Random Variables & Expectation

Random Variable

A random variable (r.v.) is a well defined rule for assigning a numerical value to all possible outcomes of an experiment.

example:

experiment:	taking a course
outcomes:	grades A, B, C, D, F
sample space S:	discrete & finite
random variable:	Y = 4 if grade is A
	Y = 3 if grade is B
	Y = 2 if grade is C
	Y = 1 if grade is D
	Y = 0 if grade is F

Properties of Probability Distributions

1. $0 \leq Pr(X=x) \leq 1$ for all x

2. $$\sum_x p(x) = 1$$

The set of ordered pairs $(x, f(x))$ is a **probability function**, **probability mass function**, or **probability distribution** of the discrete random variable X if, for each possible outcome x,

1. $f(x) \geq 0$,
2. $\sum_x f(x) = 1$,
3. $P(X = x) = f(x)$.

Experiment: throw 2 dice
What are the possible outcomes?

1,1	2,1	3,1	4,1	5,1	6,1
1,2	2,2	3,2	4,2	5,2	6,2
1,3	2,3	3,3	4,3	5,3	6,3
1,4	2,4	3,4	4,4	5,4	6,4
1,5	2,5	3,5	4,5	5,5	6,5
1,6	2,6	3,6	4,6	5,6	6,6

Define the random variable X to be the sum of the dots on the 2 dice.

For which outcomes does X = 9

1,1	2,1	3,1	4,1	5,1	6,1
1,2	2,2	3,2	4,2	5,2	6,2
1,3	2,3	3,3	4,3	5,3	6,3
1,4	2,4	3,4	4,4	5,4	6,4
1,5	2,5	3,5	4,5	5,5	6,5
1,6	2,6	3,6	4,6	5,6	6,6

For which outcomes does X = 9

1,1	2,1	3,1	4,1	5,1	6,1
1,2	2,2	3,2	4,2	5,2	6,2
1,3	2,3	3,3	4,3	5,3	6,3
1,4	2,4	3,4	4,4	5,4	6,4
1,5	2,5	3,5	4,5	5,5	6,5
1,6	2,6	3,6	4,6	5,6	6,6

What is Pr(X=9)?

1,1	2,1	3,1	4,1	5,1	6,1
1,2	2,2	3,2	4,2	5,2	6,2
1,3	2,3	3,3	4,3	5,3	6,3
1,4	2,4	3,4	4,4	5,4	6,4
1,5	2,5	3,5	4,5	5,5	6,5
1,6	2,6	3,6	4,6	5,6	6,6

Since there are 36 equally likely outcomes, each has a probability of 1/36.

So since there are 4 outcomes that yield X=9, Pr(X=9) = 4/36 =1/9

Let's calculate the probabilities of all the possible values x of the random variable X

1,1	**2,1**	**3,1**	**4,1**	**5,1**	**6,1**
1,2	**2,2**	**3,2**	**4,2**	**5,2**	**6,2**
1,3	**2,3**	**3,3**	**4,3**	**5,3**	**6,3**
1,4	**2,4**	**3,4**	**4,4**	**5,4**	**6,4**
1,5	**2,5**	**3,5**	**4,5**	**5,5**	**6,5**
1,6	**2,6**	**3,6**	**4,6**	**5,6**	**6,6**

x	Pr(X=x)

Let's calculate the probabilities of the possible values x of the random variable X

1,1	**2,1**	**3,1**	**4,1**	**5,1**	**6,1**
1,2	**2,2**	**3,2**	**4,2**	**5,2**	**6,2**
1,3	**2,3**	**3,3**	**4,3**	**5,3**	**6,3**
1,4	**2,4**	**3,4**	**4,4**	**5,4**	**6,4**
1,5	**2,5**	**3,5**	**4,5**	**5,5**	**6,5**
1,6	**2,6**	**3,6**	**4,6**	**5,6**	**6,6**

x	Pr(X=x)
2	1/36

Let's calculate the probabilities of the possible values x of the random variable X

