1 Eigenvalues and Eigenvectors

The product $\mathbf{Ax}$ of a matrix $\mathbf{A} \in \mathcal{M}_{n\times n}(\mathbb{R})$ and an n-vector $\mathbf{x}$ is itself an n-vector. Of particular interest in many settings (of which differential equations is one) is the following question:

> For a given matrix $\mathbf{A}$, what are the vectors $\mathbf{x}$ for which the product $\mathbf{Ax}$ is a scalar multiple of $\mathbf{x}$? That is, what vectors $\mathbf{x}$ satisfy the equation
>
> $$\mathbf{Ax} = \lambda\mathbf{x}$$
>
> for some scalar λ?

It should immediately be clear that, no matter what $\mathbf{A}$ and λ are, the vector $\mathbf{x} = \mathbf{0}$ (that is, the vector whose elements are all zero) satisfies this equation. With such a trivial answer, we might ask the question again in another way:

> For a given matrix $\mathbf{A}$, what are the *nonzero* vectors $\mathbf{x}$ that satisfy the equation
>
> $$\mathbf{Ax} = \lambda\mathbf{x}$$
>
> for some scalar λ?

To answer this question, we first perform some algebraic manipulations upon the equation $\mathbf{Ax} = \lambda\mathbf{x}$. We note first that, if $\mathbf{I} = \mathbf{I_n}$ (the $n \times n$ multiplicative identity in $\mathcal{M}_{n\times n}(\mathbb{R})$), then we can write

$$\begin{aligned} \mathbf{Ax} = \lambda\mathbf{x} \quad &\Leftrightarrow \quad \mathbf{Ax} - \lambda\mathbf{x} = \mathbf{0} \\ &\Leftrightarrow \quad \mathbf{Ax} - \lambda\mathbf{Ix} = \mathbf{0} \\ &\Leftrightarrow \quad (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}. \end{aligned}$$

Remember that we are looking for nonzero $\mathbf{x}$ that satisfy this last equation. But $\mathbf{A} - \lambda\mathbf{I}$ is an $n \times n$ matrix and, should its determinant be nonzero, this last equation will have exactly one solution, namely $\mathbf{x} = \mathbf{0}$. Thus our question above has the following answer:

> The equation $\mathbf{Ax} = \lambda\mathbf{x}$ has nonzero solutions for the vector x if and only if the matrix $\mathbf{A} - \lambda\mathbf{I}$ has zero determinant.

As we will see in the examples below, for a given matrix $\mathbf{A}$ there are only a few special values of the scalar λ for which $\mathbf{A} - \lambda\mathbf{I}$ will have zero determinant, and these special values are called the *eigenvalues* of the matrix $\mathbf{A}$. Based upon the answer to our question, it seems we must first be able to find the eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$ of $\mathbf{A}$ and then see about solving the individual equations $\mathbf{Ax} = \lambda_i\mathbf{x}$ for each $i = 1, \ldots, n$.

Example: Find the eigenvalues of the matrix $\mathbf{A} = \begin{bmatrix} 2 & 2 \\ 5 & -1 \end{bmatrix}$.

The eigenvalues are those λ for which $\det(\mathbf{A}-\lambda\mathbf{I})=0$. Now

$$\begin{aligned}
\det(\mathbf{A}-\lambda\mathbf{I}) &= \det\left(\begin{bmatrix} 2 & 2 \\ 5 & -1 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} 2 & 2 \\ 5 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) \\
&= \begin{vmatrix} 2-\lambda & 2 \\ 5 & -1-\lambda \end{vmatrix} \\
&= (2-\lambda)(-1-\lambda) - 10 \\
&= \lambda^2 - \lambda - 12.
\end{aligned}$$

The eigenvalues of $\mathbf{A}$ are the solutions of the quadratic equation $\lambda^2-\lambda-12=0$, namely $\lambda_1=-3$ and $\lambda_2=4$.

As we have discussed, if $\det(\mathbf{A}-\lambda\mathbf{I})=0$ then the equation $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{b}$ has either no solutions or infinitely many. When we take $\mathbf{b}=\mathbf{0}$ however, it is clear by the existence of the solution $\mathbf{x}=\mathbf{0}$ that there are infinitely many solutions (i.e., we may rule out the "no solution" case). If we continue using the matrix $\mathbf{A}$ from the example above, we can expect nonzero solutions $\mathbf{x}$ (infinitely many of them, in fact) of the equation $\mathbf{A}\mathbf{x}=\lambda\mathbf{x}$ precisely when $\lambda=-3$ or $\lambda=4$. Let us procede to characterize such solutions.

