Introduction

Before…

- Probability Functions Representing Reliability
  - Reliability Function
  - CDF and PDF
  - Hazard Function
- Summary Statistics of Reliability
  - Expected Life (Mean time to failure)
  - Median Life, B_α Life and Mode

Next…

What are the **probability models** useful in describing a failure process?

- Exponential Distribution
- Weibull Distribution
- Normal Distribution
- Lognormal Distribution

Constant Failure Rate Model

→ Exponential Distribution

Overview

1. Probability Functions

2. MTTF, Variance and Median

3. Memoryless (Lack-of-Memory) Property

4. Applications

Exponential Distribution

- Plays a central role in reliability
- The only continuous distribution with CFR (Constant Failure Rate) (The only discrete distribution with the memoryless property is the geometric distribution)
- One of the easiest distribution to analyze statistically

Probability Functions

Constant Failure Rate:

$$\lambda > 0$$

Hazard Function:

$$h(t) = \lambda$$

Reliability Function:

$$R(t) = e^{-\lambda t}$$

Cumulative Distribution Function:

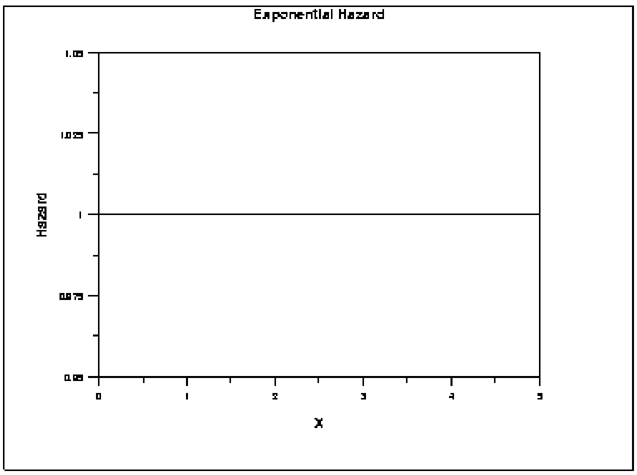
$$F(t) = 1 - R(t) = 1 - e^{-\lambda t}$$

Probability Density Function:

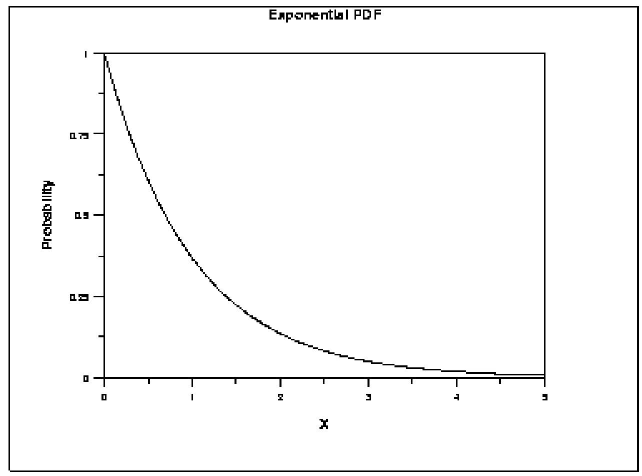
$$f(t) = \lambda e^{-\lambda t}$$

$$\text{Note: } \theta = \text{Mean} = \frac{1}{\lambda}$$

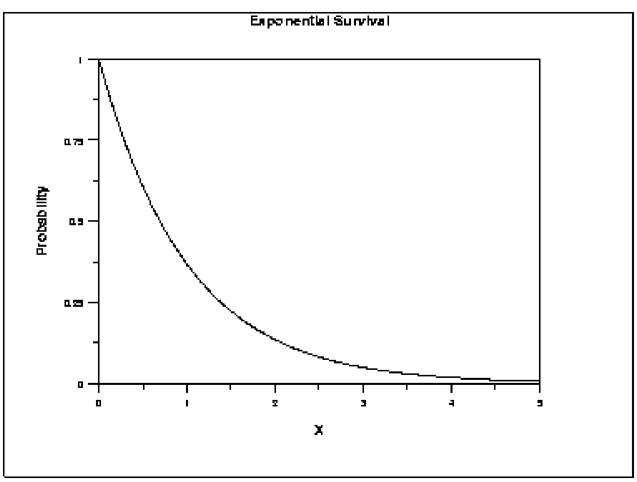
Plots of *R(t), F(t), f(t), h(t)* for Exponential distribution



