5

SHAFTS

The objective of this chapter is to introduce the concepts of shaft design. An overall shaft design procedure is presented, including consideration of bearing and component mounting and shaft dynamics. At the end of the chapter the reader should be able to scheme out a general shaft arrangement, determine deflections and critical speeds, and specify shaft dimensions for strength and fluctuating load integrity.

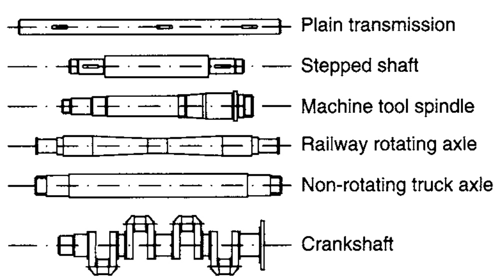

Figure 5.1 Typical shaft arrangements (adapted from Reshetov, 1978).

LEARNING OBJECTIVES

At the end of this chapter you should be able to:

- select appropriate methods for mounting and locating components on shafts;
- produce a scheme for a shaft design to locate and mount standard machine elements;
- determine the deflection of a shaft;
- calculate the first two critical frequencies of a shaft;
- determine the minimum diameter of a shaft to avoid fatigue failure.

5.1 Introduction

The term 'shaft' usually refers to a component of circular cross-section that rotates and transmits power from a driving device, such as a motor or engine, through a machine. Shafts can carry gears, pulleys and sprockets to transmit rotary motion and power via mating gears, belts and chains. Alternatively, a shaft may simply connect to another via a coupling. A shaft can be stationary and support a rotating member, such as the short shafts that support the non-driven wheels of automobiles often referred to as spindles. Some common shaft arrangements are illustrated in Figure 5.1.

Shaft design considerations include:

1. size and spacing of components (as on a general assembly drawing), tolerances
2. material selection, material treatments
3. deflection and rigidity
   - bending deflection
   - torsional deflection
   - slope at bearings
   - shear deflection
4. stress and strength
   - static strength
   - fatigue
   - reliability
5. frequency response
6. manufacturing constraints.

Shafts typically consist of a series of stepped diameters accommodating bearing mounts and providing shoulders for locating devices, such as gears, sprockets and pulleys to butt up against and keys are often used to prevent rotation, relative to the shaft, of these 'added' components. A typical

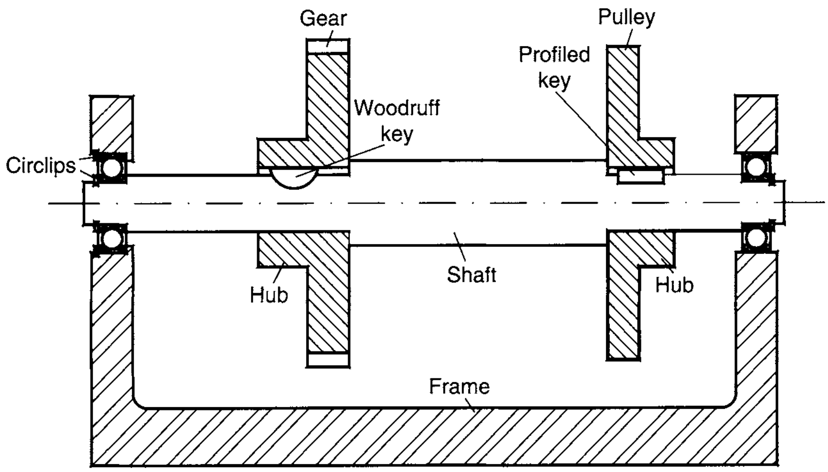

Figure 5.2 Typical shaft arrangement incorporating constant diameter sections and shoulders for locating added components.

arrangement illustrating the use of constant diameter sections and shoulders is illustrated in Figure 5.2 for a transmission shaft supporting a gear and pulley wheel.

Shafts must be designed so that deflections are within acceptable levels. Too much deflection can for example degrade gear performance, and cause noise and vibration. The maximum allowable deflection of a shaft is usually determined by limitations set on the critical speed, minimum deflections required for gear operation and bearing requirements. In general, deflections should not cause mating gear teeth to separate more than about 0.13 mm and the slope of the gear axes should not exceed about 0.03°. The deflection of the journal section of a shaft across a plain bearing should be small in comparison with the oil film thickness. Torsional and lateral deflection both contribute to lower critical speed. In addition, shaft angular deflection at rolling element bearings should not exceed 0.04°, with the exception being self-aligning rolling element bearings.

Shafts can be subjected to a variety of combinations of axial, bending and torsional loads (see Figure 5.3) which may fluctuate or vary with time. Typically a rotating shaft transmitting power is subjected to a constant torque together with a completely reversed bending load, producing a mean torsional stress and an alternating bending stress, respectively.

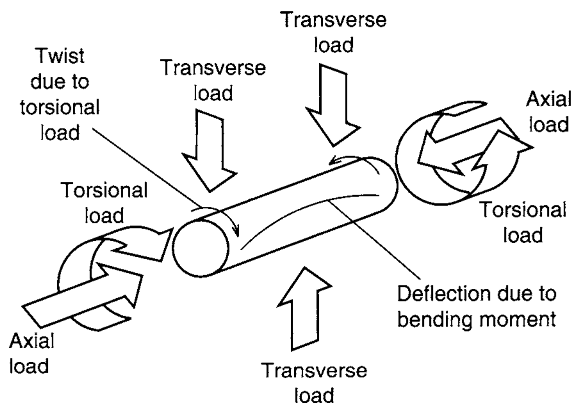

Figure 5.3 Typical shaft loading and deflection (adapted from Beswarick, 1994a).

Shafts should be designed to avoid operation at, or near, critical speeds. This is usually achieved by the provision of sufficient lateral rigidity so that the lowest critical speed is significantly above the range of operation. If torsional fluctuations are present (e.g. engine crankshafts, cam-shafts, compressors) the torsional natural frequencies of the shaft must be significantly different to the torsional input frequency. This can be achieved by providing sufficient torsional stiffness so that the shaft's lowest natural frequency is much higher than the highest torsional input frequency.

Rotating shafts must generally be supported by bearings. For simplicity of manufacture it is

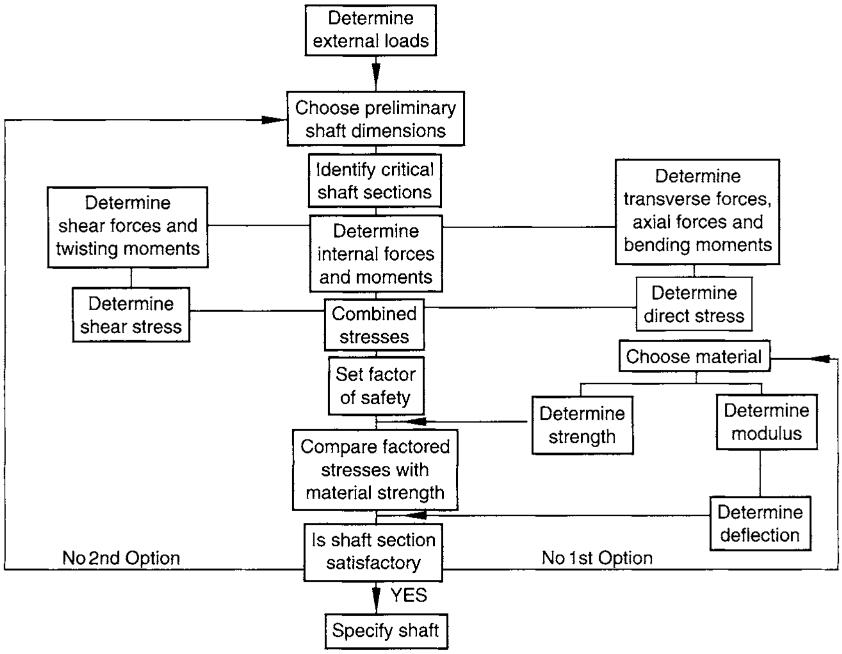

Figure 5.4 Design procedure flow chart for shaft strength and rigidity (adapted from Beswarick, 1994a).

desirable to use just two sets of bearings. If more bearings are required, precise alignment of the bearings is necessary. Provision for thrust load capability and axial location of the shaft is normally supplied by just one thrust bearing, taking thrust in each direction. It is important that the structural members supporting the shaft bearings are sufficiently strong and rigid.

The list below outlines a shaft design procedure for a shaft experiencing constant loading. The flow charts given in Figures 5.4 and 5.5 can be used to guide and facilitate design for shaft strength and rigidity and fluctuating load capability.

1. Determine the shaft rotational speed.
2. Determine the power or torque to be transmitted by the shaft.
3. Determine the dimensions of the power transmitting devices and other components mounted on the shaft and specify locations for each device.
4. Specify the locations of the bearings to support the shaft.
5. Propose a general form or scheme for the shaft geometry considering how each component will be located axially and how power transmission will take place.
6. Determine the magnitude of the torques throughout the shaft.
7. Determine the forces exerted on the shaft.
8. Produce shearing force and bending moment diagrams, so that the distribution of bending moments in the shaft can be determined.
9. Select a material for the shaft and specify any heat treatments, etc.
10. Determine an appropriate design stress taking into account the type of loading (whether smooth, shock, repeated, reversed).
11. Analyse all the critical points on the shaft and determine the minimum acceptable diameter at each point to ensure safe design.
12. Determine the deflections of the shaft at critical locations and estimate the critical frequencies.
13. Specify the final dimensions of the shaft. This is best achieved using a detailed manufacturing drawing to a recognized standard (see the *Manual of British Standards in Engineering and Drawing Design* (BSI, 1984)) and the drawing should include all the information required

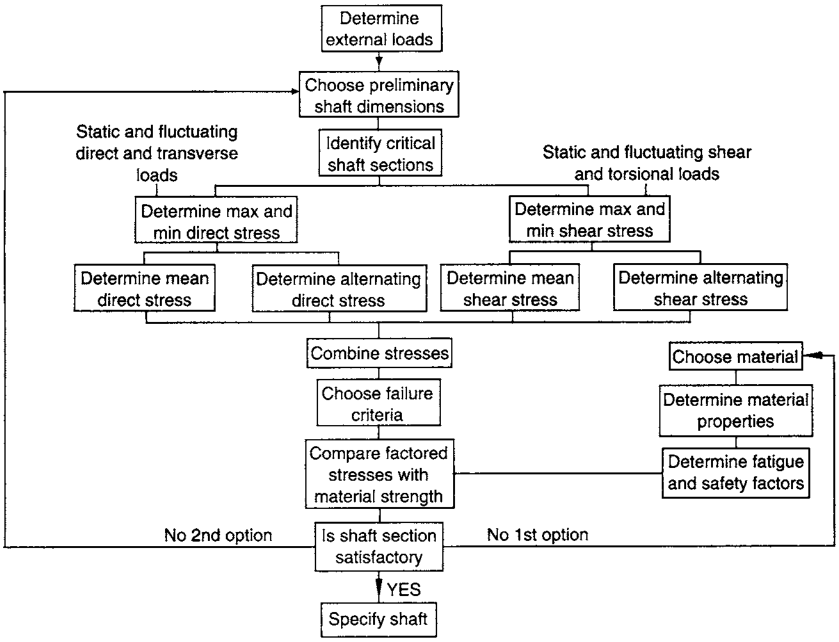

Figure 5.5 Design procedure flow chart for a shaft with fluctuating loading (adapted from Beswarick, 1994b).

to ensure the desired quality. Typically, this will include material specifications, dimensions and tolerances (bilateral, runout, datums, etc. (see Chapter 15)), surface finishes, material treatments and inspection procedures.

The following general principles should be observed in shaft design.

- Keep shafts as short as possible with the bearings close to applied loads. This will reduce shaft deflection and bending moments and increase critical speeds.
- If possible locate stress raisers away from highly stressed regions of the shaft. Use generous fillet radii and smooth surface finishes and consider using local surface strengthening processes, such as shot peening and cold rolling.
- If weight is critical use hollow shafts.

An overview of shaft hub connection methods is given in Section 5.2, shaft to shaft connection methods in Section 5.3 and the determination of critical speeds in Section 5.4. In Section 5.5, the ASME equation for the design of transmission shafts is introduced.

