, y) = 0$ defines y implicitly as a function of x say $y = f(x)$, then
$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$
2. Suppose that the equation $F(x, y, z) = 0$ implicitly defines a function $z = f(x, y)$, where f is differentiable , then
$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \text{ and } \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Example 1 : Let $y^2 - xy + 3x^2 = 0$, find $\frac{dy}{dx}$.
Solution 1: Let $F(x, y) = x^2 - xy + 3x^2$ then $F(x, y) = 0$

$F_x = -y + 6x$ and $F_y = 2y - x$.

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{(-y + 6x)}{2y - x} = \frac{y - 6x}{2y - x} .$$

Solution 2 : $y^2 - xy + 3x^2 = 0$

Differentiate both sides implicitly

$2yy' - (y + xy') + 6x = 0 \quad \Rightarrow \quad 2yy' - y - xy' + 6x = 0$

$\Rightarrow \quad 2yy' - xy' = y - 6x \quad \Rightarrow \quad (2y - x)y' = y - 6x$

$$\Rightarrow \quad \frac{dy}{dx} = y' = \frac{y - 6x}{2y - x}$$

Example 2 : Let $F(x, y, z) = x^2y + z^2 + \sin(xyz) = 0$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
Solution :

$F_x = 2xy + yz\cos(xyz)$

$F_y = x^2 + xz\cos(xyz)$

$F_z = 2z + xy\cos(xyz)$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2xy + yz\cos(xyz)}{2z + xy\cos(xyz)}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{x^2 + xz\cos(xyz)}{2z + xy\cos(xyz)}$$