First, we work with $\lambda=-3$. The equation $\mathbf{A}\mathbf{x}=\lambda\mathbf{x}$ becomes $\mathbf{A}\mathbf{x}=-3\mathbf{x}$. Writing

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and using the matrix $\mathbf{A}$ from above, we have

$$\mathbf{A}\mathbf{x} = \begin{bmatrix} 2 & 2 \\ 5 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1+2x_2 \\ 5x_1-x_2 \end{bmatrix},$$

while

$$-3\mathbf{x} = \begin{bmatrix} -3x_1 \\ -3x_2 \end{bmatrix}.$$

Setting these equal, we get

$$\begin{aligned}
\begin{bmatrix} 2x_1+2x_2 \\ 5x_1-x_2 \end{bmatrix} = \begin{bmatrix} -3x_1 \\ -3x_2 \end{bmatrix} &\Rightarrow 2x_1+2x_2=-3x_1 \qquad \text{and} \qquad 5x_1-x_2=-3x_2 \\
&\Rightarrow 5x_1=-2x_2 \\
&\Rightarrow x_1=-\frac{2}{5}x_2.
\end{aligned}$$

This means that, while there are infinitely many nonzero solutions (solution vectors) of the equation $\mathbf{A}\mathbf{x}=-3\mathbf{x}$, they all satisfy the condition that the first entry x_1 is $-2/5$ times the second entry x_2. Thus all solutions of this equation can be characterized by

$$\begin{bmatrix} 2t \\ -5t \end{bmatrix} = t\begin{bmatrix} 2 \\ -5 \end{bmatrix},$$

where t is any real number. The nonzero vectors $\mathbf{x}$ that satisfy $\mathbf{Ax} = -3\mathbf{x}$ are called *eigenvectors* associated with the eigenvalue $\lambda = -3$. One such eigenvector is

$$\mathbf{u_1} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

and all other eigenvectors corresponding to the eigenvalue (-3) are simply scalar multiples of $\mathbf{u_1}$ — that is, $\mathbf{u_1}$ spans this set of eigenvectors.

Similarly, we can find eigenvectors associated with the eigenvalue $\lambda = 4$ by solving $\mathbf{Ax} = 4\mathbf{x}$:

$$\begin{bmatrix} 2x_1 + 2x_2 \\ 5x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 4x_2 \end{bmatrix} \quad \Rightarrow \quad 2x_1 + 2x_2 = 4x_1 \quad \text{and} \quad 5x_1 - x_2 = 4x_2$$
$$\Rightarrow \quad x_1 = x_2.$$

Hence the set of eigenvectors associated with $\lambda = 4$ is spanned by

$$\mathbf{u_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Example: Find the eigenvalues and associated eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 7 & 0 & -3 \\ -9 & -2 & 3 \\ 18 & 0 & -8 \end{bmatrix}.$$

First we compute $\det(\mathbf{A} - \lambda\mathbf{I})$ via a cofactor expansion along the second column:

$$\begin{aligned} \begin{vmatrix} 7-\lambda & 0 & -3 \\ -9 & -2-\lambda & 3 \\ 18 & 0 & -8-\lambda \end{vmatrix} &= (-2-\lambda)(-1)^4 \begin{vmatrix} 7-\lambda & -3 \\ 18 & -8-\lambda \end{vmatrix} \\ &= -(2+\lambda)[(7-\lambda)(-8-\lambda) + 54] \\ &= -(\lambda+2)(\lambda^2 + \lambda - 2) \\ &= -(\lambda+2)^2(\lambda - 1). \end{aligned}$$

Thus $\mathbf{A}$ has two distinct eigenvalues, $\lambda_1 = -2$ and $\lambda_3 = 1$. (Note that we might say $\lambda_2 = -2$, since, as a root, -2 has multiplicity two. This is why we labelled the eigenvalue 1 as λ_3.)

Now, to find the associated eigenvectors, we solve the equation $(\mathbf{A} - \lambda_j\mathbf{I})\mathbf{x} = \mathbf{0}$ for $j = 1, 2, 3$. Using the eigenvalue $\lambda_3 = 1$, we have

$$\begin{aligned} (\mathbf{A} - \mathbf{I})\mathbf{x} &= \begin{bmatrix} 6x_1 - 3x_3 \\ -9x_1 - 3x_2 + 3x_3 \\ 18x_1 - 9x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ &\Rightarrow \quad x_3 = 2x_1 \quad \text{and} \quad x_2 = x_3 - 3x_1 \\ &\Rightarrow \quad x_3 = 2x_1 \quad \text{and} \quad x_2 = -x_1. \end{aligned}$$