5.2 Shaft–hub connection

Power transmitting components such as gears, pulleys and sprockets need to be mounted on shafts securely and located axially with respect to mating components. In addition, a method of transmitting torque between the shaft and the component must be supplied. The portion of the component in contact with the shaft is called the hub and can be attached to, or driven by, the shaft by keys, pins, setscrews, press and shrink fits, splines and taper bushes. Table 5.1 identifies the merits of various connection methods. Alternatively the component can be formed as an integral part of a shaft as, for example, the cam on an automotive cam-shaft.

Figure 5.6 illustrates the practical implementation of several shaft hub connection methods. Gears, for example, can be gripped axially between a shoulder on a shaft and a spacer with the torque transmitted through a key. Various configurations of keys exist including square, flat and round keys as shown in Figure 5.7. The grooves in the shaft and hub into which the key fits are called keyways or keyseats. A simpler and less expensive method for transmitting light loads

Table 5.1 Merits of various shaft–hub connections (after Hurst, 1994)

	Pin	Grub screw	Clamp	Press fit	Shrink fit	Spline	Key	Taper bush
High torque capacity	✗	✗	✓	✗	✓	✓	✓	✓
Large axial loads	✓	✗	✓	✗	✓	✗	✗	✓
Axially compact	✗	✗	✗	✓	✓	✓	✓	✓
Axial location provision	✓	✓	✓	✓	✓	✗	✗	✓
Easy hub replacement	✗	✓	✓	✗	✗	✓	✓	✓
Fatigue	✗	✗	✓	✓	✓	✗	✗	✓
Accurate angular positioning	✓	✗	✗	✗	✗	✓	✓	✓
Easy position adjustment	✗	✓	✓	✗	✗	✗	✗	✓

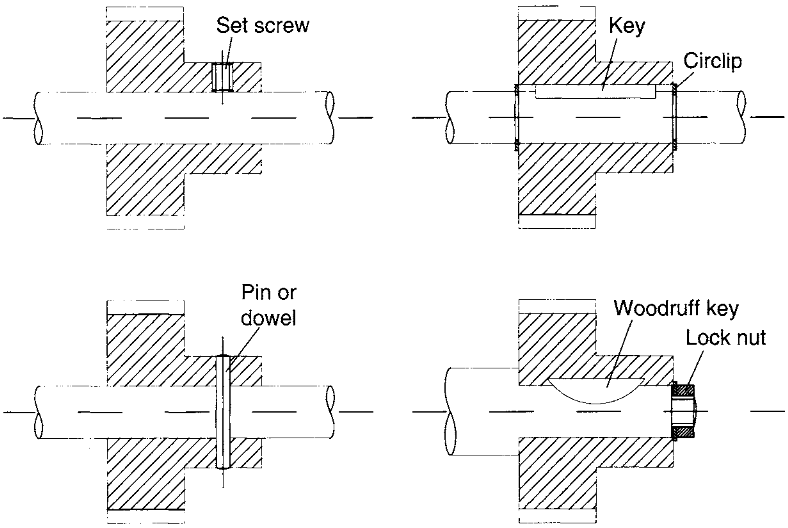

Figure 5.6 Alternative methods of shaft–hub connection.

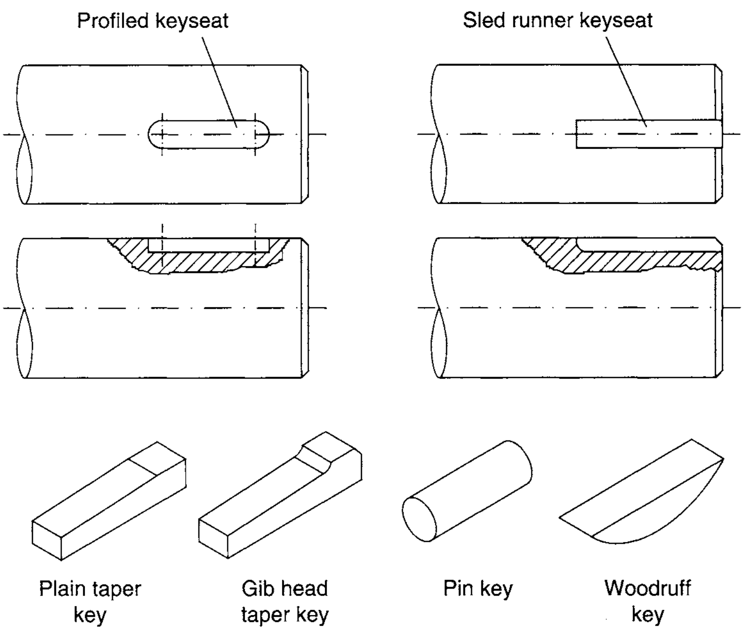

Figure 5.7 Keys for torque transmission and component location.

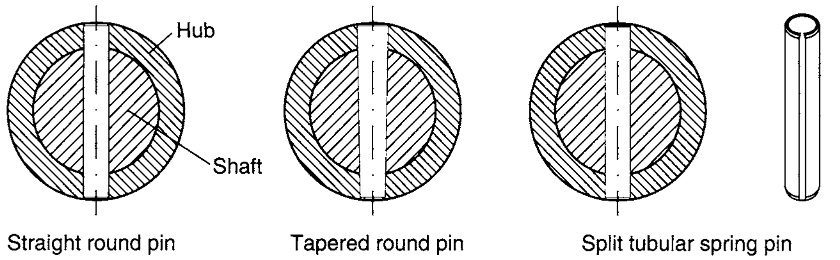

Figure 5.8 Pins for torque transmission and component location.

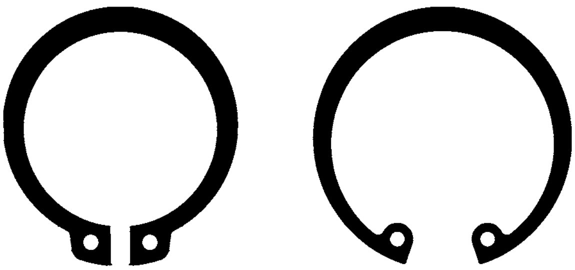

Figure 5.9 Snap rings or circlips.

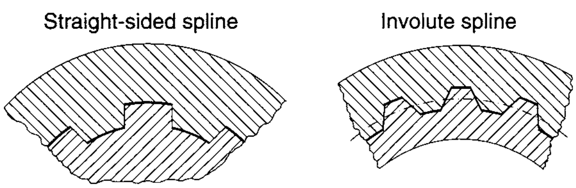

Figure 5.10 Splines.

is to use pins, and various pin types are illustrated in Figure 5.8. An inexpensive method of providing axial location of hubs and bearings on shafts is to use circlips as shown in Figures 5.2 and 5.9. One of the simplest hub–shaft attachments is to use an interference fit, where the hub bore is slightly smaller than the shaft diameter. Assembly is achieved by press fitting, or thermal expansion of the outer ring by heating and thermal contraction of the inner by use of liquid nitrogen. The design of interference fits is covered in greater detail in Section 15.2.2. Mating splines, as shown in Figure 5.10, comprise teeth cut into both the shaft and the hub and provide one of the strongest methods of transmitting torque. Both splines and keys can be designed to allow axial sliding along the shaft.

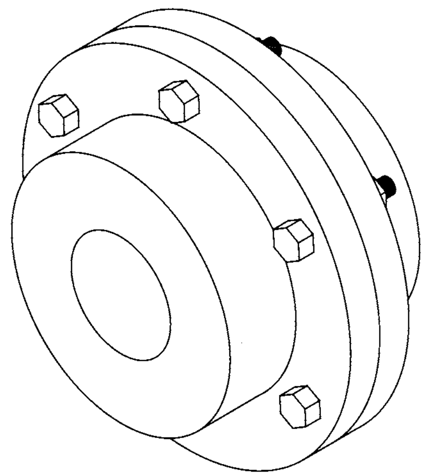

Figure 5.11 Rigid coupling.

5.3 Shaft–shaft connection – couplings

In order to transmit power from one shaft to another, a coupling or clutch can be used (for clutches see Chapter 10). There are two general types of coupling, rigid and flexible. Rigid couplings are designed to connect two shafts together so that no relative motion occurs between them (see Figure 5.11). Rigid couplings are suitable when precise alignment of two shafts is required. If significant radial or axial misalignment occurs high stresses may result which can lead to early failure. Flexible couplings (see Figure 5.12) are designed to transmit torque, whilst permitting some axial, radial and angular misalignment. Many forms of flexible coupling are available (e.g. see manufacturers' catalogues such as Turboflex and Fenner). Each

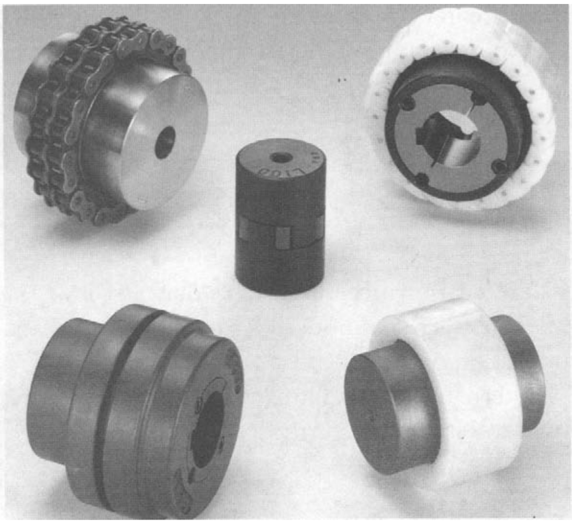

Figure 5.12 Flexible couplings (photograph courtesy of Cross and Morse).

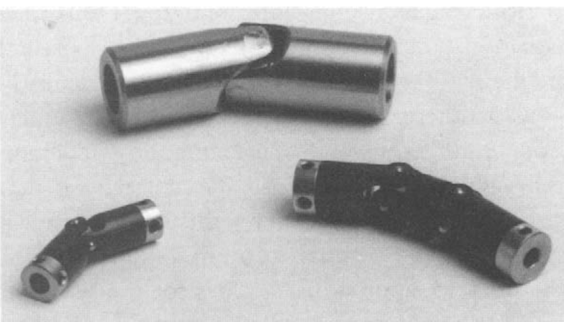

Figure 5.13 Universal joints.

coupling is designed to transmit a given limiting torque. Generally flexible couplings are able to tolerate up to $\pm 3°$ of angular misalignment and up to 0.75 mm parallel misalignment depending on their design. If more misalignment is required a universal joint can be used (see Figure 5.13). Couplings are considered in detail by Neale *et al.* (1998) and Piotrowski (1995).

5.4 Critical speeds and shaft deflection

The centre of mass of a rotating system (for example a mid-mounted disc on a shaft supported by bearings at each end) will never coincide with the centre of rotation due to manufacturing and operational constraints. As the shaft rotational speed is increased, the centrifugal force acting at the centre of mass tends to bow the shaft. The more the shaft bows, the greater the eccentricity and the greater the centrifugal force. Below the lowest critical speed of rotation the centrifugal and elastic forces balance at a finite value of shaft deflection. At the critical speed, equilibrium theoretically requires infinite deflection of the centre of mass although realistically bearing damping, internal hysteresis and windage causes equilibrium to occur at a finite displacement. This displacement can be large enough to break the shaft, damage bearings and cause destructive machine vibration and therefore deflections need to be determined along the shaft and the consequences evaluated.

The critical speed of rotation is the same as the lateral frequency of vibration, which is induced when rotation is stopped and the centre displaced laterally and suddenly released (i.e. the same as the frequency you would obtain if you struck the stationary shaft with a hammer and monitored the frequency at which it vibrated). For all shafts, except for the single concentrated mass shaft, critical speeds also occur at higher frequencies.

At the first critical speed the shaft will bend into the simplest possible shape, and at the second critical speed it will bend into the next simplest shape. For example, the shapes the shaft will bend into at the first two critical speeds (or modes) for an end-supported shaft with two masses are illustrated in Figure 5.14.