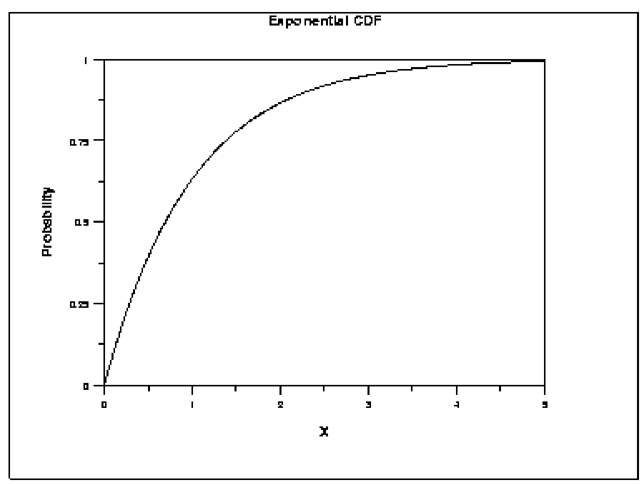
h(t)



f(t)

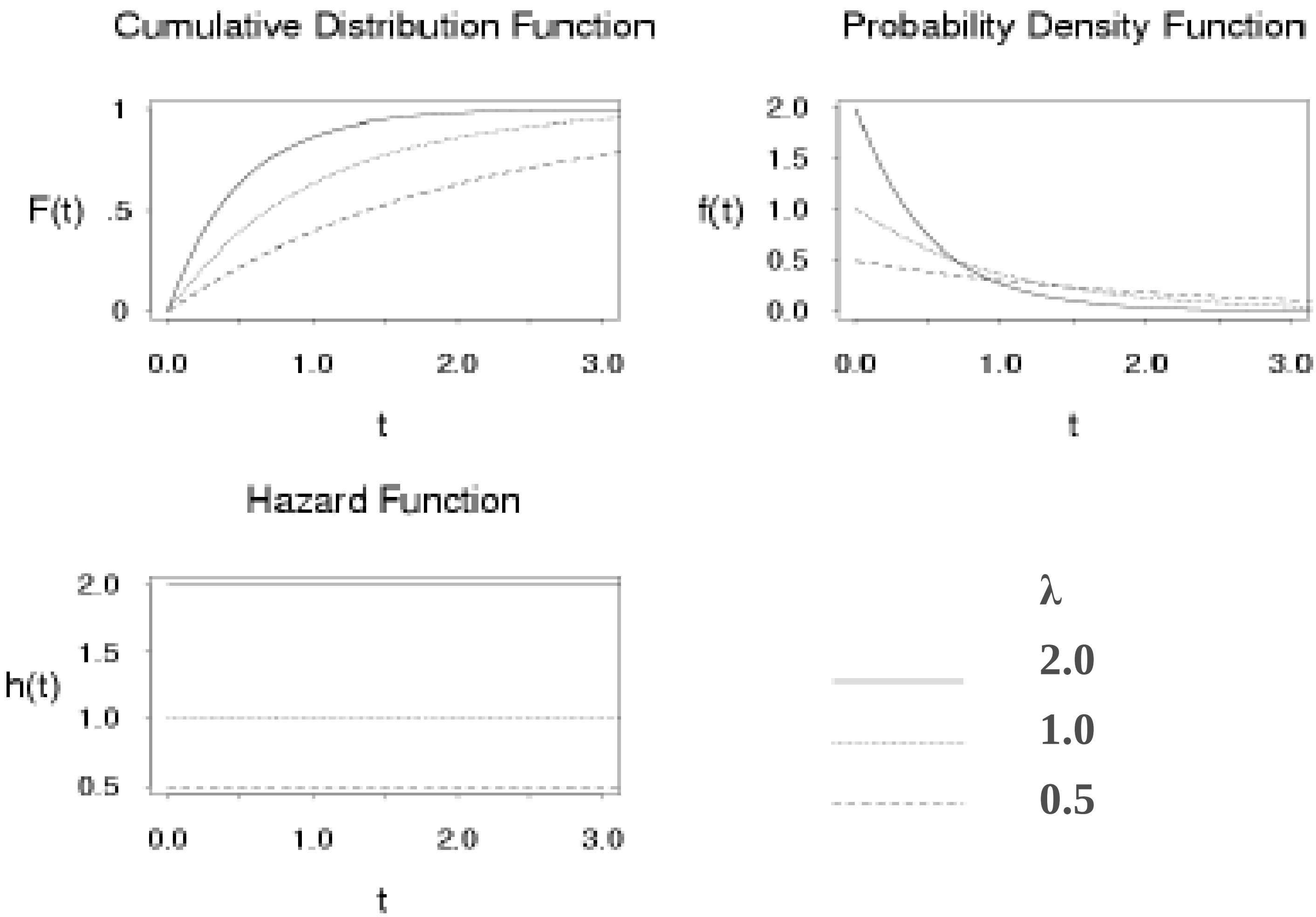


R(t)



