Chapter 3

SYSTEMS OF LINEAR EQUATIONS

3.1 Systems of Linear Equations

Consider the system of linear equations in n different variables

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \dots & + & a_{nn}x_n & = & b_n \end{array} \qquad (*)$$

Using multiplication of matrices , the above system of linear equations can be written as : $\mathbf{A}\ \mathbf{X} = \mathbf{B}$

where $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$, $\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

A is called the coefficients matrix
X is called the column vector of variables (or column vector of the unknowns)
B is called the column vector of constants (or column vector of the resultants)

Theorem : The system of linear equations (*) has a solution if $\det(\mathbf{A}) \neq 0$.

This chapter presents three metods of solving the system of linear equations (*), the first method is Cramer's rule , the second is Gauss elimination method , and the third is Gauss-Jordan method .

3.2 Cramer's rule

Consider the system of linear equations in n different variables

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \dots & + & a_{nn}x_n & = & b_n \end{array} \quad (*)$$

$$\mathbf{A}\,\mathbf{X} = \mathbf{B}$$

where $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$, $\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

If $\det(\mathbf{A}) \neq 0$ then the solution of the system (*) is given by

$x_i = \dfrac{\det(\mathbf{A}_i)}{\det(\mathbf{A})}$ for every $i = 1, 2, \cdots, n$.

Where $\mathbf{A}_i$ is the matrix formed by replacing the i^{th} column of $\mathbf{A}$ by the column vector of constants.

$\mathbf{A}_1 = \begin{pmatrix} b_1 & a_{12} & \dots & a_{1n} \\ b_2 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & \dots & a_{nn} \end{pmatrix}$, $\mathbf{A}_2 = \begin{pmatrix} a_{11} & b_1 & \dots & a_{1n} \\ a_{21} & b_2 & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & b_n & \dots & a_{nn} \end{pmatrix}$

$\mathbf{A}_n = \begin{pmatrix} a_{11} & a_{12} & \dots & b_1 \\ a_{21} & a_{22} & \dots & b_2 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & b_n \end{pmatrix}$

Example 1: Use Cramer's rule to solve the system of linear equations

$$\begin{array}{ccccc} 2x & + & 3y & = & 7 \\ -x & + & y & = & 4 \end{array}$$

Solution : In this system of linear equations

$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}$, $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$

$\det(\mathbf{A}) = \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} = (2 \times 1) - (-1 \times 3) = 2 - (-3) = 2 + 3 = 5$

$\mathbf{A}_1 = \begin{pmatrix} 7 & 3 \\ 4 & 1 \end{pmatrix} \implies \det(\mathbf{A}_1) = 7 - 12 = -5$

$\mathbf{A}_2 = \begin{pmatrix} 2 & 7 \\ -1 & 4 \end{pmatrix} \implies \det(\mathbf{A}_2) = 8 - (-7) = 15$

$x = \dfrac{\det(\mathbf{A}_1)}{\det(\mathbf{A})} = \dfrac{-5}{5} = -1$ and $y = \dfrac{\det(\mathbf{A}_2)}{\det(\mathbf{A})} = \dfrac{15}{5} = 3$

The solution of the system of linear equations is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

Example 2: Use Cramer's rule to solve the system of linear equations

$$\begin{array}{ccccccc} 2x & + & y & + & z & = & 3 \\ 4x & + & y & - & z & = & -2 \\ 2x & - & 2y & + & z & = & 6 \end{array}$$

Solution : In this system of linear equations

$\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 1 & -1 \\ 2 & -2 & 1 \end{pmatrix}$, $\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix}$

To calculate $\det(\mathbf{A})$:

$$\begin{array}{ccccc} 2 & 1 & 1 & 2 & 1 \\ 4 & 1 & -1 & 4 & 1 \\ 2 & -2 & 1 & 2 & -2 \end{array}$$