In certain circumstances, the fundamental frequency of a shaft system cannot be made higher than the shaft design speed. If the shaft can be accelerated rapidly through and beyond the first resonant critical frequency, before the vibrations have a chance to build up in amplitude, then the system can be run at speeds higher than the natural frequency. This is the case with steam and gas turbines where the size of the turbomachinery and generators give low natural frequency, but must be run at high speed due to efficiency considerations.

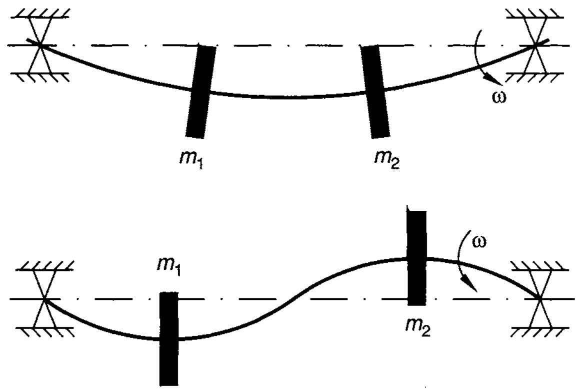

Figure 5.14 Shaft shapes for a simply supported shaft with two masses at the first and second critical speeds. Masses large in comparison with shaft mass.

As a general design principle maintaining the operating speed of a shaft below half the shaft whirl critical frequency is normally acceptable.

A complete analysis of the natural frequencies of a shaft can be performed using a finite element analysis package, such as ANSYS, called a 'nodal analysis'. This can give a large number of natural frequencies in three dimensions from the fundamental upwards. This is the sensible and easiest approach for complex systems, but a quick estimate for a simplified system can be undertaken for design purposes as outlined in this section.

The critical speed of a shaft with a single mass attached can be approximated by:

$$\omega_c = \sqrt{g/y} \qquad (5.1)$$

where ω_c is critical angular velocity (rad/s); y, static deflection at the location of the mass (m); and g, acceleration due to gravity (m/s^2).

The first critical speed of a shaft carrying several concentrated masses is approximated by the Rayleigh–Ritz equation (see Eq. 5.2). The dynamic deflections of a shaft are generally unknown. Rayleigh showed that an estimate of the deflection curve is suitable provided it represents the maximum deflection and the boundary conditions. The static deflection curve due to the shaft's own weight and the weight of any attached components gives a suitable estimate. Note that external loads are not considered in this analysis only those due to gravitation. The resulting calculation gives a value higher than the actual natural frequency by a few per cent.

$$\omega_c = \sqrt{g\sum_{i=1}^{n} W_i y_i \Big/ \sum_{i=1}^{n} W_i y_i^2} \qquad (5.2)$$

where ω_c is critical angular velocity (rad/s); W_i, mass or weight of node i (kg or N); and y_i, static deflection of W_i (m).

Alternatively the Dunkerley equation can be used to estimate the critical speed:

$$\frac{1}{\omega_c^2} = \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \frac{1}{\omega_3^2} + \cdots \qquad (5.3)$$

where ω_1 is the critical speed if only mass number 1 is present, etc.

Both the Rayleigh–Ritz and the Dunkerley equations are approximations to the first natural frequency of vibration, which is assumed nearly equal to the critical speed. The Dunkerley equation tends to underestimate and the Rayleigh–Ritz equation to overestimate the critical frequency.

Often we need to know the second critical speed. For a two mass system, as illustrated in Figure 5.14, approximate values for the first two critical speeds can be found by solving the frequency equation given in Eq. 5.4.

$$\frac{1}{\omega^4} - (a_{11}m_1 + a_{22}m_2)\frac{1}{\omega^2} + (a_{11}a_{22} - a_{12}a_{21})m_1m_2 = 0 \qquad (5.4)$$

where a_{11}, a_{12}, etc., are the influence coefficients. These correspond to the deflection of the shaft at the locations of the loads as a result of 1 N loads. The first subscript refers to location of the deflection, the second to the location of the 1 N force. For example, the influence coefficients for a simply supported shaft with two loads, as illustrated in Figure 5.15, are listed below.

- a_{11} is the deflection at the location of mass 1, that would be caused by a 1 N weight at the location of mass 1.

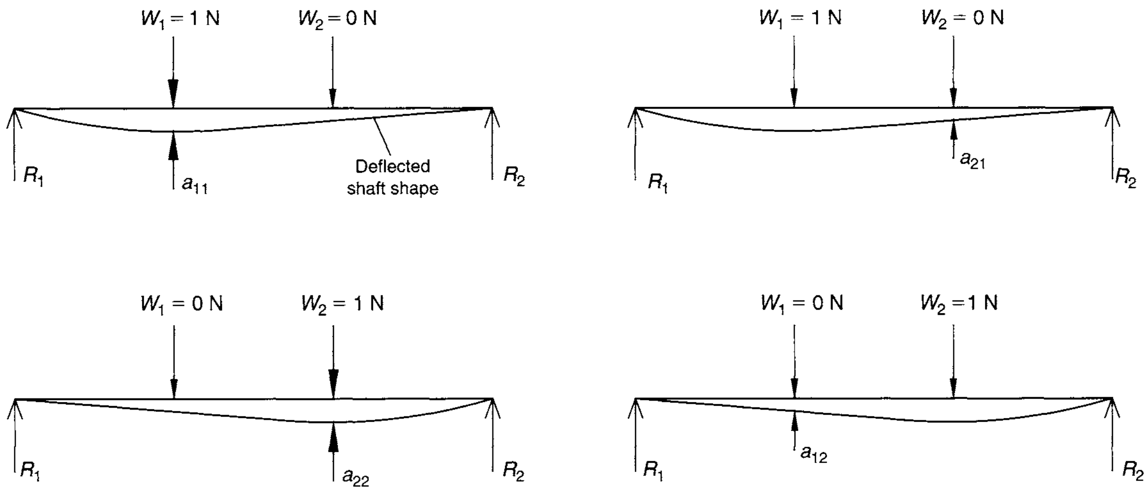

Figure 5.15 Example of influence coefficient definition for a simply supported shaft with two concentrated loads.

- a_{21} is the deflection at the location of mass 2, that would be caused by a 1 N weight at the location of mass 1.
- a_{22} is the deflection at the location of mass 2, that would be caused by a 1 N weight at the location of mass 2.
- a_{12} is the deflection at the location of mass 1, that would be caused by a 1 N weight at the location of mass 2.

Note that values for a_{np} and a_{pn} are equal by the principle of reciprocity (e.g. $a_{12} = a_{21}$).

For a multimass system the frequency equation can be obtained by equating the following determinate to zero:

$$\begin{vmatrix} \left(a_{11}m_1 - \frac{1}{\omega^2}\right) & a_{12}m_2 & a_{13}m_3 & \cdots \\ a_{21}m_1 & \left(a_{22}m_2 - \frac{1}{\omega^2}\right) & a_{23}m_3 & \cdots \\ a_{31}m_1 & a_{32}m_2 & \left(a_{33}m_3 - \frac{1}{\omega^2}\right) & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0 \tag{5.5}$$

It should be noted that lateral vibration requires an external source of energy. For example, vibrations can be transferred from another part of a machine and the shaft will vibrate in one or more lateral planes regardless of whether the shaft is rotating. Shaft whirl is a self-excited vibration caused by the shaft's rotation acting on an eccentric mass.

These analysis techniques for calculating the critical frequency require the determination of the shaft deflection. Section 5.4.1 introduces the Macaulay method which is suitable for calculating the deflection of a constant diameter shaft and Section 5.4.2 introduces the strain energy method which is suitable for more complex shafts with stepped diameters.

5.4.1 Macaulay's method for calculating the deflection of beams

Macaulay's method can be used to determine the deflection of a constant cross-section shaft. The general rules for this method are given below. Having calculated the deflections of a shaft this information can then be used to determine critical frequencies.

1. Take an origin at the left hand side of the beam.
2. Express the bending moment at a suitable section XX in the beam to include the effect of all the loads.
3. Uniformly distributed loads must be made to extend to the right hand end of the beam. Use negative loads to compensate.

4. Put in square brackets all functions of length other than those involving single powers of x.
5. Integrate as a whole any term in square brackets.
6. When evaluating the moment, slope or deflection, neglect the square brackets terms when they become negative.
7. In the moment equation, express concentrated moments in the form $M_1[x-a]^0$, where M_1 is the concentrated moment and $x-a$ is its point of application relative to the section XX.

Macaulay's method for calculating the deflection of beams and shafts is useful in that it is relatively simple to use and easily programmed.

Example 5.1

As part of the preliminary design of a machine shaft, a check is undertaken to determine that the critical speed is significantly higher than the design speed of 7000 rpm. The components can be represented by three point masses as shown in Figure 5.16. Assume the bearings are stiff and act as simple supports. The shaft diameter is 40 mm and the material is steel with a Young's modulus of $200\times10^9\,N/m^2$.

Solution

Macaulay's method is used to determine the shaft deflections: Resolving vertically:

$$R_1 + R_2 = W_1 + W_2 + W_3.$$

Clockwise moments about O:

$$W_1L_1 + W_2(L_1+L_2) - R_2(L_1+L_2+L_3) + W_3(L_1+L_2+L_3+L_4) = 0$$

Hence

$$R_2 = \frac{W_1L_1 + W_2(L_1+L_2) + W_3(L_1+L_2+L_3+L_4)}{L_1+L_2+L_3}$$

Calculating the moment at XX:

$$M_{xx} = -R_1x + W_1[x-L_1] + W_2[x-(L_1+L_2)] - R_2[x-(L_1+L_2+L_3)] \quad (5.6)$$

The relationship for the deflection y of a beam subjected to a bending moment M is given by Eq. 5.7.

$$EI\frac{d^2y}{dx^2} = M \quad (5.7)$$

where E is the modulus of elasticity or Young's modulus (N/m^2); I, second moment of area (m^4); Y, deflection (m); x, distance from the end of the beam to the location at which the deflection is to be determined (m); M, moment (N m).

Eq. 5.7 can be integrated once to find the slope dy/dx and twice to find the deflection y.

Integrating Eq. 5.6 to find the slope:

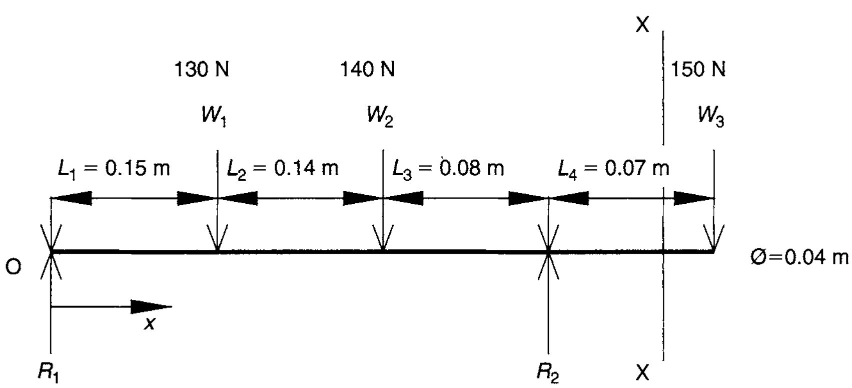

Figure 5.16 Machine shaft example.

$$EI\frac{dy}{dx} = -R_1\frac{x^2}{2} + \frac{W_1}{2}[x - L_1]^2 + \frac{W_2}{2}[x - (L_1 + L_2)]^2 - \frac{R_2}{2}[x - (L_1 + L_2 + L_3)]^2 + A$$

Integrating again to find the deflection equation gives:

$$EIy = -R_1\frac{x^3}{6} + \frac{W_1}{6}[x - L_1]^3 + \frac{W_2}{6}[x - (L_1 + L_2)]^3 - \frac{R_2}{6}[x - (L_1 + L_2 + L_3)]^3 + Ax + B.$$

Note that Macaulay's method requires that terms within square brackets be ignored when the sign of the bracket goes negative.

It is now necessary to substitute boundary conditions to find the constants of integration. In the case of a shaft the deflection at the bearings may be known in which case these can be used as boundary conditions.