1,1	**2,1**	**3,1**	**4,1**	**5,1**	**6,1**
1,2	**2,2**	**3,2**	**4,2**	**5,2**	**6,2**
1,3	**2,3**	**3,3**	**4,3**	**5,3**	**6,3**
1,4	**2,4**	**3,4**	**4,4**	**5,4**	**6,4**
1,5	**2,5**	**3,5**	**4,5**	**5,5**	**6,5**
1,6	**2,6**	**3,6**	**4,6**	**5,6**	**6,6**

x	Pr(X=x)
2	1/36
3	2/36

Let's calculate the probabilities of the possible values x of the random variable X

1,1	**2,1**	**3,1**	**4,1**	**5,1**	**6,1**
1,2	**2,2**	**3,2**	**4,2**	**5,2**	**6,2**
1,3	**2,3**	**3,3**	**4,3**	**5,3**	**6,3**
1,4	**2,4**	**3,4**	**4,4**	**5,4**	**6,4**
1,5	**2,5**	**3,5**	**4,5**	**5,5**	**6,5**
1,6	**2,6**	**3,6**	**4,6**	**5,6**	**6,6**

x	Pr(X=x)
2	1/36
3	2/36
4	3/36

Let's calculate the probabilities of the possible values x of the random variable X

1,1	**2,1**	**3,1**	**4,1**	**5,1**	**6,1**
1,2	**2,2**	**3,2**	**4,2**	**5,2**	**6,2**
1,3	**2,3**	**3,3**	**4,3**	**5,3**	**6,3**
1,4	**2,4**	**3,4**	**4,4**	**5,4**	**6,4**
1,5	**2,5**	**3,5**	**4,5**	**5,5**	**6,5**
1,6	**2,6**	**3,6**	**4,6**	**5,6**	**6,6**

x	Pr(X=x)
2	1/36
3	2/36
4	3/36
5	4/36

Let's calculate the probabilities of the possible values x of the random variable X

1,1	**2,1**	**3,1**	**4,1**	**5,1**	**6,1**
1,2	**2,2**	**3,2**	**4,2**	**5,2**	**6,2**
1,3	**2,3**	**3,3**	**4,3**	**5,3**	**6,3**
1,4	**2,4**	**3,4**	**4,4**	**5,4**	**6,4**
1,5	**2,5**	**3,5**	**4,5**	**5,5**	**6,5**
1,6	**2,6**	**3,6**	**4,6**	**5,6**	**6,6**

x	Pr(X=x)
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36

Let's calculate the probabilities of the possible values x of the random variable X

1,1	**2,1**	**3,1**	**4,1**	**5,1**	**6,1**
1,2	**2,2**	**3,2**	**4,2**	**5,2**	**6,2**
1,3	**2,3**	**3,3**	**4,3**	**5,3**	**6,3**
1,4	**2,4**	**3,4**	**4,4**	**5,4**	**6,4**
1,5	**2,5**	**3,5**	**4,5**	**5,5**	**6,5**
1,6	**2,6**	**3,6**	**4,6**	**5,6**	**6,6**

x	Pr(X=x)
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36

Let's calculate the probabilities of the possible values x of the random variable X

1,1	**2,1**	**3,1**	**4,1**	**5,1**	**6,1**
1,2	**2,2**	**3,2**	**4,2**	**5,2**	**6,2**
1,3	**2,3**	**3,3**	**4,3**	**5,3**	**6,3**
1,4	**2,4**	**3,4**	**4,4**	**5,4**	**6,4**
1,5	**2,5**	**3,5**	**4,5**	**5,5**	**6,5**
1,6	**2,6**	**3,6**	**4,6**	**5,6**	**6,6**

x	Pr(X=x)
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36

Let's calculate the probabilities of the possible values x of the random variable X

1,1	**2,1**	**3,1**	**4,1**	**5,1**	**6,1**
1,2	**2,2**	**3,2**	**4,2**	**5,2**	**6,2**
1,3	**2,3**	**3,3**	**4,3**	**5,3**	**6,3**
1,4	**2,4**	**3,4**	**4,4**	**5,4**	**6,4**
1,5	**2,5**	**3,5**	**4,5**	**5,5**	**6,5**
1,6	**2,6**	**3,6**	**4,6**	**5,6**	**6,6**

x	Pr(X=x)
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36

Let's calculate the probabilities of the possible values x of the random variable X