$\det(\mathbf{A}) = (2 - 2 - 8) - (2 + 4 + 4) = -8 - 10 = -18$

$\mathbf{A}_1 = \begin{pmatrix} 3 & 1 & 1 \\ -2 & 1 & -1 \\ 6 & -2 & 1 \end{pmatrix}$

To calculate $\det(\mathbf{A}_1)$:

$$\begin{array}{ccccc} 3 & 1 & 1 & 3 & 1 \\ -2 & 1 & -1 & -2 & 1 \\ 6 & -2 & 1 & 6 & -2 \end{array}$$

$\det(\mathbf{A}_1) = (3 - 6 + 4) - (6 + 6 - 2) = 1 - 10 = -9$

$\mathbf{A}_2 = \begin{pmatrix} 2 & 3 & 1 \\ 4 & -2 & -1 \\ 2 & 6 & 1 \end{pmatrix}$

To calculate $\det(\mathbf{A}_2)$:

$$\begin{array}{ccccc} 2 & 3 & 1 & 2 & 3 \\ 4 & -2 & -1 & 4 & -2 \\ 2 & 6 & 1 & 2 & 6 \end{array}$$

$\det(\mathbf{A}_2) = (-4 - 6 + 24) - (-4 - 12 + 12) = 14 + 4 = 18$

$\mathbf{A}_3 = \begin{pmatrix} 2 & 1 & 3 \\ 4 & 1 & -2 \\ 2 & -2 & 6 \end{pmatrix}$

To calculate $\det(\mathbf{A}_3)$:

$$\begin{array}{ccccc} 2 & 1 & 3 & 2 & 1 \\ 4 & 1 & -2 & 4 & 1 \\ 2 & -2 & 6 & 2 & -2 \end{array}$$

$\det(\mathbf{A}_3) = (12 - 4 - 24) - (6 + 8 + 24) = 1 - 10 = -16 - 38 = -54$

$$x = \frac{\det(\mathbf{A}_1)}{\det(\mathbf{A})} = \frac{-9}{-18} = \frac{1}{2}$$

$$y = \frac{\det(\mathbf{A}_2)}{\det(\mathbf{A})} = \frac{18}{-18} = -1$$

$$z = \frac{\det(\mathbf{A}_3)}{\det(\mathbf{A})} = \frac{-54}{-18} = 3$$

The solution of the system of linear equations is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -1 \\ 3 \end{pmatrix}$

3.3 Gauss elimination method

Consider the system of linear equations in n different variables

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \dots & + & a_{nn}x_n & = & b_n \end{array} \quad (*)$$

$$\mathbf{A}\ \mathbf{X} = \mathbf{B}$$

where $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$, $\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

To solve the system of linear equations (*) by Gauss elimination method :

1. Construct the augmented matrix $[\mathbf{A}|\mathbf{B}]$

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{array}\right)$$

2. Use **elementary row operations** on the augmented matrix to transform the matrix **A** to an upper triangular matrix with leading coeficient of each row equals 1.

   (Note: the leading coefficient of a row is the leftmost non-zero element of that row).

$$\left(\begin{array}{cccccc|c} 1 & c_{12} & c_{13} & c_{14} & \dots & c_{1n} & d_1 \\ 0 & 1 & c_{23} & c_{24} & \dots & a_{2n} & d_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & c_{(n-1)n} & d_{n-1} \\ 0 & 0 & 0 & \dots & 0 & 1 & d_n \end{array}\right)$$

3. From the last augmented matrix , $x_n = d_n$ and the rest of the unknowns can be calculated by backward substitution.

Example 1: Use Gauss elimination method to solve the system

$$\begin{array}{ccccccc} x & - & 2y & + & z & = & 4 \\ -x & + & 2y & + & z & = & -2 \\ 4x & - & 3y & - & z & = & -4 \end{array}$$