Assuming that the deflection at the bearing is zero then substituting $y = 0$ at $x = 0$ into the equation for deflection above gives $B = 0$. At $x = L_1 + L_2 + L_3$, $y = 0$.

$$0 = -\frac{R_1}{6}(L_1 + L_2 + L_3)^3 + \frac{W_1}{6}(L_2 + L_3)^3 + \frac{W_2}{6}(L_3)^3 + A(L_1 + L_2 + L_3)$$

Hence

$$A = \frac{1}{L_1 + L_2 + L_3}\left[\frac{R_1}{6}(L_1 + L_2 + L_3)^3 - \frac{W_1}{6}(L_2 + L_3)^3 - \frac{W_2}{6}(L_3)^3\right]$$

The second moment of area for a solid round shaft is given by

$$I = \frac{\pi d^4}{64}$$

so

$$I = \frac{\pi 0.04^4}{64} = 1.2566 \times 10^{-7}\ \text{m}^4$$

With $W_1 = 130$ N, $W_2 = 140$ N, $W_3 = 150$ N, $\phi = 0.04$ m, and $E = 200 \times 10^9$ N/m^2, substitution of these values into the above equations gives:

$$R_1 = 79.19\ \text{N}$$
$$R_2 = 340.8\ \text{N}$$
$$A = 1.151\ \text{N m}^2$$

At $x = 0.15$ m, $y = 5.097 \times 10^{-6}$ m.
At $x = 0.29$ m, $y = 2.839 \times 10^{-6}$ m.
At $x = 0.44$ m, $y = -1.199 \times 10^{-6}$ m.

Use *absolute* values for the displacement in the Rayleigh–Ritz equation:

$$\omega_c = \sqrt{\frac{9.81\left[\left(130 \times 5.097 \times 10^{-6}\right) + \left(140 \times 2.839 \times 10^{-6}\right) + \left(150 \times \left|-1.199 \times 10^{-6}\right|\right)\right]}{\left(130 \times \left(5.097 \times 10^{-6}\right)^2\right) + \left(140 \times \left(2.839 \times 10^{-6}\right)^2\right) + \left(150 \times \left(\left|-1.199 \times 10^{-6}\right|\right)^2\right)}}$$

$$= \sqrt{\frac{9.81 \times 1.240 \times 10^{-3}}{4.722 \times 10^{-9}}} = 1605\ \text{rad/s}$$

so the critical speed is $1605 \times 2\pi/60 = 15\,530$ rpm.

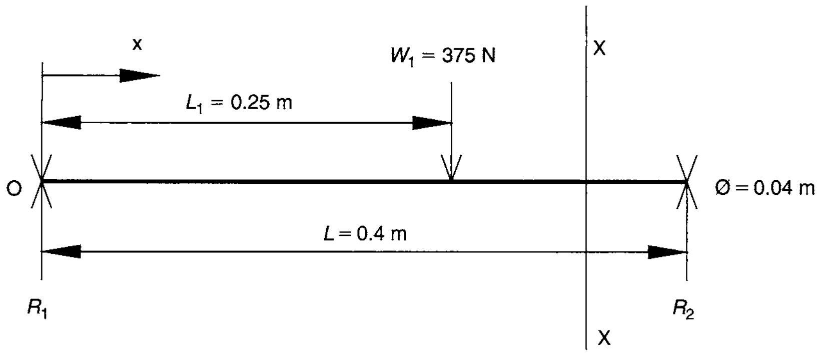

Figure 5.17 Simple shaft with single concentrated mass.

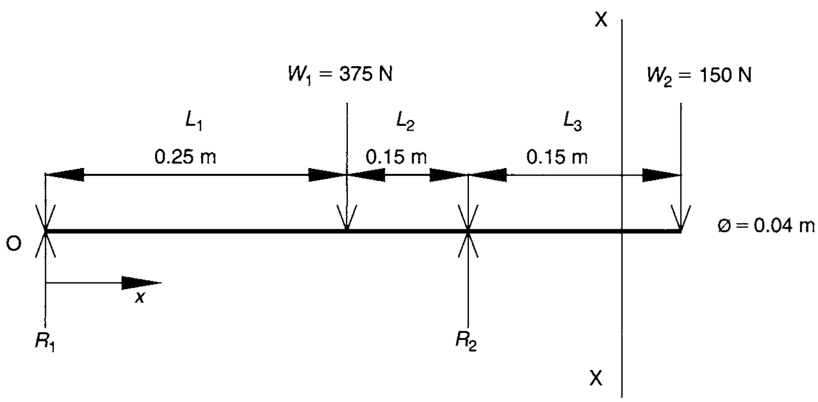

Figure 5.18 Simple shaft with a single concentrated mass between bearing and overhang mass.

Example 5.2

This example explores the influence on the critical frequency of adding an overhang load to a beam.

1. Determine the critical frequency for a 0.4 m long steel shaft running on bearings with a single point load as illustrated in Figure 5.17. Assume that the bearings are rigid and act as simple supports. Ignore the self-weight of the shaft. Take Young's modulus of elasticity as $200 \times 10^9\,\text{N/m}^2$.
2. Calculate the critical frequency if an overhang load of 150 N is added at a distance of 0.15 m from the right hand bearing as shown in Figure 5.18.

Solution

1. Resolving vertically: $R_1 + R_2 = W_1$. Clockwise moments about O: $375 \times 0.25 - R_2 0.4 = 0$.

Hence $R_2 = 234.375\,\text{N}, R_1 = 140.625\,\text{N}$.

$$EI\frac{d^2y}{dx^2} = -R_1x + W_1[x - L_1]$$

$$EI\frac{dy}{dx} = -R_1\frac{x^2}{2} + \frac{W_1}{2}[x - L_1]^2 + A$$

$$EIy = -R_1\frac{x^3}{6} + \frac{W_1}{6}[x - L_1]^3 + Ax + B.$$

Boundary conditions:

Assuming that the deflection of the shaft is zero at the bearings, then substituting $y = 0$ and $x = 0$ into the above equation gives $B = 0$. At $x = L$, $y = 0$.

Hence

$$A = \frac{1}{L}\left(R_1\frac{L^2}{6} - \frac{W_1}{6}(L - L_1)^3\right)$$

$$= \frac{1}{0.4}(1.5 - 0.2109) = 3.22266.$$

For a solid circular shaft

$$I = \frac{\pi d^4}{64} = \frac{\pi 0.04^4}{64}$$
$$= 1.2566 \times 10^{-7}\ \text{m}^4$$
$$IE = 200 \times 10^9\ \text{N/m}^2$$
$$EIy = -140.625 \times \frac{0.25^3}{6}$$
$$+ (3.22266 \times 0.25)$$
$$y = 1.7485 \times 10^{-5}\ \text{m}.$$

Application of Macaulay's method to the geometry given in Figure 5.17 yields $y_1 = 1.74853 \times 10^{-5}$ m. So from Eq. 5.1:

$$\omega_c = \sqrt{\frac{g}{y_1}} = 749.03\ \text{rad/s} = 7152\ \text{rpm}$$

2. $$EI\frac{d^2y}{dx^2} = -R_1x + W_1[x - L_1]$$
$$-R_2[x - (L_1 + L_2)]$$
$$EI\frac{dy}{dx} = -R_1\frac{x^2}{2} + \frac{W_1}{2}[x - L_1]^2$$
$$-\frac{R^2}{2}[x - (L_1 + L_2)]^2 + A$$
$$EIy = -R_1\frac{x^3}{6} + \frac{W_1}{6}[x - L_1]^3$$
$$-\frac{R_2}{6}[x - (L_1 + L_2)]^3 + Ax + B$$

Boundary conditions:

Assuming negligible deflection at the bearings. At $x = 0$, $y = 0$. Hence $B = 0$.

At $x = L_1 + L_2$, $y = 0$. Hence

$$A = \frac{1}{L_1 + L_2}\left(\frac{R_1}{6}(L_1 + L_2)^3 - \frac{W_1}{6}L_2^3\right).$$

Resolving vertically: $R_1 + R_2 = W_1 + W_2$.

Moments about O: $W_1L_1 + W_2(L_1 + L_2 + L_3) - R_2(L_1 + L_2) = 0$.

Hence

$$R_2 = \frac{1}{L_1 + L_2}\left(W_1L_1 + W_2(L_1 + L_2 + L_3)\right)$$
$$R_1 = W_1 + W_2 - R_2$$
$$R_1 = 84.375\ \text{N}$$
$$R_2 = 440.625\ \text{N}.$$

Substitution for A gives $A = 1.722656\ \text{N m}^2$.

Evaluating y at $x = L_1$ gives $y = 8.3929381 \times 10^{-6}$ m.

Evaluating y at $x = L_1 + L_2 + L_3$ gives $y = 1.8884145 \times 10^{-6}$ m.

Hence

$$\omega_c = \sqrt{\frac{9.81\left(3.147 \times 10^{-3} + 2.8326 \times 10^{-4}\right)}{2.64155 \times 10^{-8} + 5.34061 \times 10^{-10}}}$$
$$= \sqrt{\frac{9.81 \times 3.4306 \times 10^{-3}}{2.695 \times 10^{-8}}} = 1117.5\ \text{rad/s}$$
$$\omega_c = 10671\ \text{rpm}.$$

Comparison of the results from 1 and 2 shows that the addition of an overhang weight can have a significant effect in raising the critical frequency from 7150 to 10 670 rpm in this case. This technique of adding overhang masses to increase the critical speed is sometimes used in plant machinery.

Example 5.3

1. Calculate the first two critical speeds for the loadings on a steel shaft indicated in Figure 5.19. Assume the mass of the shaft can be ignored for the purpose of this calculation and that the bearings can be considered to be stiff simple supports. The internal and external diameters of the shaft are 0.03 and 0.05 m, respectively. Take Young's modulus as 200 GN/m^2.
2. The design speed for the shaft is nominally 5000 rpm; what would be the implication of the bearing data given in Table 5.2 on the

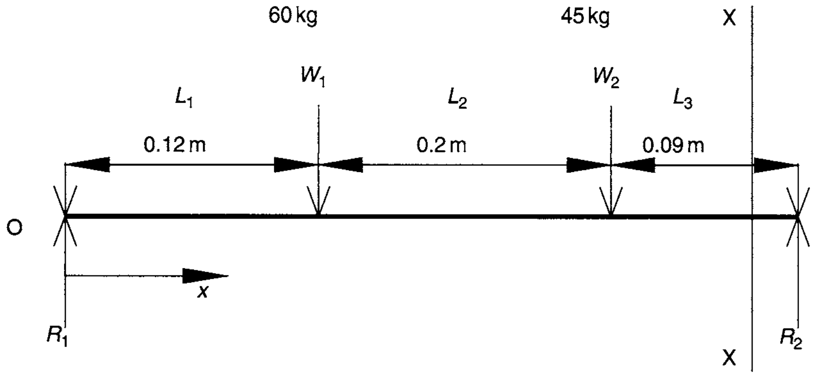

Figure 5.19 Simple shaft.

Table 5.2 Example bearing data for deflection versus load

Load on bearing (N)	Deflection at bearing (mm)
50	0.003
100	0.009
200	0.010
300	0.0115
400	0.0126
500	0.0136
1000	0.0179
2000	0.0245

shaft operation assuming the bearings allow flexibility only in directions perpendicular to the shaft axis?

Solution

1. Applying Macaulay's method to find the influence coefficients: first calculate the deflections at L_1 and L_2 with $W_1 = 1$ N and $W_2 = 0$ N to give a_{11} and a_{21}. Then calculating the deflections at L_1 and L_2 with $W_1 = 0$ N and $W_2 = 1$ N to give a_{12} and a_{22}.

$$EIy = -R_1\frac{x^3}{6} + \frac{W_1}{6}[x - L_1]^3 + \frac{W_2}{6}[x - (L_1 + L_2)]^3 + Ax + B\,.$$

With $W_1 = 1$ N, $W_2 = 0$ N, $R_1 = 0.707317$ N, $R_2 = 0.292683$ N, $A = 9.90244 \times 10^{-3}$ N m^2 ($B = 0$).

Hence

$$a_{11} = 1.84355 \times 10^{-8}\,\text{m},$$
$$a_{21} = 1.19688 \times 10^{-8}\,\text{m}.$$

With $W_1 = 0$ N, $W_2 = 1$ N, $R_1 = 0.2195122$ N, $R_2 = 0.7804878$ N, $A = 5.853659 \times 10^{-3}$ N m^2.