1,1	**2,1**	**3,1**	**4,1**	**5,1**	**6,1**
1,2	**2,2**	**3,2**	**4,2**	**5,2**	**6,2**
1,3	**2,3**	**3,3**	**4,3**	**5,3**	**6,3**
1,4	**2,4**	**3,4**	**4,4**	**5,4**	**6,4**
1,5	**2,5**	**3,5**	**4,5**	**5,5**	**6,5**
1,6	**2,6**	**3,6**	**4,6**	**5,6**	**6,6**

x	Pr(X=x)
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36

Let's calculate the probabilities of the possible values x of the random variable X

1,1	**2,1**	**3,1**	**4,1**	**5,1**	**6,1**
1,2	**2,2**	**3,2**	**4,2**	**5,2**	**6,2**
1,3	**2,3**	**3,3**	**4,3**	**5,3**	**6,3**
1,4	**2,4**	**3,4**	**4,4**	**5,4**	**6,4**
1,5	**2,5**	**3,5**	**4,5**	**5,5**	**6,5**
1,6	**2,6**	**3,6**	**4,6**	**5,6**	**6,6**

x	Pr(X=x)
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36

Let's calculate the probabilities of the possible values x of the random variable X

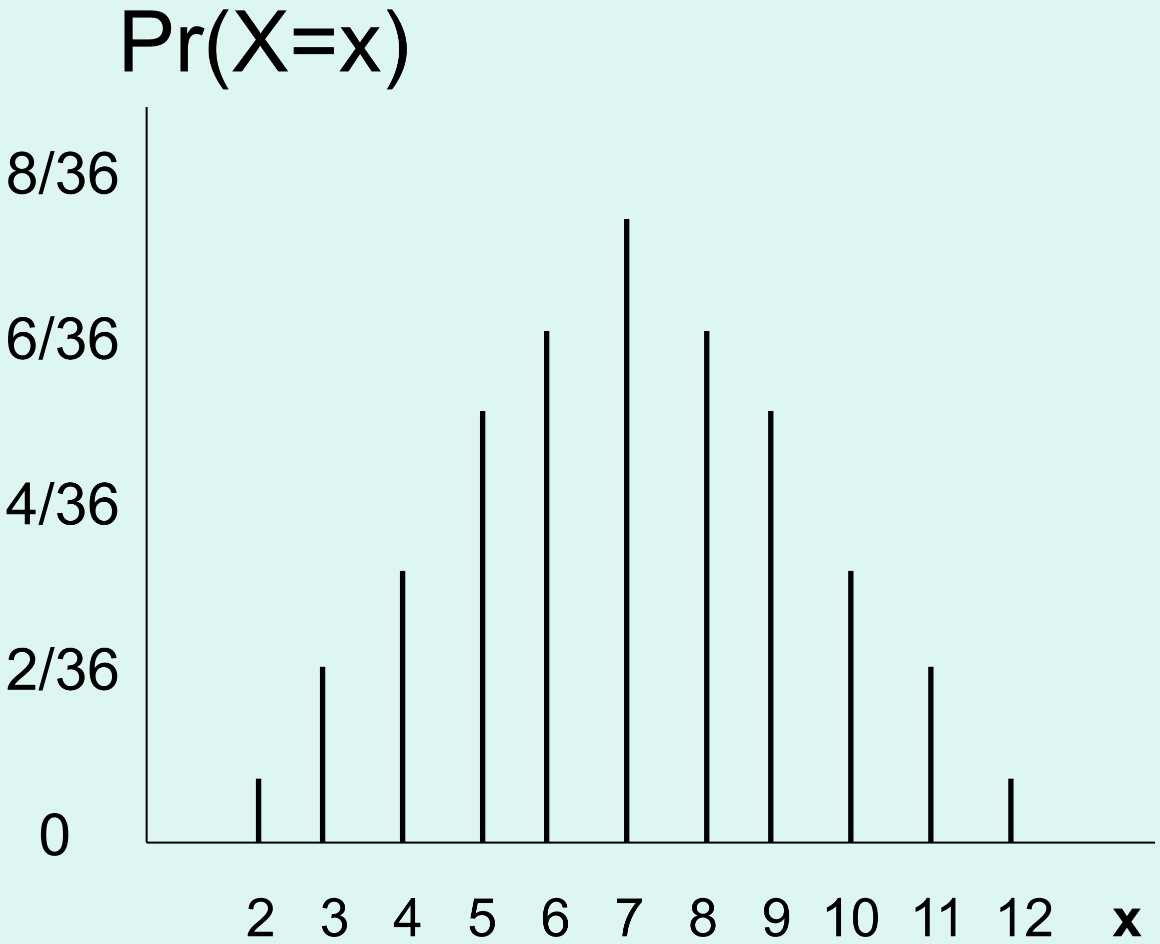
1,1	**2,1**	**3,1**	**4,1**	**5,1**	**6,1**
1,2	**2,2**	**3,2**	**4,2**	**5,2**	**6,2**
1,3	**2,3**	**3,3**	**4,3**	**5,3**	**6,3**
1,4	**2,4**	**3,4**	**4,4**	**5,4**	**6,4**
1,5	**2,5**	**3,5**	**4,5**	**5,5**	**6,5**
1,6	**2,6**	**3,6**	**4,6**	**5,6**	**6,6**

