CHAPTER FIVE

DEFLECTION OF STRUCTURES

CONJUGATE BEAM METHOD

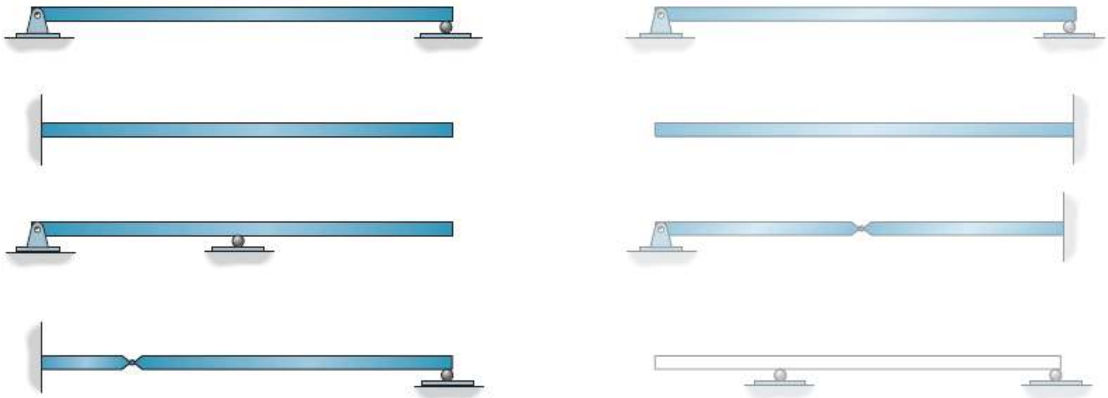
The conjugate beam method was developed by H. Müller-Breslau in 1865. Is used for finding out the slope and deflection of cantilevers and simply supported beams.
The conjugate beam is the *imaginary beam* of length equal to the *same length* of original beam and *loaded* by the usual *bending moment diagram divided by EI.*

The slope and deflection are found according to the Mohr's theorems which it state that:

1. *The **shear force** at any section in conjugate beam is equal and corresponding to the **slope** at the same section in actual beam.*
2. *The **bending moment** at any section in conjugate beam is equal and corresponding to the **deflection** at the same section in actual beam*

Conjugate-Beam Supports.

When drawing the conjugate beam, it is important that the shear and moment developed at the supports of the conjugate beam account for the corresponding slope and displacement of the real beam at its supports according to consequence of theorm1 and 2 as shown in the following Figures.



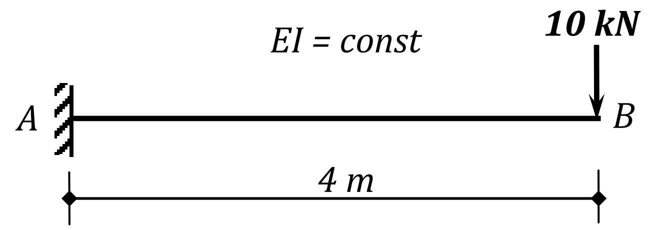
Real Beam		Conjugate Beam	
θ $\Delta = 0$	pin	V $M = 0$	pin
θ $\Delta = 0$	roller	V $M = 0$	roller
$\theta = 0$ $\Delta = 0$	fixed	$V = 0$ $M = 0$	free
θ Δ	free	V M	fixed
θ $\Delta = 0$	internal pin	V $M = 0$	hinge
θ $\Delta = 0$	internal roller	V $M = 0$	hinge
θ Δ	hinge	V M	internal roller

Sign Convention

	Actual Beam	**Conjugate Beam**
Moment & Load	***Moment***	***Load (M/EI)***
	Positive (sag)	upward ↑
	Negative (hog)	Downward ↓
Shear & Rotation	***Rotation***	***Shear***
	Counterclockwise ↺	upward ↑
	Clockwise ↻	Downward ↓
Moment & Deflection	***Deflection***	***Moment***
	upward ↑	Positive (sag)
	Downward ↓	Negative (hog)

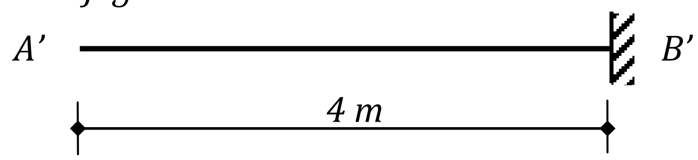
Example 1:

Find the rotation and deflection at the free end for the cantilever shown in Figure below.