Hence

$$a_{22} = 1.26264 \times 10^{-8}\,\text{m},$$
$$a_{12} = 1.19688 \times 10^{-8}\,\text{m}.$$

For this example, $a_{11} = 1.84355 \times 10^{-8}$ m, $a_{21} = 1.19688 \times 10^{-8}$ m, $a_{12} = 1.19688 \times 10^{-8}$ m, $a_{22} = 1.26264 \times 10^{-8}$ m.

Solving

$$\frac{1}{\omega^4} - (a_{11}m_1 + a_{22}m_2)\frac{1}{\omega^2} + (a_{11}a_{22} - a_{12}a_{21})m_1m_2 = 0$$

by multiplying through by ω^4 and solving as a quadratic:

$$\omega^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

with

$$a = (a_{11}a_{22} - a_{12}a_{21})m_1m_2,$$
$$b = -(a_{11}m_1 + a_{22}m_2),$$
$$c = 1,$$

gives $\omega_{c1} = 812.5$ rad/s (7759 rpm) and $\omega_{c2} = 2503.4$ rad/s (23 905 rpm).

2. The deflection at the bearings should also be taken into account in determining the critical frequency of the shaft. The reaction at each of the bearings is approximately 500 N (R_1 = 513.2 N and R_2 = 516.8 N). The deflection at each bearing can be assumed roughly equal (so from Table 5.2, $y_{\text{bearing}} = 0.0136$ mm). This deflection can be simply added to the static deflection of the shaft due to bending by superposition. If the deflections were not equal, similar triangles could be used to calculate the deflection at the mass locations.

The influence coefficients can also be used to determine the deflection under the actual loads applied. $W_1 = 60 \times 9.81 = 588.6$ N, $W_2 = 45 \times 9.81 = 441.45$ N:

$$y_{1\,\text{Load}} = W_1 a_{11} + W_2 a_{12} = 1.613 \times 10^{-5} \text{ m},$$

$$y_{2\,\text{Load}} = W_1 a_{21} + W_2 a_{22} = 1.26 \times 10^{-5} \text{ m},$$

$$y_{1\,\text{Total}} = 1.61 \times 10^{-5} + 1.36 \times 10^{-5} = 2.97 \times 10^{-5} \text{ m},$$

$$y_{2\,\text{Total}} = 1.26 \times 10^{-5} + 1.36 \times 10^{-5} = 2.62 \times 10^{-5} \text{ m}.$$

These deflections give (using the Rayleigh– Ritz equation) a first critical frequency of:

$$\omega_c = \sqrt{\frac{9.81 \times 2.96 \times 10^{-3}}{8.38 \times 10^{-8}}} = 588.6 \text{ rad/s} = 5620 \text{ rpm}.$$

This is very close to the design speed. Different bearings or a stiffer shaft would be advisable.

5.4.2 Castigliano's theorem for calculating shaft deflections

The strain energy, U, in a straight beam subjected to a bending moment M is:

$$U = \int \frac{M^2 dx}{2EI} \tag{5.8}$$

Castigliano's strain energy equation can be applied to problems involving shafts of non-constant section to calculate deflections, and hence critical speeds.

Example 5.4

Determine the deflection of the stepped shaft illustrated in Figure 5.20 under a single concentrated load.

Solution

The total strain energy due to bending is

$$U = \int \frac{M^2 dx}{2EI} = U_1 + U_2$$

where U_1 is the energy from $x = 0$ to L_1, and U_2 is the energy from $x = L_1$ to L.

In general, consider the strain energy for each section of a beam, i.e. between loads and between any change of shaft section.

Resolving vertical loads:

$$R_1 + R_2 = W.$$

Taking moments about the left hand bearing:

$$R_2 = \frac{WL_1}{L}.$$

Hence

$$R_1 = W\left(1 - \frac{L_1}{L}\right).$$

Splitting the beam between $x = 0$ and L_1 the moment can be expressed as $R_1 x$.

$$U_1 = \int_0^{L_1} \frac{(R_1 x)^2}{2EI_1} dx = \left[\frac{(W(1 - (L_1/L)))^2 x^3}{6EI_1}\right]_0^{L_1} = \frac{(W(1 - (L_1/L)))^2 L_1^3}{6EI_1}.$$

Splitting the beam between $x = L_1$ and L the moment can be expressed as $R_2(L - x)$.

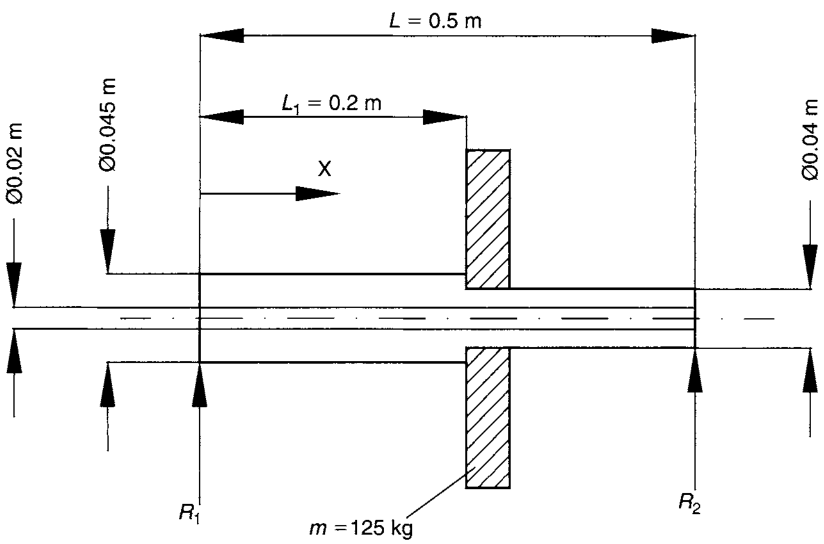

Figure 5.20 Stepped shaft with single load.

$$U_2 = \int_{L_1}^{L} \frac{(R_2(l - X))^2}{2EI} dx$$

$$= \left[\frac{(WL_1/L)^2 (L^2 x - Lx^2 + x^3/3}{2EI_2} \right]_{L_1}^{L}$$

$$= \frac{W^2 L_1^2}{2EI_2 L^2} \left(L^3 - L^3 + \frac{L^3}{3} - L^2 L_1 + LL_1^2 - \frac{L_1^3}{3} \right).$$

$$U = U_1 + U_2$$

$$= \frac{(W(1 - (L_1/L)))^2 L_1^3}{6EI_1} + \frac{W^2 L_1^2}{2EI_2 L^2} \left(L^3 - L^3 + \frac{L^3}{3} - L^2 L_1 + LL_1^2 - \frac{L_1^3}{3} \right).$$

Differentiating the above expression with respect to W gives the deflection at W:

$$\frac{\partial U}{\partial W} = \frac{L_1^3}{6EI_1} \left(2W - \frac{4WL_1}{L} + \frac{2WL_1^2}{L^2} \right) + \frac{2WL_1^2}{2EI_2 L^2} \left(\frac{L^3}{3} - L^2 L_1 + LL_1^2 - \frac{L_1^3}{3} \right).$$

$$I_1 = \frac{\pi(d_o^4 - d_i^4)}{64} = \frac{\pi(0.045^4 - 0.02^4)}{64} = 1.9343 \times 10^{-7}\ \text{m}^4.$$

$$I_2 = \frac{\pi(0.04^4 - 0.02^4)}{64} = 1.1781 \times 10^{-7}\ \text{m}^4. \tag{5.9}$$

Substitution of $W = 125 \times 9.81$ N, $L_1 = 0.2$ m, $L = 0.5$ m, $E = 200 \times 10^9$ N/m^2, I_1 and I_2 into Eq. 5.9 gives a deflection $\partial U / \partial W = 1.05372 \times 10^{-4}$ m and hence a first critical frequency using the approximation of Eq. 5.1 of:

$$\sqrt{\frac{9.81}{1.05372 \times 10^{-4}}} = 305.121 \text{ rad/s} = 2913 \text{ rpm}.$$

5.5 ASME design code for transmission shafting

According to Fuchs and Stephens (1980) between 50 and 90 per cent of all mechanical failures are

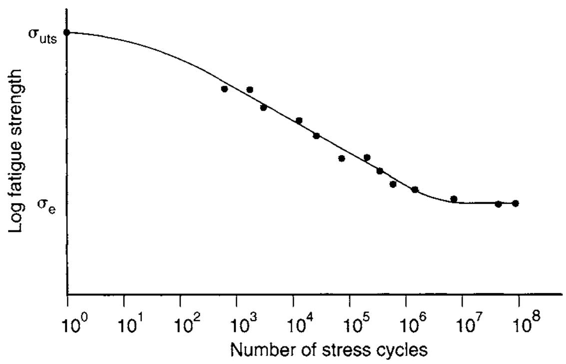

Figure 5.21 A typical strength cycle diagram for various steels.

fatigue failures. This challenges the engineer to consider the possibility of fatigue failure at the design stage. Figure 5.21 shows the characteristic variation of fatigue strength for steel with the number of stress cycles. For low strength steels a 'levelling off' occurs in the graph between 10^6 and 10^7 cycles under non-corrosive conditions and, regardless of the number of stress cycles beyond this, the component will not fail due to fatigue. The value of stress corresponding to this levelling off is called the 'endurance stress' or the 'fatigue limit'.

In order to design hollow or solid rotating shafts under combined cyclic bending and steady torsional loading for limited life the ASME (American Society of Mechanical Engineers) design code for the design of transmission shafting can be used. The ASME procedure ensures that the shaft is properly sized to provide adequate service life, but the designer must ensure that the shaft is stiff enough to limit deflections of power transfer elements, such as gears, and to minimize misalignment through seals and bearings. In addition the shaft's stiffness must be such that it avoids unwanted vibrations through the running range.

The equation for determining the diameter for a solid shaft is given by

$$d = \left[\frac{32 n_s}{\pi} \sqrt{\left(\frac{M}{\sigma_e}\right)^2 + \frac{3}{4}\left(\frac{T}{\sigma_y}\right)^2} \right]^{1/3} \qquad (5.10)$$

where d, diameter (m); n_s, factor of safety; M, bending moment (N m); σ_e, endurance limit of the item (N/m^2); T, torque (N m); and σ_y, yield strength (N/m^2).

There is usually some uncertainty regarding what level a component will actually be loaded to, how strong a material is and how accurate the modelling methods are. Factors of safety are frequently used to account for these uncertainties. The value of a factor of safety is usually based on experience of what has given acceptable performance in the past. The level is also a function of the consequences of component failure and the cost of providing an increased safety factor. As a guide, typical values for the factor of safety based on strength recommended by Vidosek (1957) are:

- 1.25–1.5 for reliable materials under controlled conditions subjected to loads and stresses known with certainty;
- 1.5–2 for well-known materials under reasonably constant environmental conditions subjected to known loads and stresses;
- 2–2.5 for average materials subjected to known loads and stresses;
- 2.5–3 for less well-known materials under average conditions of load, stress and environment;
- 3–4 for untried materials under average conditions of load, stress and environment;
- 3–4 for well-known materials under uncertain conditions of load, stress and environment.

The endurance limit, σ_e, for a mechanical element can be estimated by Eq. 5.11 (after Marin, 1962). Here a series of modifying factors are applied to the endurance limit of a test specimen for various effects such as size, load and temperature.

$$\sigma_e = k_a k_b k_c k_d k_e k_f k_g \sigma'_e \qquad (5.11)$$

where k_a, surface factor; k_b, size factor; k_c, reliability factor; k_d, temperature factor; k_e, duty cycle factor; k_f, fatigue stress concentration factor; k_g, miscellaneous effects factor; σ'_e, endurance limit of test specimen (N/m^2).

Table 5.3 Surface finish factors (Noll and Lipson, 1946)

Surface finish	*a* (MPa)	*b*
Ground	1.58	−0.085
M/c or cold-drawn	4.51	−0.265
Hot rolled	57.7	−0.718
Forged	272.0	−0.995

Table 5.4 Reliability factors for use in the ASME transmission shaft equation

Shaft nominal reliability	k_c
0.5	1.0
0.9	0.897
0.99	0.814
0.999	0.753

Source, ANSI/ASME B106. 1 M-1985.