F(t)

Examples of Exponential Distribution



MTTF, Variance and Median

- MTTF and Variance

$$MTTF = \frac{1}{\lambda}$$

$$\sigma^2 = \frac{1}{\lambda^2} \Rightarrow \sigma = \frac{1}{\lambda}$$

  - MTTF (time per failure) is the reciprocal of the failure rate (failures per unit time)
  - The standard deviation equals to the MTTF, implying the variability of the failure time increases as the reliability (MTTF) increases

- Median

$$R(B_{50}) = 0.5 = e^{-\lambda B_{50}}$$

$$\Rightarrow B_{50} = -\frac{1}{\lambda}\ln 0.5 = 0.69315 \text{ MTTF} < \text{MTTF}$$

  - The median is always less than the mean → Exponential distribution is skewed to the right

Examples

Example 1

A 5-ton truck has a MMBF (mean mile between failures) of 1,750 miles when used in cross country terrain. Assuming constant failure rate, what is the reliability for a 75-mile mission?

Solution

$$\lambda = \frac{1}{\theta} = \frac{1}{\text{MMBF}} = \frac{1}{1750}$$

$$R(T) = e^{-\lambda t} = e^{-75/1750} = 0.9580$$

Examples

Example 2

An automobile is commonly warranted for 12,000 miles. What system MMBF is required that no more than 1% of the vehicles sold will require warranty work? (Assuming constant failure rate.)

Solution

$$\because R(12000) = 0.99 = e^{-12000\lambda}$$

$$\text{MMBF} = \frac{1}{\lambda} = 1193990$$

Examples

Example 3

A sighting system consists of a north seeking gyro, laser designator, A/D converter, digital computer, and transmitter. This entire system is assumed to have an exponential distributed time-to-failure with a mean of 130 hours.

(1) What is the failure rate for the system?

(2) What is the probability that the system were surviving 10 hours of use?

(3) How long can the system be used such that there is 90% reliability?

Solution

$$(1)\quad \lambda = \frac{1}{\theta} = \frac{1}{130}$$

$$(2)\ R(10) = e^{-\lambda t} = e^{-\frac{10}{130}} = 0.93$$

$$(3)\ R(t) = e^{-\lambda t} = 0.9 \Rightarrow t = 13.7$$

Examples

Example 4

What is the probability of an item surviving until t = 100 units if the item is exponentially distributed with a mean time between failure of 80 units? Given that the item survived to 200 units, what is the probability of survival until t = 300 units? What is the value of the hazard function at 200 units, 300 units?

The probability of survival until t = 100 units is

$$R(100) = e^{-\left(\frac{100}{80}\right)} = 0.2865$$

The probability of survival until t = 300 units given survival until t = 200 units is

$$R(300,200) = \frac{R(300)}{R(200)} = \frac{e^{-300/80}}{e^{-200/80}} = 0.2865$$

Note that this is equal to the probability of failure in the interval from t=0 to t=100.

The value of the hazard function is equal to the failure rate and is constant
h(t)= 1/80 = 0.125

Examples

Example 5

The lifetime (in hours) of an electrical component can be described by the exponential distribution

$f(t) = \lambda \cdot \exp(-\lambda \cdot t) \qquad t \geq 0; \quad \lambda = 1/(500h)$.

1.What is the probability that the component does not fail before the time $t_1 = 200\ h$?

2.What is the probability that the component fails before $t_2 = 100\ h$?

3.What is the probability that the component fails between the times $t_3 = 200\ h$ and $t_4 = 300\ h$?

4.How long, t_5, can the component survive with exactly 90% safety and which range of time can the component survive with at least 90% safety?

5.What value must the parameter λ have for a lifetime distribution where the probability is 90% so that the lifetime of a component is at least 50 h?