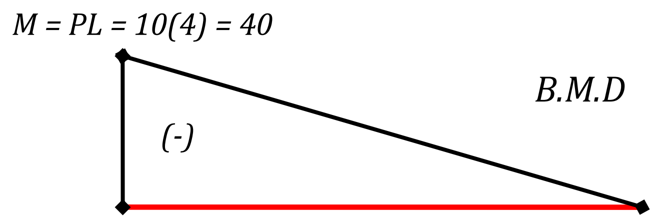


Solution:

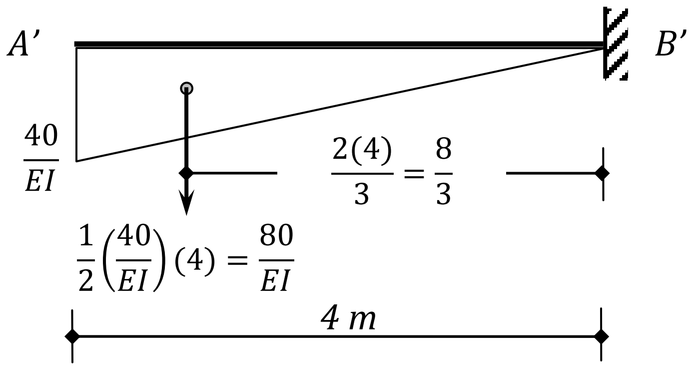
- *Draw the conjugate beam:*



- *Draw the bending moment diagram for actual beam:*



- *Draw the M/EI diagram for conjugate beam:*



- *For rotation at B, find the shear force at conjugate beam at B:*

$$V_B = \frac{80}{EI}, \text{ downward } \downarrow$$

So:

$$\theta_B = \frac{80}{EI} \ \circlearrowright$$

- *For deflection at B, find the moment at conjugate beam at B:*

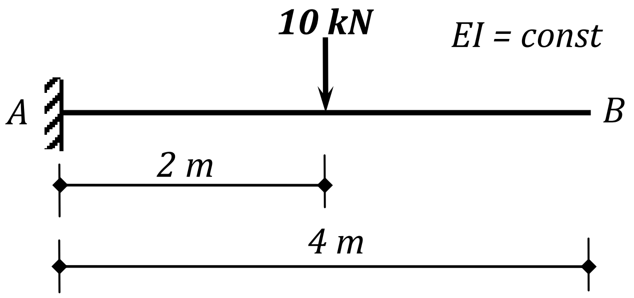
$$M_B = -\frac{80}{EI}\left(\frac{8}{3}\right) = -\frac{640}{3EI}, \text{ (hog)}$$

So:

$$\delta_B = \frac{640}{3EI}, \text{ downward } \downarrow$$

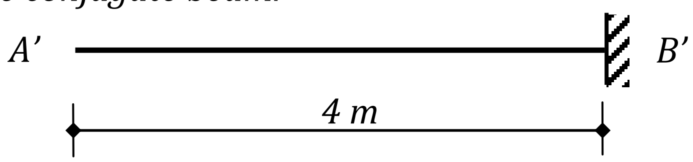
Example 2:

Find the rotation and deflection at the free end for the cantilever shown in Figure below.

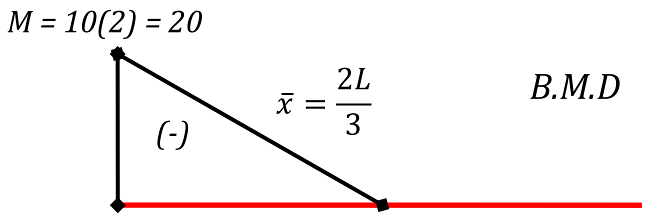


Solution:

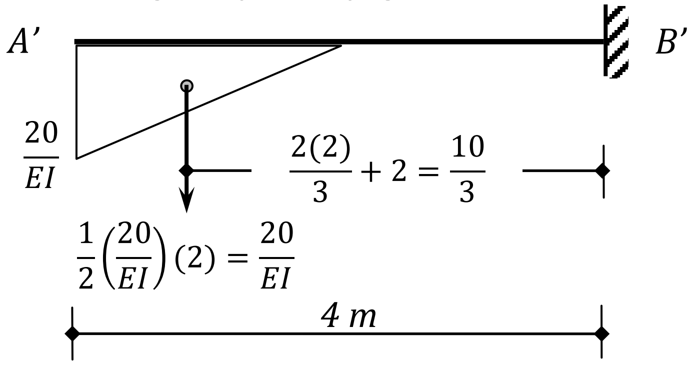
- *Draw the conjugate beam:*



- *Draw the bending moment diagram for actual beam:*



- *Draw the M/EI diagram for conjugate beam:*



- *For rotation at B, find the shear force at conjugate beam at B:*

$$V_B = \frac{20}{EI}\text{, downward} \downarrow$$

So:

$$\theta_B = \frac{20}{EI} \circlearrowright$$

- *For deflection at B, find the moment at conjugate beam at B:*

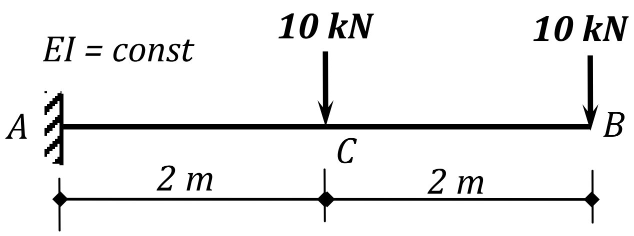
$$M_B = -\frac{20}{EI}\left(\frac{10}{3}\right) = -\frac{200}{3EI}\text{, (hog)} \circlearrowright$$

So:

$$\delta_B = \frac{200}{3EI}\text{, downward} \downarrow$$

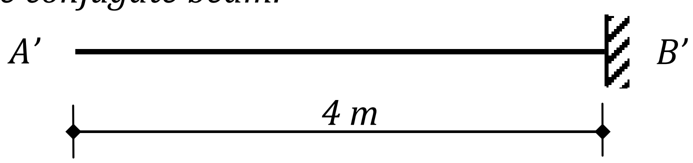
Example 3:

Find the rotation and deflection at the free end B and at joint C for the cantilever shown in Figure below.

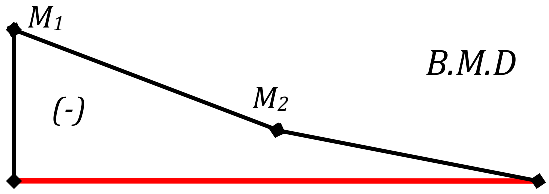


Solution:

- *Draw the conjugate beam:*

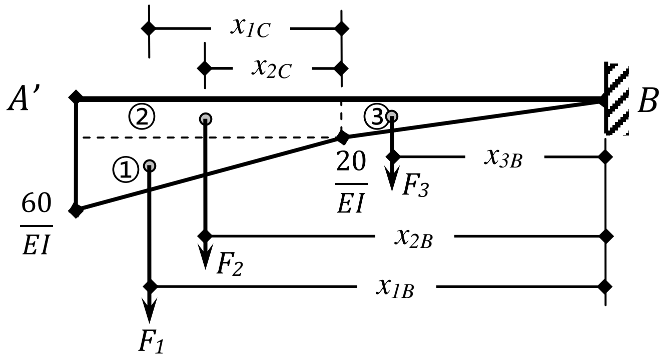


- *Draw the bending moment diagram for actual beam:*



$M_1 = 10(4) + 10(2) = 60$ kN.m, $M_2 = 10(2) = 20$ kN.m

- *Draw the M/EI diagram for conjugate beam:*



	F	Shear at C	Shear at B
①	$\frac{1}{2}\left(\frac{60}{EI}-\frac{20}{EI}\right)(2)=\frac{40}{EI}$	$F_1 + F_2$ $\frac{40}{EI}+\frac{40}{EI}=\frac{80}{EI}$	$F_1 + F_2 + F_3$ $\frac{40}{EI}+\frac{40}{EI}+\frac{20}{EI}=\frac{100}{EI}$
②	$\left(\frac{20}{EI}\right)(2)=\frac{40}{EI}$		
③	$\frac{1}{2}\left(\frac{20}{EI}\right)(2)=\frac{20}{EI}$		

	F	x to C	x to B
①	$\frac{40}{EI}$	$x_{1C}=\frac{2}{3}(2)=\frac{4}{3}$	$x_{1B}=\frac{2}{3}(2)+2=\frac{10}{3}$
②	$\frac{40}{EI}$	$x_{2B}=\frac{1}{2}(2)=1$	$x_{2B}=\frac{1}{2}(2)+2=3$
③	$\frac{20}{EI}$		$x_{3B}=\frac{2}{3}(2)=\frac{4}{3}$
Moment		$-\frac{40}{EI}\left(\frac{4}{3}\right)-\frac{40}{EI}(1)=-\frac{280}{3EI}$	$-\frac{40}{EI}\left(\frac{10}{3}\right)-\frac{40}{EI}(3)-\frac{20}{EI}\left(\frac{4}{3}\right)=-\frac{280}{EI}$