If the stress at the location under consideration is greater than σ_e, then the component will fail eventually due to fatigue (i.e. the component has a limited life).

Mischke (1987) has determined the following approximate relationships between the endurance limit of test specimens and the ultimate tensile strength of the material (for steels only):

$$\sigma'_e = 0.504\sigma_{uts} \quad \text{for } \sigma_{uts} \le 1400 \text{ MPa} \tag{5.12}$$

$$\sigma'_e = 700 \text{ MPa} \quad \text{for } \sigma_{uts} \ge 1400 \text{ MPa} \tag{5.13}$$

The surface finish factor is given by

$$k_a = a\sigma_{uts}^b \tag{5.14}$$

Values for *a* and *b* can be found in Table 5.3.

The size factor k_b can be calculated from Eq. 5.15 or 5.16 (Kuguel, 1969).

$$k_b = \left(\frac{d}{7.62}\right)^{-0.1133} \quad (\text{for } d < 50 \text{ mm}) \tag{5.15}$$

$$k_b = 1.85d^{-0.19} \quad (\text{for } d > 50 \text{ mm}) \tag{5.16}$$

The reliability factor k_c is given in Table 5.4 as a function of the nominal reliability desired.

For temperatures between −57°C and 204°C, the temperature factor, k_d, can be taken as 1. The ASME standard documents values to use outside this range.

The duty cycle factor k_e is used to account for cycle loading experienced by the shaft, such as stops and starts, transient overloads, shock loading, etc., and requires prototype fatigue testing for its quantification. k_e is taken as 1 in the examples presented here.

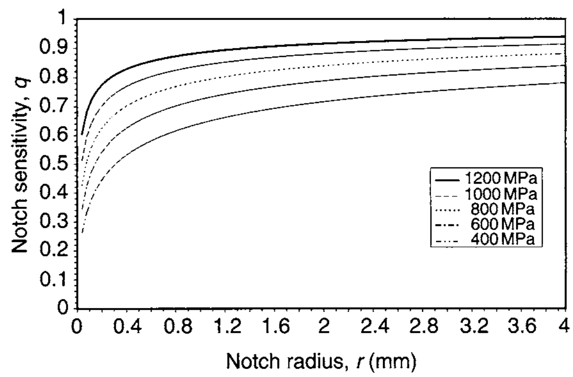

Figure 5.22 Notch sensitivity index versus notch radius for a range of steels subjected to reversed bending or reversed axial loads (data from Kuhn and Handrath, 1952).

The fatigue stress concentration factor k_f is used to account for stress concentration regions, such as notches, holes, keyways and shoulders. It is given by

$$k_f = \frac{1}{K_f} \tag{5.17}$$

where K_f is the component fatigue stress concentration factor which is given by:

$$K_f = 1 + q(k_t - 1) \tag{5.18}$$

where q is notch sensitivity; and k_t is geometric stress concentration factor.

Values for the notch sensitivity and typical geometric stress concentration factors are given in Figures 5.22–5.25 and Table 5.5.

The miscellaneous factor k_g is used to account for residual stresses, heat treatment, corrosion,

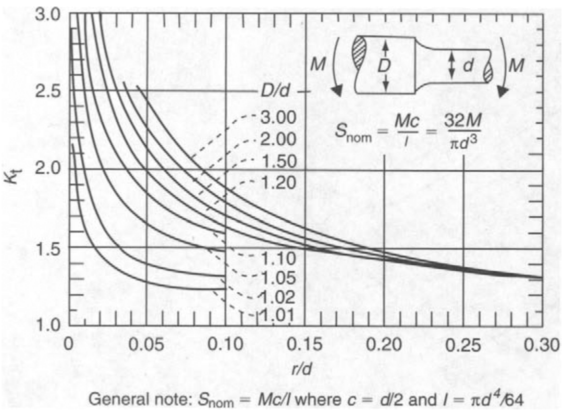

Figure 5.23 Stress concentration factors for a shaft with a fillet subjected to bending (reproduced from Peterson, 1974).

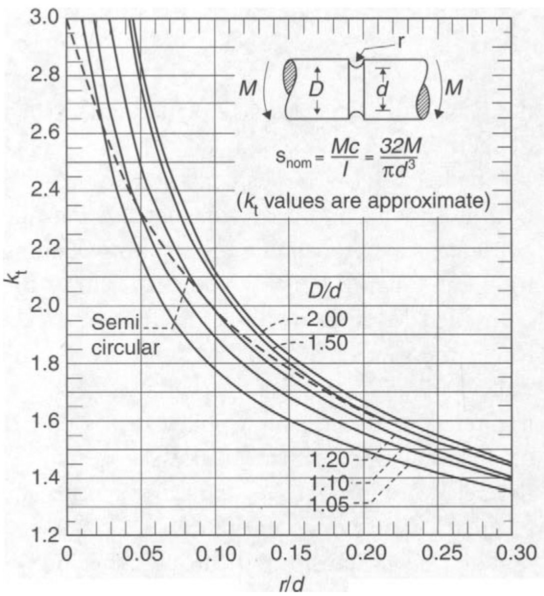

Figure 5.24 Stress concentration factors for a shaft with a groove subjected to bending (reproduced from Peterson, 1974).

surface coatings, vibration, environment and unusual loadings. k_g is taken as 1 here.

In general, the following principles should be used when designing to avoid fatigue failures.

- Calculations should allow an appropriate safety factor, particularly where stress concentrations occur (e.g. keyways, notches, change of section).

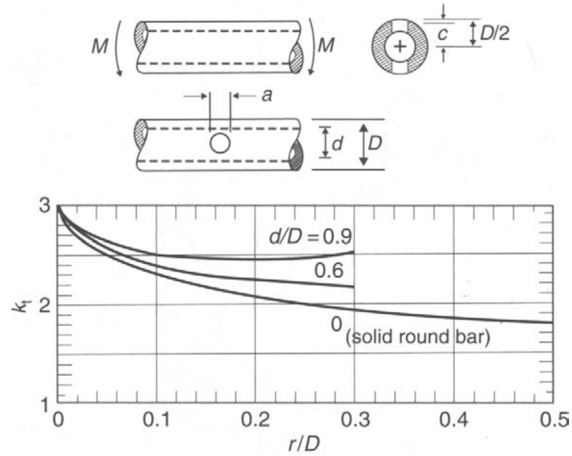

Figure 5.25 Stress concentration factors for a shaft with a transverse hole subjected to bending (reproduced from Peterson, 1974).

Table 5.5 Fatigue stress concentration factors, k_f for keyways in steel shafts (Juvinall, 1967)

	Keyway bending stress	
Steel	Profiled	Sled runner
Annealed < 200 BHN	0.63	0.77
Quenched and drawn > 200 BHN	0.5	0.63

- Provide generous radii at changes of section and introduce stress relief grooves, etc.
- Choose materials if possible that have limiting fatigue stresses, e.g. most steels.
- Provide for suitable forms of surface treatment, e.g. shot-peening, work-hardening, nitriding. Avoid treatments which introduce residual tensile stresses, such as electroplating.
- Specify fine surface finishes.
- Avoid corrosive conditions.
- Stress relieving should be used where possible, particularly for welded structures.

Example 5.5

Using the ASME equation for the design of transmission shafting determine a sensible

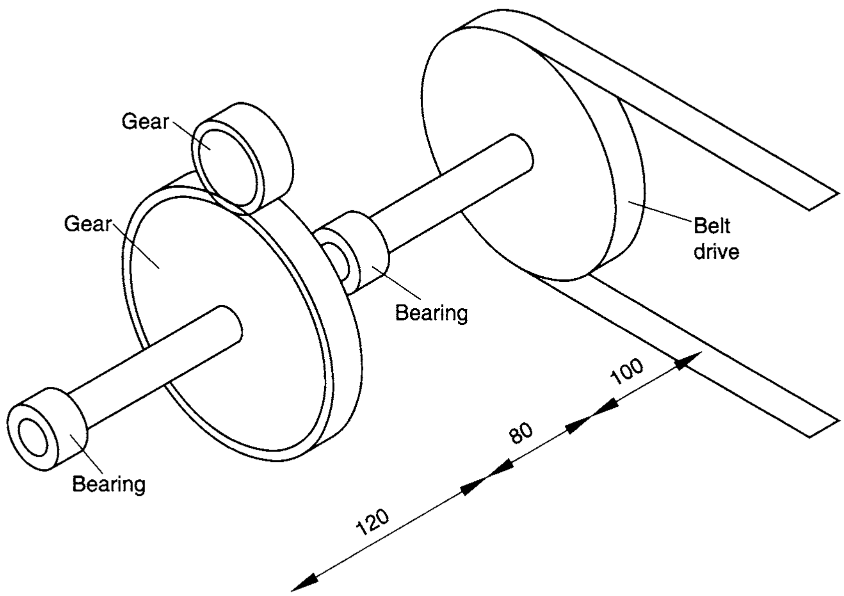

Figure 5.26 Transmission drive shaft.

minimum nominal diameter for the drive shaft illustrated in Figure 5.26 consisting of a mid-mounted spur gear and overhung pulley wheel. The shaft is to be manufactured using 817M40 hot rolled alloy steel with $\sigma_{uts} = 1000\,\text{MPa}$, $\sigma_y = 770\,\text{MPa}$ and Brinell Hardness approximately 220 BHN. The radius of the fillets at the gear and pulley shoulders is 3 mm. The power to be transmitted is 8 kW at 900 rpm. The pitch circle diameter of the 20° pressure angle spur gear is 192 mm and the pulley diameter is 250 mm. The masses of the gear and pulley are 8 and 10 kg, respectively. The ratio of belt tensions should be taken as 2.5. Profiled keys are used to transmit torque through the gear and pulley. A shaft nominal reliability of 90 per cent is desired. Assume the shaft is of constant diameter for the calculation.

Solution

In order to use the ASME design equation for transmission shafts, the maximum combination of torque and bending moment must be determined. A sensible approach is to determine the overall bending moment diagram for the shaft as this information may be of use in other design calculations such as calculating the shaft deflection.

The loading on the shaft can be resolved into both horizontal and vertical planes and must be considered in determining the resulting bending moments on the shaft. Figure 5.27 shows the combined loadings on the shaft from the gear forces, the gear's mass, the belt tensions and the pulley's mass. The mass of the shaft itself has been ignored. The tension on a pulley belt is tighter on the 'pulling' side than on the 'slack' side and the relationship of these tensions is normally given as a ratio, which in this case is 2.5:1.

The power transmitted through the shaft is 8000 W.

The torque is given by

$$\text{Torque} = \frac{\text{Power}}{\omega} = \frac{8000}{(2\pi/60) \times 900}$$
$$= 84.9\ \text{N m}.$$

Belts are introduced in more detail in Chapter 8. Here only the tensions will be considered as necessary for the development of the solution.

The ratio of belt tensions is $T_1/T_2 = 2.5$, so $T_1 = 2.5T_2$.

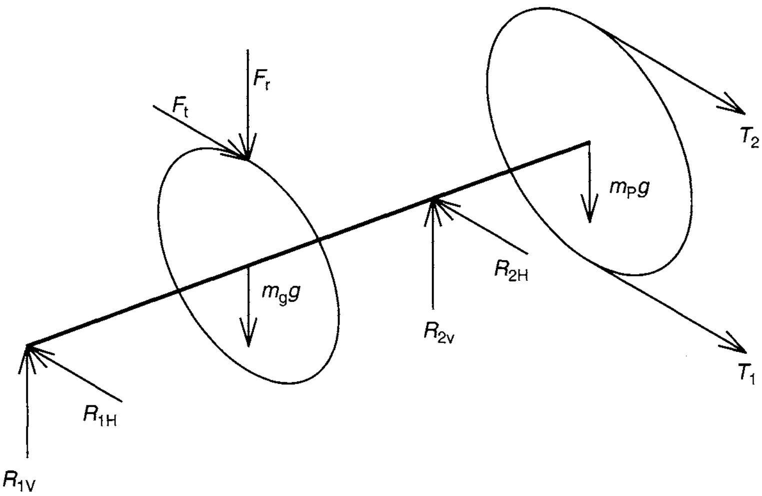

Figure 5.27 Shaft loading diagram.