So the eigenvectors associated with $\lambda_3 = 1$ are all scalar multiples of

$$\mathbf{u_3} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

Now, to find eigenvectors associated with $\lambda_1 = -2$ we solve $(\mathbf{A} + 2\mathbf{I})\mathbf{x} = \mathbf{0}$. We have

$$\begin{aligned} (\mathbf{A} + 2\mathbf{I})\mathbf{x} &= \begin{bmatrix} 9x_1 - 3x_3 \\ -9x_1 + 3x_3 \\ 18x_1 - 6x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ &\Rightarrow \quad x_3 = 3x_1. \end{aligned}$$

Something different happened here in that we acquired no information about x_2. In fact, we have found that x_2 can be chosen arbitrarily, and independently of x_1 and x_3 (whereas x_3 cannot be chosen independently of x_1). This allows us to choose two linearly independent eigenvectors associated with the eigenvalue $\lambda = -2$, such as $\mathbf{u_1} = (1, 0, 3)$ and $\mathbf{u_2} = (1, 1, 3)$. It is a fact that all other eigenvectors associated with $\lambda_2 = -2$ are in the span of these two; that is, all others can be written as linear combinations $c_1\mathbf{u_1} + c_2\mathbf{u_2}$ using an appropriate choices of the constants c_1 and c_2.

Example: Find the eigenvalues and associated eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}.$$

We compute

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} -1 - \lambda & 2 \\ 0 & -1 - \lambda \end{vmatrix} \\ &= (\lambda + 1)^2. \end{aligned}$$

Setting this equal to zero we get that $\lambda = -1$ is a (repeated) eigenvalue. To find any associated eigenvectors we must solve for $\mathbf{x} = (x_1, x_2)$ so that $(\mathbf{A} + \mathbf{I})\mathbf{x} = \mathbf{0}$; that is,

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad x_2 = 0.$$

Thus, the eigenvectors corresponding to the eigenvalue $\lambda = -1$ are the vectors whose second component is zero, which means that we are talking about all scalar multiples of $\mathbf{u} = (1, 0)$.

Notice that our work above shows that there are no eigenvectors associated with $\lambda = -1$ which are linearly independent of $\mathbf{u}$. This may go against your intuition based upon the results of the example before this one, where an eigenvalue of multiplicity two had two linearly independent associated eigenvectors. Nevertheless, it is a (somewhat disparaging) fact that eigenvalues can have fewer linearly independent eigenvectors than their multiplicity suggests.

Example: Find the eigenvalues and associated eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}.$$

We compute

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} \\ &= (\lambda - 2)^2 + 1 \\ &= \lambda^2 - 4\lambda + 5. \end{aligned}$$

The roots of this polynomial are $\lambda_1 = 2+i$ and $\lambda_2 = 2-i$; that is, the eigenvalues are not real numbers. This is a common occurrence, and we can press on to find the eigenvectors just as we have in the past with real eigenvalues. To find eigenvectors associated with $\lambda_1 = 2+i$, we look for $\mathbf{x}$ satisfying

$$\begin{aligned} (\mathbf{A} - (2+i)\mathbf{I})\mathbf{x} = \mathbf{0} \quad &\Rightarrow \quad \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &\Rightarrow \quad \begin{bmatrix} -ix_1 - x_2 \\ x_1 - ix_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &\Rightarrow \quad x_1 = ix_2. \end{aligned}$$

Thus all eigenvectors associated with $\lambda_1 = 2+i$ are scalar multiples of $\mathbf{u_1} = (i, 1)$. Proceeding with $\lambda_2 = 2-i$, we have

$$\begin{aligned} (\mathbf{A} - (2-i)\mathbf{I})\mathbf{x} = \mathbf{0} \quad &\Rightarrow \quad \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &\Rightarrow \quad \begin{bmatrix} ix_1 - x_2 \\ x_1 + ix_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &\Rightarrow \quad x_1 = -ix_2, \end{aligned}$$

which shows all eigenvectors associated with $\lambda_2 = 2-i$ to be scalar multiples of $\mathbf{u_2} = (-i, 1)$.

Notice that $\mathbf{u_2}$, the eigenvector associated with the eigenvalue $\lambda_2 = 2-i$ in the last example, is the complex conjugate of $\mathbf{u_1}$, the eigenvector associated with the eigenvalue $\lambda_1 = 2+i$. It is indeed a fact that, if $\mathbf{A} \in \mathcal{M}_{n\times n}(\mathbb{R})$ has a nonreal eigenvalue $\lambda_1 = \lambda + i\mu$ with corresponding eigenvector ξ_1, then it also has eigenvalue $\lambda_2 = \lambda - i\mu$ with corresponding eigenvector $\xi_2 = \bar{\xi}_1$.