x	Pr(X=x)
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36

X	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**SUM**
Pr(x)	**1/36**	**2/36**	**3/36**	**4/36**	**5/36**	**6/36**	**5/36**	**4/36**	**3/36**	**2/36**	**1/36**	**1**

Let's graph the probability distribution of X.

x	Pr(X=x)
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36

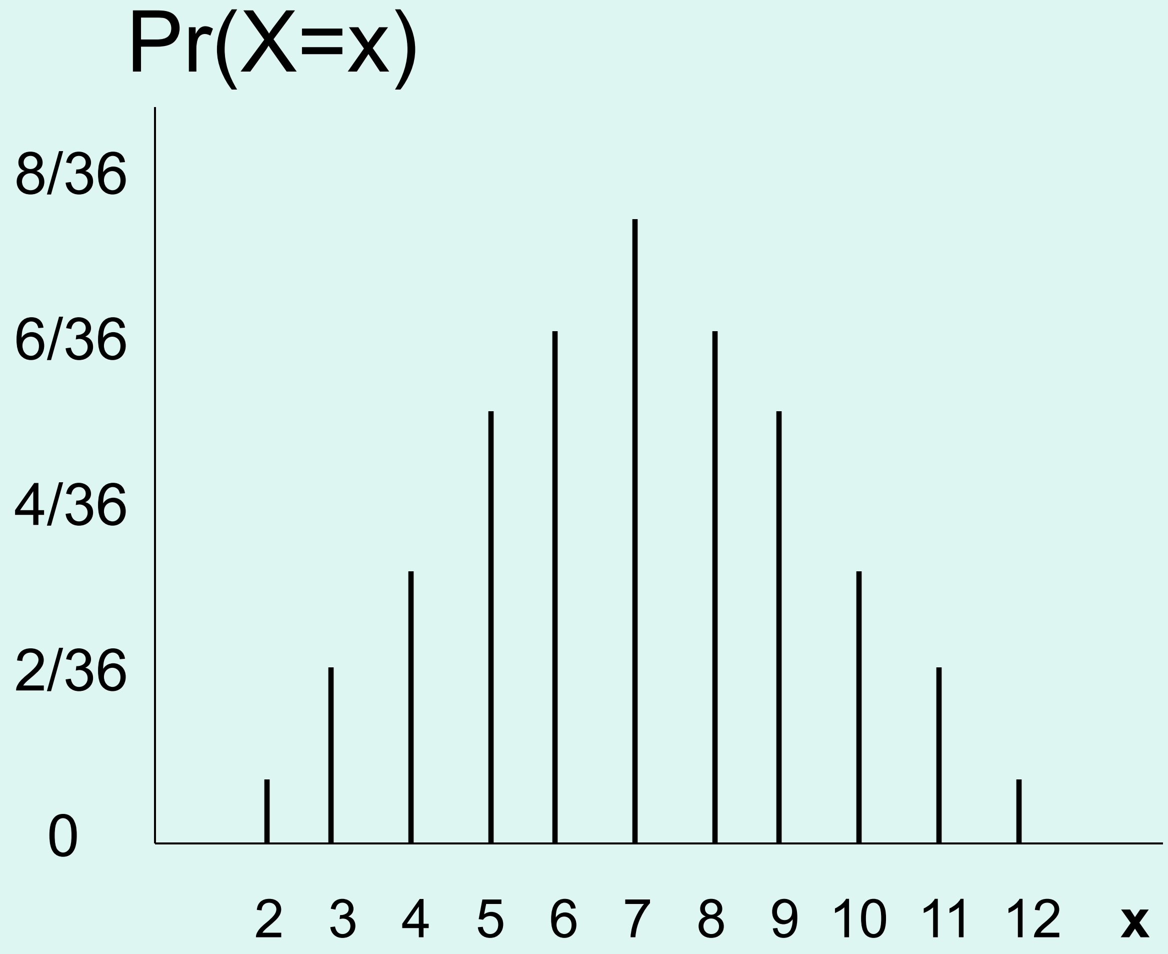


$$Pr(X=x) = f(x) = p(x)$$

as described in this table or graph is called the

probability distribution or *probability mass function (p.m.f.)*

x	Pr(X=x)
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36



Define the random variable X to be the subtraction of the dots on the 2 dice.

الفرق بين القيمتين

X=0,1,2,3,4,5

Example 2

A coin is tossed three sequential times, If X is the random variable that the Head occurs, What is the probability Distributions of that variable?

The Sample space will be :

S={*HHH, HHT, HTH, HTT,TTT,TTH,THT,THH*},

and *X*= {*0,1,2,3*}

حيث 0 تعني عدم ظهور الصورة و 1تعني ظهور الصورة مرة و 2 تعني ظهور الصورة مرتين و 3 تعني ظهور الصورة ثلاث مرات

The probability distribution function as follows;

$P(0)= P(X=0)= P(TTT)= 1/8$

$P(1)=P(X=1)=P\{THT,TTH,HTT\}= 3/8$

$P(2)=P(X=2)=P\{HHT,HTH,THH\}= 3/8$

$P(3)=P(X=3)=P\{HHH\}= 1/8$

ويمكن كتابة المتغير العشوائي X ودالة الاحتمال القابلة لة كالاتي:

X	0	1	2	3	SUM
P(X)	1/8	3/8	3/8	1/8	1

Cumulative Mass Function

$$F(x_0) = \Pr(X \leq x_0) = \sum_{x \leq x_0} p(x)$$

In the previous example the Cumulative Mass Function will be as follows:

X	0	1	2	3	SUM
P(X)	1/8	3/8	3/8	1/8	1

If $X < 0$ Then $F(x)=0$

If $0 \le X < 1$ Then $F(x)=F(0)=P(0)=1/8$

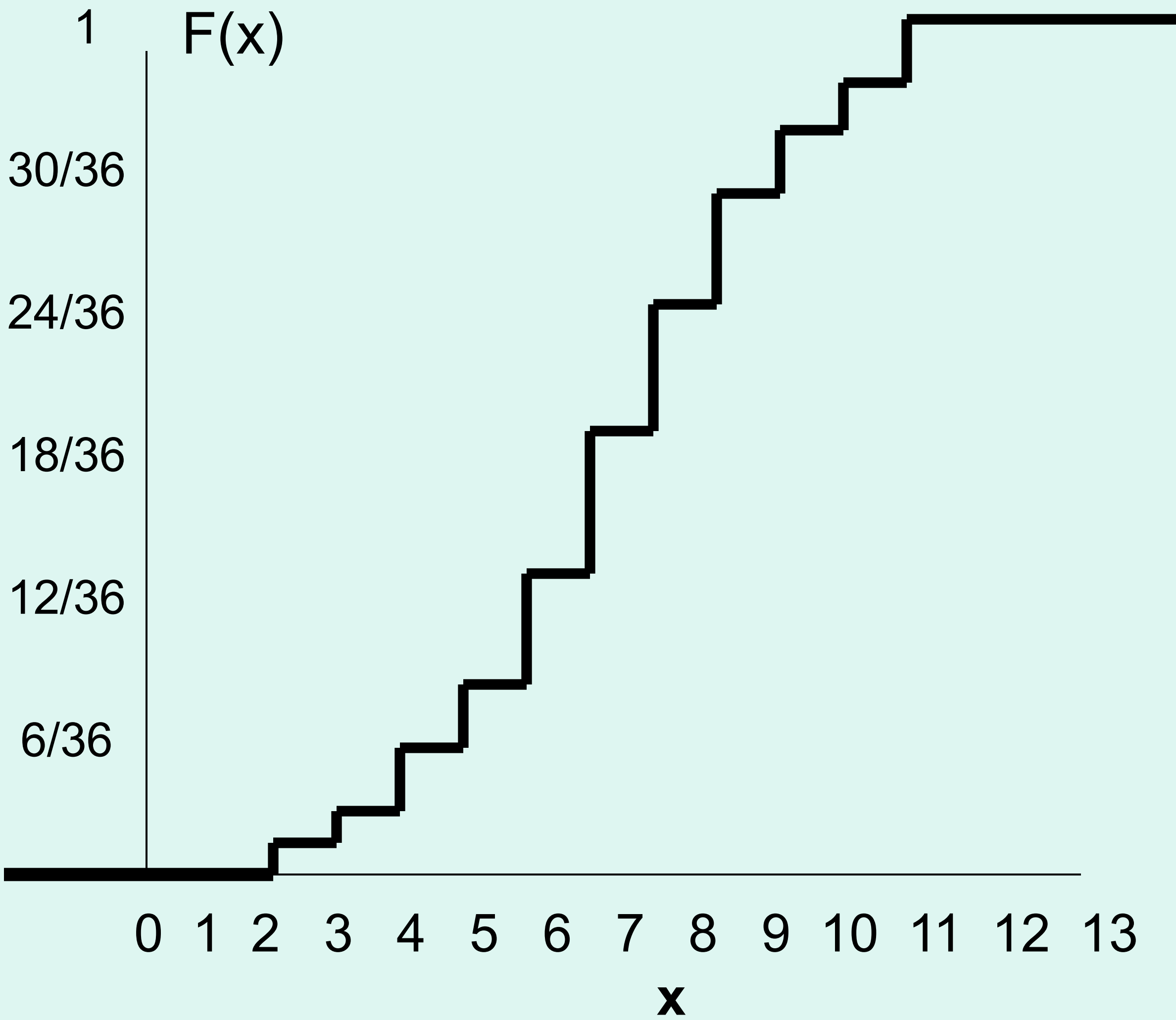
If $1 \le X<2$ Then $F(x)=F(1)= P(0)+P(1)=4/8$

If $2 \le X<3$ Then $F(x)=F(2)=P(0)+P(1)+P(2)= 7/8$

If $3 \le X$ Then $F(x)=F(3)= P(0)+P(1)+P(2)+P(3)= 1$

Cumulative Mass Function (2 dice problem)

x	Pr(X=x)	Pr(X≤x)
2	1/36	1/36
3	2/36	3/36
4	3/36	6/36
5	4/36	10/36
6	5/36	15/36
7	6/36	21/36
8	5/36	26/36
9	4/36	30/36
10	3/36	33/36
11	2/36	35/36
12	1/36	1



Example 2 : Find F(0) , F(1) ,F(2) , P(0 $\leq X \leq 1$)

X	0	1	2	Total
$P\ (X)$	$^4/_9$	$^4/_9$	$^1/_9$	1

$$F(0) = P(X \leq 0) = P(0) = \frac{4}{9}$$

$$F(1) = P(X \leq 1) = P(0) + P(1) = \frac{4}{9} + \frac{4}{9} = \frac{8}{9}$$

$$F(2) = P(X \leq 2) = P(0) + P(1) + P(2)$$

$$= \frac{4}{9} + \frac{4}{9} + \frac{1}{9} = \frac{9}{9}$$

$$P(0 \leq X \leq 1) = P(0) + P(1) = \frac{4}{9} + \frac{4}{9} = \frac{8}{9}$$

The function $f(x)$ is a **probability density function** (pdf) for the continuous random variable X, defined over the set of real numbers, if

1. $f(x) \geq 0$, for all $x \in R$.
2. $\int_{-\infty}^{\infty} f(x)\, dx = 1$.
3. $P(a < X < b) = \int_a^b f(x)\, dx$.