The torque on the pulley in terms of the belt tensions is given by

$$\begin{aligned}\text{Torque} &= T_1 r_p - T_2 r_p \\ &= 2.5T_2 r_p - T_2 r_p = 1.5T_2 r_p.\end{aligned}$$

The belt tensions are

$$T_2 = 84.9/(1.5 \times 0.125) = 452.8\,\text{N},$$
$$T_1 = 2.5T_2 = 1131.8\,\text{N}.$$

Forces on spur gears are considered in detail in Chapter 6. Here the relevant relationships will be stated as necessary and used in the development of the solution. A spur gear experiences both radial and tangential loading as shown in Figure 5.24. The tangential load is given by F_t = Torque/pitch circle radius.

$$F_t = \frac{\text{Torque}}{r_g} = \frac{84.9}{0.192/2} = 884.4\,\text{N}.$$

The radial load is given by $F_r = F_t \tan\phi$, where ϕ is the pressure angle.

$$F_r = F_t \tan\phi = 884.4 \times \tan 20 = 321.9\,\text{N}.$$

Note that when determining the horizontal and vertical bending moment diagrams both

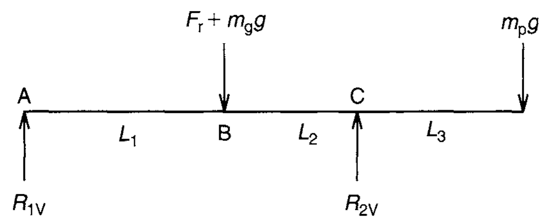

Figure 5.28 Vertical loading diagram.

transmitted and gravitational loads should be included.

The vertical loads on the shaft are illustrated in Figure 5.28.

Moments about A:

$$(F_r + m_g g)L_1 - R_{2V}(L_1 + L_2) + m_p g(L_1 + L_2 + L_3) = 0.$$

$$R_{2V} = \frac{1}{L_1 + L_2}\left[\begin{array}{l}(F_r + m_g g)L_1 \\ + m_p g(L_1 + L_2 + L_3)\end{array}\right]$$
$$\frac{1}{0.12 + 0.08}\left[\begin{array}{l}(321.9 + (8 \times 9.81))0.12 \\ + (10 \times 9.81)0.3\end{array}\right]$$
$$= 387.4\,\text{N}.$$

Resolving vertical forces

$$R_{1V} + R_{2V} = F_r + m_g g + m_p g.$$

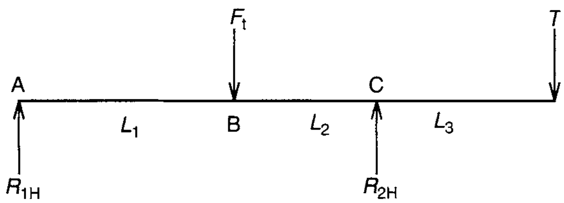

Figure 5.29 Horizontal loading diagram.

Hence

$$R_{1V} = 321.9 + 78.5 + 98.1 - 387.4 = 111.1\,\text{N}.$$

The horizontal loads on the shaft are illustrated in Figure 5.29. The total tension T is

$$T_1 + T_2 = 1131.8 + 452.8 = 1585\,\text{N}.$$

Moments about A

$$F_t L_1 - R_{2H}(L_1 + L_2) + T(L_1 + L_2 + L_3) = 0.$$

Hence

$$R_{2H} = \frac{F_t L_1 + T(L_1 + L_2 + L_3)}{L_1 + L_2} = \frac{(884.4 \times 0.12) + (1585 \times 0.3)}{0.2} = 2908\,\text{N}.$$

Resolving horizontal forces

$$R_{1H} + R_{2H} = F_t + T.$$

Hence

$$R_{1H} = 884.4 + 1585 - 2908 = -438.5\,\text{N}.$$

The bending moment diagrams can now be determined. You may wish to recall that the bending moment is the algebraic sum of the moments of the external forces to one side of the section about an axis through the section.

The vertical bending moments at B and C (Figure 5.30) are given by:

$$M_{BV} = R_{1V} L_1 = 111.1 \times 0.12 = 13.3\,\text{Nm},$$
$$M_{CV} = R_{1V}(L_1 + L_2) - ((F_r + m_g g)L_2) = 111.1(0.12 + 0.08) - (321.9 + 8 \times 9.81)0.08 = -9.810\,\text{Nm}.$$

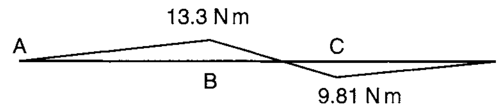

Figure 5.30 Vertical bending moment diagram.

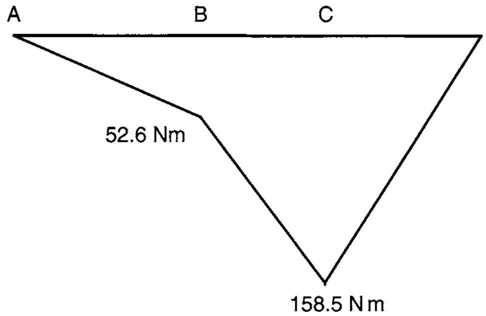

Figure 5.31 Horizontal bending moment diagram.

The horizontal bending moments at B and C (Figure 5.31) are given by:

$$M_{BH} = R_{1H} L_1 = -438.5 \times 0.12 = -52.62\,\text{Nm},$$
$$M_{CH} = R_{1H}(L_1 + L_2) - (F_t L_2) = -438.5 \times 0.2 - 884.4 \times 0.08 = -158.5\,\text{N m}.$$

The resultant bending moment diagram can be determined by calculating the resultant bending moments at each point.

$$|M_B| = \sqrt{(13.3)^2 + (-52.62)^2} = 54.27\,\text{Nm}.$$
$$|M_C| = \sqrt{(-9.81)^2 + (-158.5)^2} = 158.8\,\text{Nm}.$$

From Figure 5.32, it can be seen that the maximum bending moment occurs at the right hand bearing. The value of the resultant bending moment here should be used in the ASME design equation.

The next task is to determine the endurance limit of the shaft and the modifying factors. The shaft material is hot rolled steel and the endurance limit of the test specimen if unknown can be estimated using $\sigma'_e = 0.504\sigma_{uts}$. The ultimate tensile strength for 817M40 is

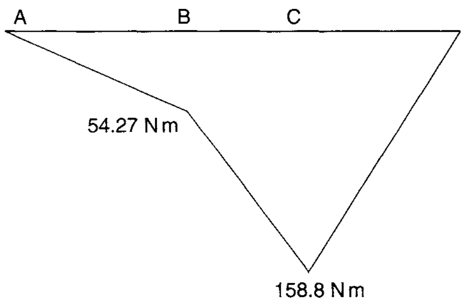

Figure 5.32 Resultant bending moment diagram.

1000 MPa and σ_y is 770 MPa. So $\sigma'_e = 0.504 \times 1000 = 504$ MPa.

The material is hot rolled, so from Eq. 5.14 and Table 5.3, $k_a = a\sigma^b_{uts} = 57.7(1000)^{-0.718} = 0.405$.

Assuming that the diameter will be about 30 mm, the size factor can be estimated using $k_b = (d/7.62)^{-0.1133} = (30/7.62)^{-0.1133} = 0.856$. If the shaft is significantly different in size to 30 mm, then the calculation should be repeated until convergence between the assumed and the final calculated value for the shaft diameter is achieved.

The desired nominal reliability is 90 per cent, so $k_c = 0.897$.

The operating temperature is not stated, so a value of temperature factor of $k_d = 1$ is assumed. The duty cycle factor is assumed to be $k_e = 1$.

The fillet radius at the shoulders is 3 mm. The ratio of diameters $D/d = (3 + 3 + 30)/30 = 36/30 = 1.2$. $r/d = 3/30 = 0.1$. So from Figure 5.23 the geometric stress concentration factor is $k_t = 1.65$. From Figure 5.22 the notch sensitivity index for a 1000 MPa strength material with notch radius of 3 mm is $q = 0.9$.

$$K_f = 1 + q(k_t - 1) = 1 + 0.9(1.65 - 1) = 1.59$$

$$k_f = 1/K_f = 0.629.$$

The miscellaneous factor is taken as $k_g = 1$.

The endurance limit can now be calculated from

$$\sigma_e = k_a k_b k_c k_d k_e k_f k_g \sigma'_e.$$

$$\sigma_e = 0.405 \times 0.856 \times 0.897 \times 1 \times 1 \times 0.629 \times 1 \times 504 = 98.6 \text{ MPa}.$$

As a well-known material has been selected subject to known loads the factor of safety can be taken as $n_s = 2$. The diameter can now be calculated from the ASME equation.

$$d = \left[\frac{32 n_s}{\pi}\sqrt{\left(\frac{M}{\sigma_e}\right)^2 + \frac{3}{4}\left(\frac{T}{\sigma_y}\right)^2}\right]^{1/3}$$

$$= \left[\frac{32 \times 2}{\pi}\sqrt{\left(\frac{158.8}{98.6 \times 10^6}\right)^2 + 0.75\left(\frac{84.9}{770 \times 10^6}\right)^2}\right]^{1/3}$$

$$= 0.032 \text{ m}$$

As this value is close to the assumed value used to evaluate the size and fatigue stress factors further iteration is not necessary. For manufacturing convenience, it may be necessary to modify this diameter to the nearest standard size as used within the company or taking into account materials readily available from suppliers, in this case 35 mm. Standard sizes can be found in texts such as the *Machinery's Handbook* (Schubert, 1982). An advantage of using standard sizes is that standard stock bearings can be selected to fit.

5.6 Conclusions

Shaft design involves consideration of the layout of features and components to be mounted on the shaft, specific dimensions and allowable tolerances, materials, deflection, frequency response, life and manufacturing constraints. This chapter has introduced the use of steps and shoulders and miscellaneous devices to locate components, methods to calculate the deflection of a shaft and its critical speeds, and a method to determine the minimum safe diameter for a shaft experiencing torque and bending for a given life.

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At the time of going to press the internet contains useful information relating to this chapter at the following sites.

www.allpowertrans.com
www.ameridrives.com
www.couplingcorp.com
www.heli-cal.com
www.magnaloy.com
www.mayrcorp.com
www.peterstubs.com
www.rw-america.com
www.servometer.com
www.zero-max.com

Nomenclature

Generally, preferred SI units have been stated.

a influence coefficient (m)
d diameter (m)
E Young's modulus (N/m^2)
F force (N)
g acceleration due to gravity (m^2/s)
I second moment of area (m^4)
k_a surface factor
k_b size factor
k_c reliability factor
k_d temperature factor
k_e duty cycle factor
k_f fatigue stress concentration factor
K_f component fatigue stress concentration factor

k_g miscellaneous effects factor
k_t geometric stress concentration factor
L length (m)
m mass (kg)
M moment (N m)
n_s factor of safety
q notch sensitivity
R reaction (N)
T torque (N m), tension (N)
U strain energy (J)
W_i mass (kg) or weight of node i (N)
x coordinate (m)
y static deflection (m)
y_i static deflection of W_i (m)
σ_e endurance limit of the item (N/m^2)
σ'_e endurance limit of test specimen (N/m^2)
σ_{uts} ultimate tensile strength (N/m^2)
σ_y yield strength (N/m^2)
ω frequency
ω_c critical speed (rad/s)

Worksheet

5.1 An air compressor consists of four compressor discs mounted on a steel shaft 50 mm apart as shown in Figure 5.33. The shaft is simply supported at either end. Each compressor wheel mass is 18 kg. Calculate the deflection at each wheel using Macaulay's method and calculate, using the Rayleigh–Ritz equation, the first critical frequency of the shaft. The shaft outer diameter is 0.05 m, the inner diameter is 0.03 m.

5.2 A 22.63 kg compressor impeller wheel is driven by a 13.56 kg turbine mounted on a common shaft (see Figure 5.34) manufactured from steel with Young's modulus $E = 207\,\text{GN/m}^2$. The design speed is 10 000 rpm. Determine the shaft diameter so that the first critical speed is 12 000 rpm giving a safety margin of 2000 rpm. Use Rayleigh's equation and assume rigid bearings and a massless shaft.