Examples

Solution

a) Searched for:

$$\underline{\underline{P(t_1 \le t)}} = 1 - F(t_1) = R(t_1) = exp(-\lambda \cdot t_1) = exp(-\frac{200\text{h}}{500\text{h}}) = 0.6703 \mathrel{\hat{=}} \underline{\underline{67.03\%}}$$

b) Searched for:

$$\underline{\underline{P(t_2 \ge t)}} = F(t_2) = 1 - exp(-\lambda \cdot t_2) = 1 - exp(-\frac{100}{500}) = 0.1813 \mathrel{\hat{=}} \underline{\underline{18.13\%}}$$

c) Searched for:

$$\underline{\underline{P(t_3 \le t \le t_4)}} = F(t_4) - F(t_3) = 1 - exp(-\lambda \cdot t_4) - 1 + exp(-\lambda \cdot t_3)$$

$$= -exp(-\frac{300}{500}) + exp(-\frac{200}{500}) = -0.5488 + 0.6703 = 0.1215$$

$$\mathrel{\hat{=}} \underline{\underline{12.15\%}}$$

d) Required condition:

$$P(t_5 < t) = 1 - P(t_5 \ge t) = 1 - P(t_5 \ge t) = 1 - F(t_5)$$

$$= R(t_5) = exp(-\lambda \cdot t_5) \overset{!}{=} 0.9$$

$$\Rightarrow \underline{\underline{t_5}} = -\frac{ln(0.9)}{\lambda} = -ln(0.9) \cdot 500\text{h} = \underline{\underline{52.68\text{h}}}$$

With at least 90%: all times $t \le t_5$

Examples

Solution

e) Required condition: $P(50 \le t) = R(50) = exp(-\lambda \cdot 50) \overset{!}{=} 0.9$

$$\Rightarrow \underline{\underline{\lambda}} = -\frac{ln(0.9)}{50h} = +\underline{\underline{0.0021072\%}}$$

Examples

Example 6

In a factory, a device that works effectively as good as new during its operating life, has failure rate of 0.008 failures per day. If the probability of failure for this device is independent of running time, find the following:

1. The probability that this device will fail before 100 days of running time
2. The probability that this device will last for more than 80 days
3. The probability that this device will not run for 40 days before failing
4. The probability that this device will fail before the 10 days that follow the first 100 days of running time
5. The probability that this device will last for more than 60 days and less than 120 days
6. The probability that this device will fail after 50 days of working and before 100 days of running time
7. The probability that this device will fail during the 10 days that follow the first 100 days of running time

Examples

Example 7

The reliability of a technical component is given by the equation:

$$R(t) = exp(-(\lambda \cdot t)^2) \text{ for } t \geq$$

Calculate the failure density, the failure probability and the failure rate. Show the results graphically

Solution

$$F(t) = 1 - R(t) = 1 - \exp\left(-(\lambda \cdot t)^2\right) \qquad t \geq 0$$

$$f(t) = \frac{dF(t)}{dt} \overset{\text{chain rule}}{=} \frac{d\left(-(\lambda \cdot t)^2\right)}{dt} \cdot \frac{dF(t)}{d(-(\lambda \cdot t)^2)} =$$

$$= -2 \cdot (\lambda \cdot t) \cdot \lambda \cdot \left(-exp(-(\lambda \cdot t)^2\right) = 2 \cdot \lambda^2 \cdot t \cdot \left(exp(-(\lambda \cdot t)^2\right)$$

$$\lambda(t) = \frac{f(t)}{R(t)} = 2 \cdot \lambda^2 \cdot t \qquad \text{(linear increasing failure rate)}$$

Examples

Example 8

An electrical meter times to failure are described by the following probability density function:

$$f(t) = \lambda \exp(-\lambda t)$$

Where: λ = 0.0005

Calculate the hazard rate and the motor *MTTF*.

Examples

Example 9

A mechanical device times to failure are described by the following probability density function:

$$f(t) = 2\lambda\, e^{(-2\lambda t)}$$

Where: $\lambda = 0.0004$

Calculate the failure rate and the device MTTF.