Solution : The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ -1 & 2 & 1 & -2 \\ 4 & -3 & -1 & -4 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ -1 & 2 & 1 & -2 \\ 4 & -3 & -1 & -4 \end{array}\right) \xrightarrow{R_1+R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 0 & 2 & 2 \\ 4 & -3 & -1 & -4 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 4 & -3 & -1 & -4 \\ 0 & 0 & 2 & 2 \end{array}\right) \xrightarrow{-4R_1+R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 5 & -5 & -20 \\ 0 & 0 & 2 & 2 \end{array}\right)$$

$$\xrightarrow{\frac{1}{5}R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 2 & 2 \end{array}\right) \xrightarrow{\frac{1}{2}R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Therefore, $z = 1$.
$y - z = -4 \Rightarrow y - 1 = -4 \Rightarrow y = -4 + 1 = -3$
$x - 2y + z = 4 \Rightarrow x - 2(-3) + 1 = 4 \Rightarrow x + 6 + 1 = 4 \Rightarrow x = 4 - 7 = -3$

The solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \\ 1 \end{pmatrix}$

Example 2: Use Gauss elimination method to solve the system

$$\begin{array}{ccccccccc} 2x & - & y & + & z & + & 3w & = & 8 \\ x & + & 3y & + & 2z & - & w & = & -2 \\ 3x & + & y & - & z & - & 2w & = & 3 \\ x & + & y & + & z & - & w & = & 0 \end{array}$$

Solution : The augmented matrix is

$$\left(\begin{array}{cccc|c} 2 & -1 & 1 & 3 & 8 \\ 1 & 3 & 2 & -1 & -2 \\ 3 & 1 & -1 & -2 & 3 \\ 1 & 1 & 1 & -1 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 2 & -1 & 1 & 3 & 8 \\ 1 & 3 & 2 & -1 & -2 \\ 3 & 1 & -1 & -2 & 3 \\ 1 & 1 & 1 & -1 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 1 & 3 & 2 & -1 & -2 \\ 3 & 1 & -1 & -2 & 3 \\ 2 & -1 & 1 & 3 & 8 \end{array}\right)$$

$$\xrightarrow{-R_1+R_2} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 3 & 1 & -1 & -2 & 3 \\ 2 & -1 & 1 & 3 & 8 \end{array}\right) \xrightarrow{-3R_1+R_3} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & -2 & -4 & 1 & 3 \\ 2 & -1 & 1 & 3 & 8 \end{array}\right)$$

$$\xrightarrow{-2R_1+R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & -2 & -4 & 1 & 3 \\ 0 & -3 & -1 & 5 & 8 \end{array}\right) \xrightarrow{R_2+R_3} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & -3 & -1 & 5 & 8 \end{array}\right)$$

$$\xrightarrow{2R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & -6 & -2 & 10 & 16 \end{array}\right) \xrightarrow{3R_2+R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & 0 & 1 & 10 & 10 \end{array}\right)$$

$$\xrightarrow{3R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & 0 & 3 & 30 & 30 \end{array}\right) \xrightarrow{R_3+R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & 0 & 0 & 31 & 31 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}R_2} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & -1 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & 0 & 0 & 31 & 31 \end{array}\right) \xrightarrow{-\frac{1}{3}R_3} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & -1 \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 31 & 31 \end{array}\right)$$

$$\xrightarrow{\frac{1}{31}R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & -1 \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

Therefor, $w = 1$

$z - \dfrac{1}{3}w = -\dfrac{1}{3} \Rightarrow z - \dfrac{1}{3} = -\dfrac{1}{3} \Rightarrow z = 0$

$y + \dfrac{1}{2}z = -1 \Rightarrow y + \dfrac{1}{2}(0) = -1 \Rightarrow y = -1$

$x + y + z - w = 0 \Rightarrow x - 1 + 0 - 1 = 0 \Rightarrow x = 2$

The solution is $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}$

3.4 Gauss-Jordan method

Consider the system of linear equations in n different variables

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \dots & + & a_{nn}x_n & = & b_n \end{array} \quad (*)$$

$$\mathbf{A}\ \mathbf{X} = \mathbf{B}$$

where $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$, $\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

To solve the system of linear equations (*) by Gauss-Jordan method :