5.3 A research turbocharger shaft is simply supported by bearings as shown in Figure 5.35. Assuming there is no deflection at the bearings and that the shaft mass is negligible, calculate the first two critical frequencies of the shaft. The compressor mass is 3 kg, the turbine mass 2.5 kg. The steel shaft diameter is 0.02 m. Take Young's modulus for steel as 200 GN/m^2.

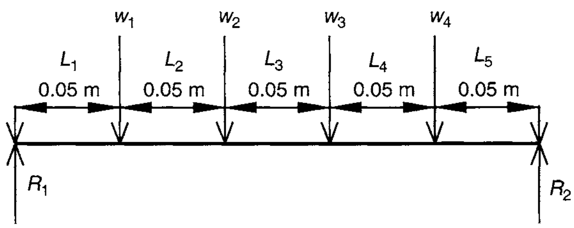

Figure 5.33 Air compressor shaft.

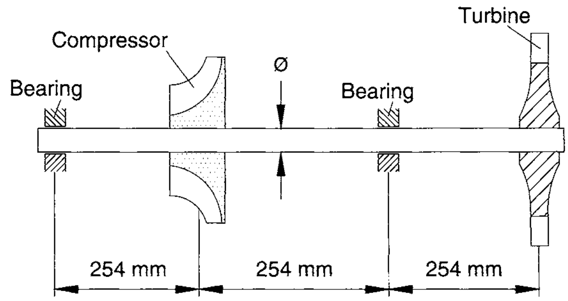

Figure 5.34 Turbine shaft.

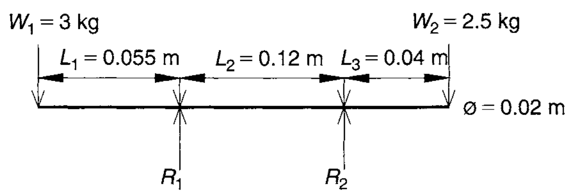

Figure 5.35 Turbocharger.

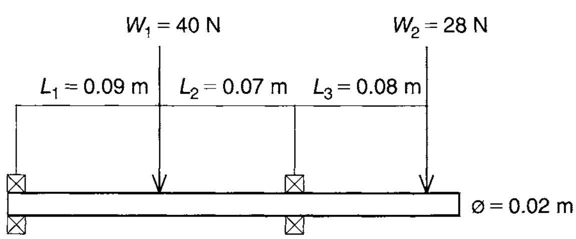

Figure 5.36 Turbocharger shaft.

5.4 As part of a preliminary design for a turbocharger determine the first two critical frequencies, stating any assumptions made, for the shaft arrangement shown in Figure 5.36. To assist with your calculation, a_{22} may be taken as 3.2594926×10^{-7} m. Young's modulus should be taken as 200 GN/m^2.

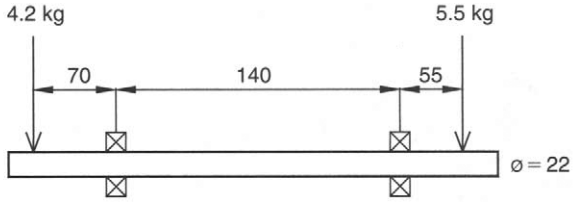

Figure 5.37 Turbocharger shaft (dimensions in mm).

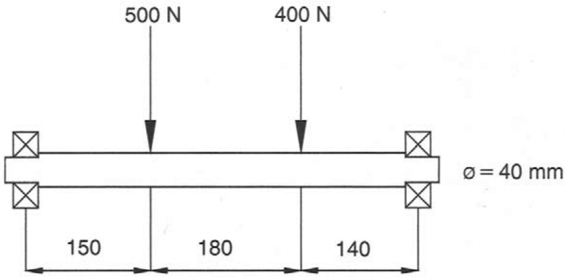

Figure 5.38 Shaft loading diagram.

5.5 As an initial step in the design of an experimental turbocharger, calculate the first two critical frequencies, stating any assumptions made, for the steel shaft shown in Figure 5.37. Young's modulus can be taken as 200 GN/m^2.

5.6 Determine the first two critical speeds for the shaft shown in Figure 5.38 stating and discussing the practical implications for any assumptions made. Young's modulus for steel should be taken as 200 GN/m^2.

5.7 Using Castigliano's theorem for calculating deflections, determine the critical frequency of the steel shaft shown in Figure 5.39. Neglect the shaft mass.

5.8 Determine the diameter of the drive shaft for a chain conveyor, which has the loading parameters illustrated in Figure 5.40. A roller chain sprocket of 500 mm pitch diameter, weighing 90 kg will be mid-mounted between two bearings. A 400 mm, 125 kg roller chain sprocket will be mounted overhung. The drive shaft is to be manufactured using cold drawn 070M20 steel. Operating temperatures are not expected to exceed 65°C and the operating environment is non-corrosive. The shaft is to be designed for non-limited life of greater than 10^8 cycles with a 90 per cent survival rate. The shaft will carry a steady driving torque of 1600 N m and rotate at 36 rpm. A sled runner keyway will be used for the overhung pulley and a profile keyway for the mid-mounted pulley (example adapted from ASME B106.1 M, 1985).

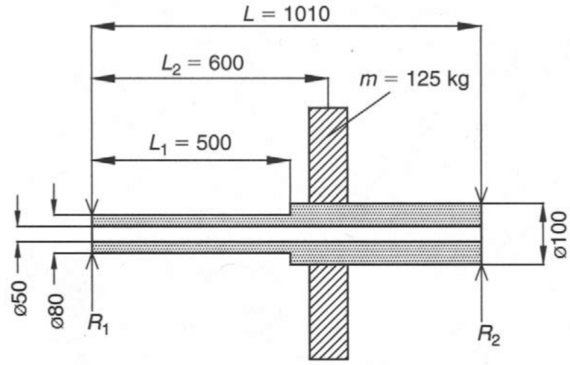

Figure 5.39 Simple stepped shaft and load (dimensions in mm).

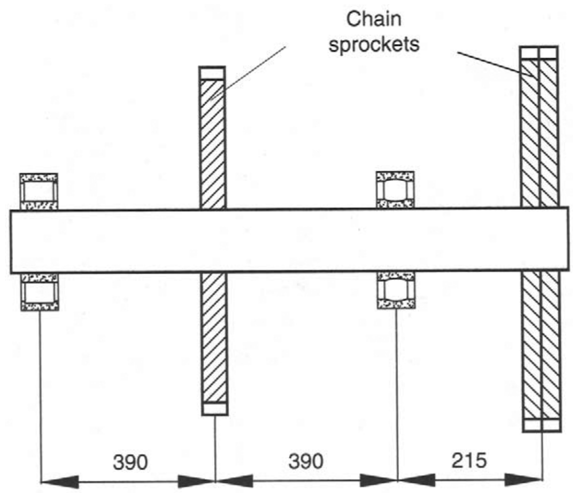

Figure 5.40 Chain conveyor drive shaft (dimensions in mm).

5.9 The geometry and loading for the drive shaft of a snow-track mobile is given in Figure 5.41. Discuss and outline, with sketches as appropriate, the design decisions that should be taken in scheming out a general arrangement for the shaft (after Juvinall and Marshek, 1991).

5.10 A shaft is required for the chain drive illustrated in Figure 5.42. The power transmitted is 12 kW at 100 rpm. The masses of the sprockets are 5 and 16.3 kg, with respective diameters of 254.6 and 460.8 mm. The material selected for the shaft is quenched and drawn 817M40 with σ_{uts} = 850 MPa and σ_y = 700 MPa. The torque is

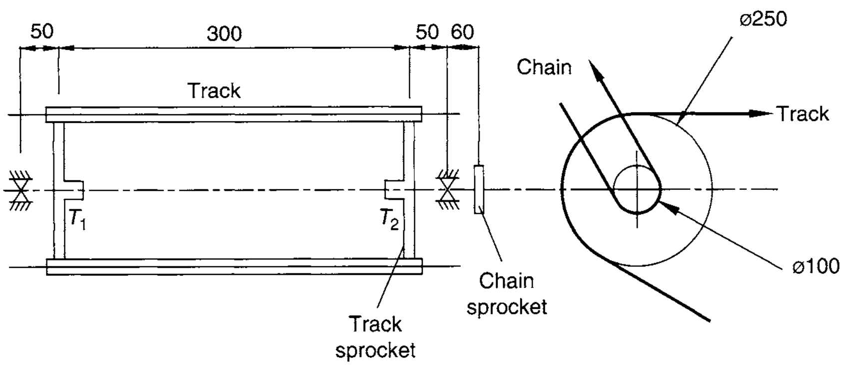

Figure 5.41 Schematic representation of the geometry and loading for a snow-track mobile drive shaft (dimensions in mm) (reproduced from Juvinall and Marshek, 1991).

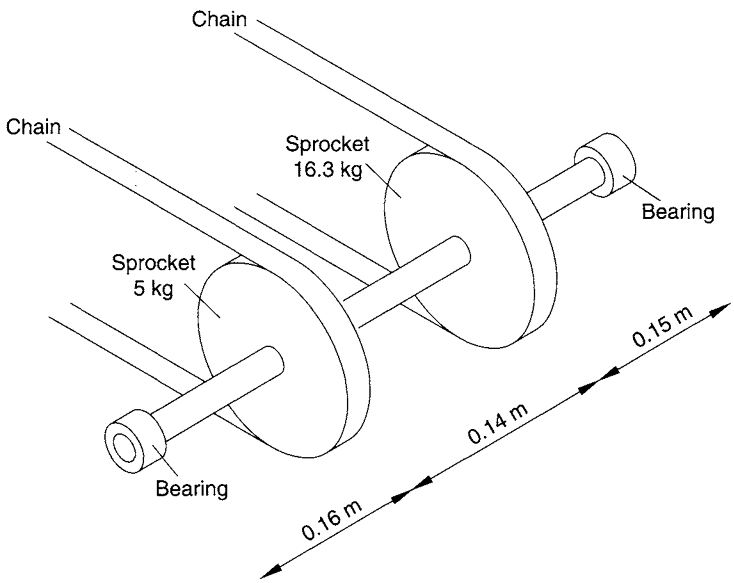

Figure 5.42 Chain drive.

transmitted between the shaft and sprockets by means of keys in profiled keyways. Determine the minimum diameter for the shaft based on the ASME equation for the design of transmission shafting if the desired reliability is 90 per cent. Use an initial estimate for the shaft diameter of 60 mm.

Answers

5.1 $y_{w1} = 1.929 \times 10^{-6}$m, $y_{w2} = 3.100 \times 10^{-6}$m, $y_{w3} = 3.100 \times 10^{-6}$m, $y_{w4} = 1.929 \times 10^{-6}$m, $\omega = 1923.8$ rad/s = 18 371 rpm.

5.2 $d = 67$ mm.

5.3 $\omega_1 = 1660$ rad/s, $\omega_2 = 3017$ rad/s.

5.4 $\omega_1 = 985.3$ rad/s, $\omega_2 = 2949$ rad/s.

5.5 $\omega_1 = 1153$ rad/s, $\omega_2 = 1703$ rad/s.

5.6 $\omega_1 = 1521.6$ rad/s, $\omega_2 = 432.8$ rad/s.

5.7 $\omega_c = 482.3$ rad/s.

5.8 74 mm.

5.9 No unique solution.

5.10 58 mm.

Learning objectives achievement

Can you identify different methods for mounting and locating components on shafts?	☐	✔ ↓	✘ → Section 5.2
Can you select an appropriate method for mounting and locating given components on a shaft?	☐	✔ ↓	✘ → Section 5.2
Can you produce a scheme of a shaft design to locate and mount standard machine elements?	☐	✔ ↓	✘ → Section 5.1
Can you determine the deflection of a shaft?	☐	✔ ↓	✘ → Sections 5.4.1, 5.4.2
Can you calculate the first two critical frequencies of a shaft?	☐	✔ ↓	✘ → Section 5.4
Can you determine the diameter for a shaft using the ASME equation for the design of transmission shafting?	☐	✔ ↓	✘ → Section 5.5