The **cumulative distribution function** $F(x)$ of a continuous random variable X with density function $f(x)$ is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\, dt, \quad \text{for } -\infty < x < \infty.$$

Expectation, Expected Value, or Mean of a Random Variable

$$\mu = E(X) = \sum_{x} x p(x)$$

Notice the similarity of the definitions of the mean of a random variable & the mean of a frequency distribution for a population

$$\mu = E(X) = \sum_{x} xp(x)$$

$$\text{pop. freq. distrib.:}\ \ \mu = (1/N)\sum_{i=1}^{c} x_i f_i = \sum_{i=1}^{c} x_i \left(\frac{f_i}{N} \right)$$

Recall that probability [p(x)] is the relative frequency [f/N] with which something occurs over the long run.

So these definitions are saying the same thing.

Example: Suppose that a stock broker wants to estimate the price of a certain stock one year from now. If the probability mass function of the price in a year is as given, determine the expected price.

x = price in one year	p(x)
94	0.25
98	0.25
102	0.25
106	0.25

Example: Suppose that a stock broker wants to estimate the price of a certain stock one year from now. If the probability mass function of the price in a year is as given, determine the expected price.

x = price in one year	p(x)
94	0.25
98	0.25
102	0.25
106	0.25
	1.00

Example: Suppose that a stock broker wants to estimate the price of a certain stock one year from now. If the probability mass function of the price in a year is as given, determine the expected price.

x = price in one year	p(x)	xp(x)
94	0.25	23.5
98	0.25	24.5
102	0.25	25.5
106	0.25	26.5
	1.00	

Example: Suppose that a stock broker wants to estimate the price of a certain stock one year from now. If the probability mass function of the price in a year is as given, determine the expected price.

x = price in one year	p(x)	xp(x)
94	0.25	23.5
98	0.25	24.5
102	0.25	25.5
106	0.25	26.5
	1.00	100.0

Notice that you do ***NOT*** divide by the number of observations when you're done adding.

Also, the probabilities do not have to be equal; they just have to add up to one.

Theorem: Suppose that g(X) is a function of a random variable X, & the probability mass function of X is $p_x(x)$. Then the expected value of g(X) is

$$E[g(X)] = \sum_{x} g(x) p_x(x)$$

Definition:
Variance of a random variable X

$$\sigma^2 = V(X) = E[(X - \mu)^2]$$

$$= \sum_x (X - \mu)^2 p(x)$$

Theorem:
The variance of X can also be calculated as follows:

$$\sigma^2 = \mathrm{V(X)} = \mathrm{E(X^2)} - [\mathrm{E(X)}]^2$$

Standard Deviation of a random variable X

$$\sigma = \sqrt{\sigma^2} = \sqrt{V(X)}$$

Example: Suppose sales at a donut shop are distributed as below. Calculate (a) the mean number of donuts sold, (b) the variance (using both the definition of the variance & the theorem), & (c) the standard deviation.

x	p(x)						
1	0.08						
2	0.27						
4	0.10						
6	0.33						
12	0.22						

First, the mean….

x	p(x)	xp(x)					
1	0.08	0.08					
2	0.27	0.54					
4	0.10	0.40					
6	0.33	1.98					
12	0.22	2.64					

First, the mean….

x	p(x)	xp(x)					
1	0.08	0.08					
2	0.27	0.54					
4	0.10	0.40					
6	0.33	1.98					
12	0.22	2.64					
		μ=5.64					

Next, the variance using the definition:

$$\sigma^2 = V(X) = E[(X-\mu)^2] = \sum_x (X-\mu)^2 p(x)$$

x	p(x)	xp(x)	x-μ				
1	0.08	0.08	-4.64				
2	0.27	0.54	-3.64				
4	0.10	0.40	-1.64				
6	0.33	1.98	0.36				
12	0.22	2.64	6.36				
		μ=5.64					

Next, the variance using the definition:

$$\sigma^2 = V(X) = E[(X-\mu)^2] = \sum_x (X-\mu)^2 p(x)$$

x	p(x)	xp(x)	x-μ	$(x-\mu)^2$			
1	0.08	0.08	-4.64	21.53			
2	0.27	0.54	-3.64	13.25			
4	0.10	0.40	-1.64	2.69			
6	0.33	1.98	0.36	0.13			
12	0.22	2.64	6.36	40.45			
		μ=5.64					