Lack-of-Memory Property

$$P[T \geq t + s \mid T \geq s] = P[T \geq t], t \geq 0, s \geq 0$$

- The probability that an item will operate for the next t=1000 hr is the same regardless of whether has been operating for s=0 hr, s=500 hr, or s=2500 hr, etc.

$$P[T \geq 1000] = P[T \geq 1000 + 2500 \mid T \geq 2500]$$

- The time of failure of an item is not dependent on how long the item has been operating

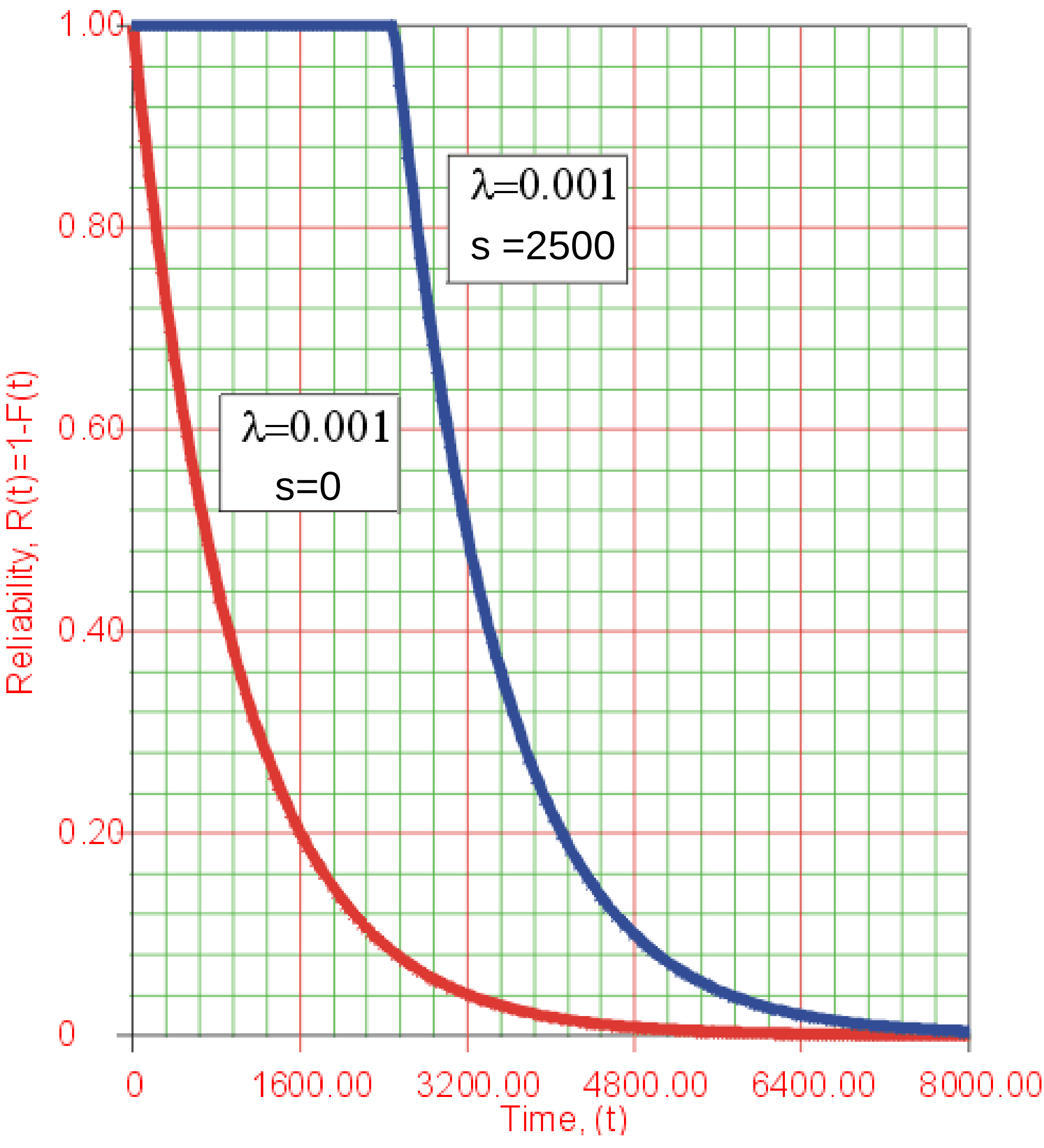
  → no wear out or aging effect, no infant mortality

  → "Used as good as new"

- It is consistent with the completely random and independent nature of the failure process

  → For example, when random environmental stresses are the primary cause of failures, the failure history of an item will not be relevant.

Lack-of-Memory Property



Applications of Exponential Distribution

- Failures due to completely random or chance events will follow this distribution.

- It should dominate during the useful life of a system or component.

- Limited application due to the assumption of constant failure rate.