1. Construct the augmented matrix $[\mathbf{A}|\mathbf{B}]$

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{array}\right)$$

2. Use elementary row operations on the augmented matrix to transform the matrix $\mathbf{A}$ to the identity matrix .

$$\left(\begin{array}{ccccc|c} 1 & 0 & \dots & 0 & 0 & d_1 \\ 0 & 1 & \dots & 0 & 0 & d_2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & d_{n-1} \\ 0 & 0 & \dots & 0 & 1 & d_n \end{array}\right)$$

3. From the last augmented matrix , $x_i = d_i$ for every $i = 1, 2, \cdots, n$

Example 1: Use Gauss-Jordan method to solve the system

$$\begin{array}{ccccccc} x & + & y & + & z & = & 2 \\ x & - & y & + & 2z & = & 0 \\ 2x & & & + & z & = & 2 \end{array}$$

Solution : The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 2 & 0 \\ 2 & 0 & 1 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 2 & 0 \\ 2 & 0 & 1 & 2 \end{array}\right) \xrightarrow{-R_1+R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & -2 \\ 2 & 0 & 1 & 2 \end{array}\right)$$

$$\xrightarrow{-2R_1+R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & -2 \\ 0 & -2 & -1 & -2 \end{array}\right) \xrightarrow{-R_2+R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & -2 \\ 0 & 0 & -2 & 0 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{2}R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & -2 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{-R_3+R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{-R_3+R_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{-\frac{1}{2}R_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{-R_2+R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Therefore , $x = 1$, $y = 1$ and $z = 0$.

The solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Example 2: Use Gauss-Jordan method to solve the system

$$\begin{array}{ccccccccc} 2x & - & y & + & z & + & 3w & = & 8 \\ x & + & 3y & + & 2z & - & w & = & -2 \\ 3x & + & y & - & z & - & 2w & = & 3 \\ x & + & y & + & z & - & w & = & 0 \end{array}$$

Solution : (Note : This is example 2 in Gauss elimination method)
The augmented matrix is

$$\left(\begin{array}{cccc|c} 2 & -1 & 1 & 3 & 8 \\ 1 & 3 & 2 & -1 & -2 \\ 3 & 1 & -1 & -2 & 3 \\ 1 & 1 & 1 & -1 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 2 & -1 & 1 & 3 & 8 \\ 1 & 3 & 2 & -1 & -2 \\ 3 & 1 & -1 & -2 & 3 \\ 1 & 1 & 1 & -1 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 1 & 3 & 2 & -1 & -2 \\ 3 & 1 & -1 & -2 & 3 \\ 2 & -1 & 1 & 3 & 8 \end{array}\right)$$

$$\xrightarrow{-R_1+R_2} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 3 & 1 & -1 & -2 & 3 \\ 2 & -1 & 1 & 3 & 8 \end{array}\right) \xrightarrow{-3R_1+R_3} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & -2 & -4 & 1 & 3 \\ 2 & -1 & 1 & 3 & 8 \end{array}\right)$$

$$\xrightarrow{-2R_1+R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & -2 & -4 & 1 & 3 \\ 0 & -3 & -1 & 5 & 8 \end{array}\right) \xrightarrow{R_2+R_3} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & -3 & -1 & 5 & 8 \end{array}\right)$$

$$\xrightarrow{2R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & -6 & -2 & 10 & 16 \end{array}\right) \xrightarrow{3R_2+R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & 0 & 1 & 10 & 10 \end{array}\right)$$

$$\xrightarrow{3R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & 0 & 3 & 30 & 30 \end{array}\right) \xrightarrow{R_3+R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & 0 & 0 & 31 & 31 \end{array}\right)$$

$$\xrightarrow{\frac{1}{31}R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{-R_4+R_3} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{R_4+R_1} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{-\frac{1}{3}R_3} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{-R_3+R_2} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{-R_3+R_1} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}R_2} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{-R_2+R_1} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

Therefore, $x = 2$, $y = -1$, $z = 0$ and $w = 1$.

The solution is $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}$