Next, the variance using the definition:

$$\sigma^2 = V(X) = E[(X-\mu)^2] = \sum_x (X-\mu)^2 p(x)$$

x	p(x)	xp(x)	x-μ	(x-μ)^2	(x-μ)^{2}p(x)		
1	0.08	0.08	-4.64	21.53	1.72		
2	0.27	0.54	-3.64	13.25	3.58		
4	0.10	0.40	-1.64	2.69	0.27		
6	0.33	1.98	0.36	0.13	0.04		
12	0.22	2.64	6.36	40.45	8.90		
		μ=5.64					

Next, the variance using the definition:

$$\sigma^2 = V(X) = E[(X-\mu)^2] = \sum_x (X-\mu)^2 p(x)$$

x	p(x)	xp(x)	x-μ	$(x-\mu)^2$	$(x-\mu)^2 p(x)$		
1	0.08	0.08	-4.64	21.53	1.72		
2	0.27	0.54	-3.64	13.25	3.58		
4	0.10	0.40	-1.64	2.69	0.27		
6	0.33	1.98	0.36	0.13	0.04		
12	0.22	2.64	6.36	40.45	8.90		
		μ=5.64			σ^2 =14.51		

Now, the variance using the theorem: $V(X) = E(X^2)-[E(X)]^2$.

x	p(x)	xp(x)	x-μ	$(x-\mu)^2$	$(x-\mu)^2 p(x)$	x^2	
1	0.08	0.08	-4.64	21.53	1.72	1	
2	0.27	0.54	-3.64	13.25	3.58	4	
4	0.10	0.40	-1.64	2.69	0.27	16	
6	0.33	1.98	0.36	0.13	0.04	36	
12	0.22	2.64	6.36	40.45	8.90	144	
		μ=5.64			σ^2 =14.51		

Now, the variance using the theorem: $V(X) = E(X^2)-[E(X)]^2$.

x	p(x)	xp(x)	x-μ	$(x-\mu)^2$	$(x-\mu)^2p(x)$	x^2	$x^2p(x)$
1	0.08	0.08	-4.64	21.53	1.72	1	0.08
2	0.27	0.54	-3.64	13.25	3.58	4	1.08
4	0.10	0.40	-1.64	2.69	0.27	16	1.60
6	0.33	1.98	0.36	0.13	0.04	36	11.88
12	0.22	2.64	6.36	40.45	8.90	144	31.68
		μ=5.64			σ^2 =14.51		

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1	0.08	0.08	-4.64	21.53	1.72	1	0.08
2	0.27	0.54	-3.64	13.25	3.58	4	1.08
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		$\mu=5.64$			$\sigma^2=14.51$		$E(X^2)=46.32$

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12	0.22	2.64	6.36	40.45	8.90	144	31.68
		μ=5.64			σ^2 =14.51		$E(X^2)$=46.32
$\sigma^2 = V(X) = E(X^2) – [E(X)]^2 = 46.32 – (5.64)^2 = 14.51$							

And lastly, the standard deviation, by taking the square root of the variance.

x	$p(x)$	$xp(x)$	$x-\mu$	$(x-\mu)^2$	$(x-\mu)^2p(x)$	x^2	$x^2p(x)$
1	0.08	0.08	-4.64	21.53	1.72	1	0.08
2	0.27	0.54	-3.64	13.25	3.58	4	1.08
4	0.10	0.40	-1.64	2.69	0.27	16	1.60
6	0.33	1.98	0.36	0.13	0.04	36	11.88
12	0.22	2.64	6.36	40.45	8.90	144	31.68
		$\mu=5.64$			$\sigma^2=14.51$		$E(X^2)=46.32$
$\sigma^2 = V(X) = E(X^2) - [E(X)]^2 = 46.32 - (5.64)^2 = 14.51$							$\boldsymbol{\sigma = 3.81}$

In the experiment of throw 2 dice

What are the possible outcomes? When x is the sum of the two dice write X, P(X), F(X) of all possible outcomes and Draw Them?