- *For rotation at B:*

$$V_B = \frac{100}{EI}\text{, downward} \downarrow$$

So:

$$\theta_B = \frac{100}{EI} \circlearrowright$$

- *For deflection at B:*

$$M_B = \frac{280}{EI}\text{, (hog)}$$

So:

$$\delta_B = \frac{280}{EI}\text{, downward} \downarrow$$

- *For rotation at C:*

$$V_C = \frac{80}{EI}\text{, downward} \downarrow$$

So:

$$\theta_C = \frac{80}{EI} \circlearrowright$$

- *For deflection at C:*

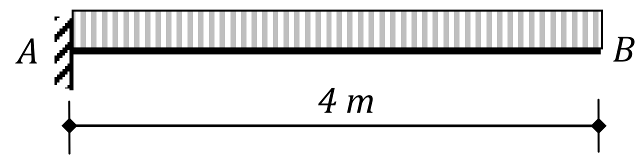
$$M_C = \frac{280}{3EI}\text{, (hog)}$$

So:

$$\delta_C = \frac{280}{3EI}\text{, downward} \downarrow$$

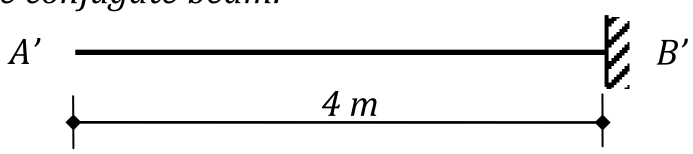
Example 4:

Find the rotation and deflection at the free end for the cantilever shown in Figure below.

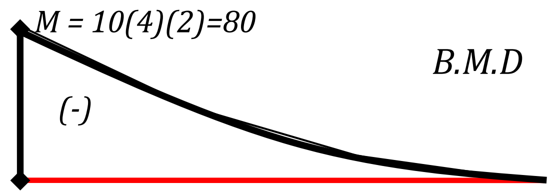


Solution:

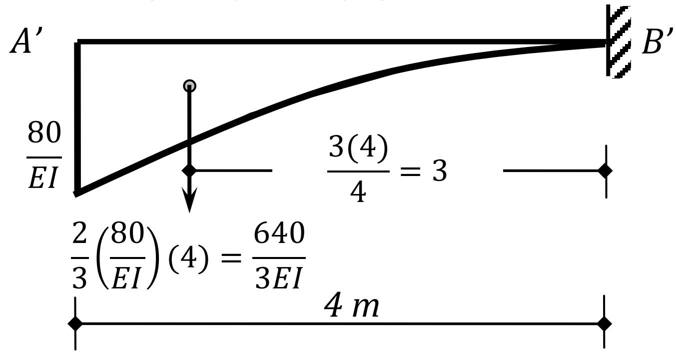
- *Draw the conjugate beam:*



- *Draw the bending moment diagram for actual beam:*



- *Draw the M/EI diagram for conjugate beam:*



- *For rotation at B, find the shear force at conjugate beam at B:*

$$V_B = \frac{640}{3EI}\text{, downward} \downarrow$$

So:

$$\theta_B = \frac{640}{3EI}$$

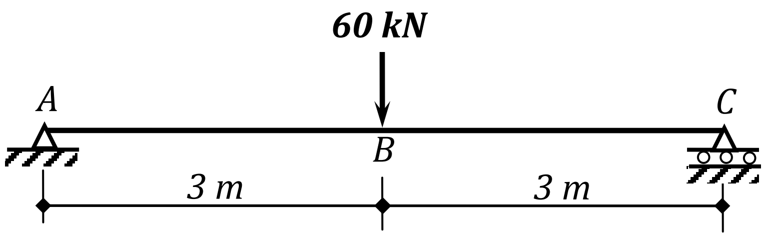
- *For deflection at B, find the moment at conjugate beam at B:*

$$M_B = -\frac{640}{3EI}(3) = -\frac{640}{EI}\text{, (hog)}$$

So:

$$\delta_B = \frac{640}{EI}\text{, downward} \downarrow$$

Example 5:



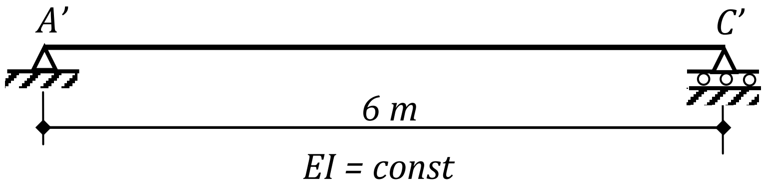
EI = const

For the beam shown in Fig. find:

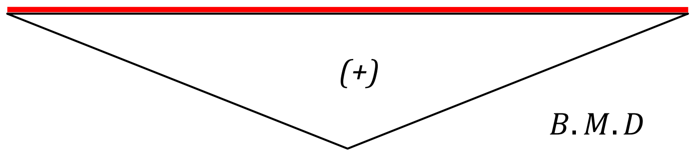
1. *the rotations at joints A and C.*
2. *the displacements at joint B.*

Solution:

- *Draw the conjugate beam:*

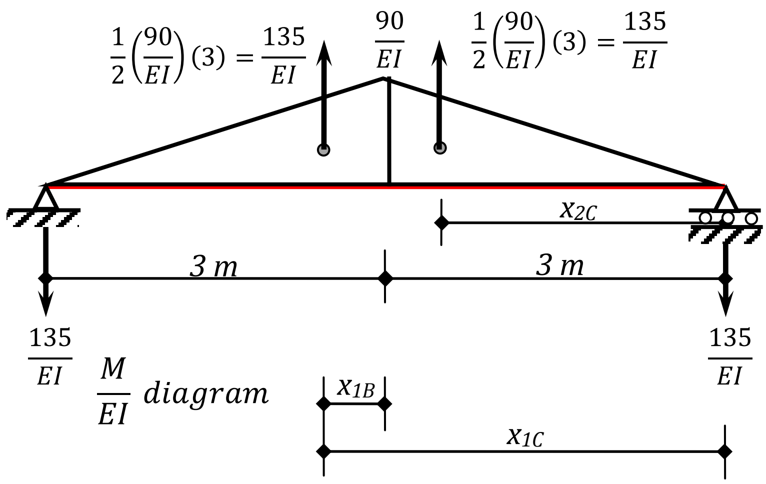


- *Draw the bending moment diagram for actual beam:*



$$M = \frac{PL}{4} = \frac{60(6)}{4} = 90$$

- *Draw the M/EI diagram for conjugate beam:*



$$x_{1B} = \frac{1}{3}(3) = 1\ m$$

$$x_{1C} = \frac{1}{3}(3) + 3 = 4\ m$$

$$x_{2C} = \frac{2}{3}(3) = 2\ m$$

- *For rotation at A, find the shear force at conjugate beam at A:*

$$V_A = \frac{135}{EI}\text{, downward} \downarrow$$

So:

$$\theta_A = \frac{135}{EI} \curvearrowright$$

- *For rotation at C, find the shear force at conjugate beam at C:*

$$V_C = \frac{135}{EI}\text{, upward} \uparrow$$

So:

$$\theta_C = \frac{135}{EI} \;\; \circlearrowleft$$

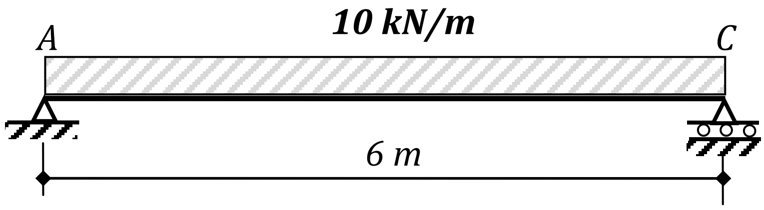
- *For deflection at B, find the moment at conjugate beam at B:*

$$M_B = -\frac{135}{EI}(3) + \frac{135}{EI}(x_{1B}) = -\frac{135}{EI}(3) + \frac{135}{EI}(1) = -\frac{270}{EI}, \text{ (hog)}$$

So:

$$\delta_B = \frac{270}{EI}, \text{ downward } \downarrow$$

Example 6:



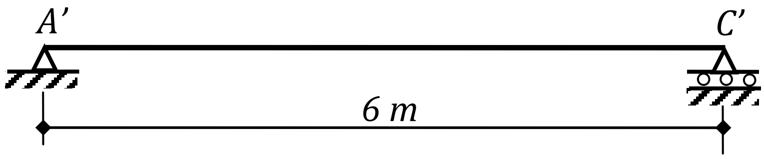
EI = const

For the beam shown in Fig. find:

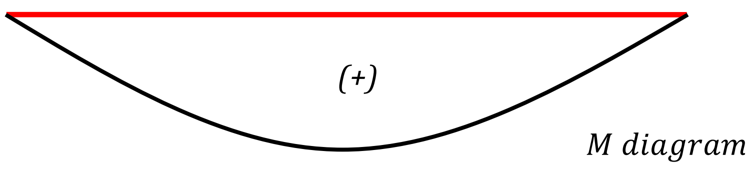
1. *the rotations at joints A and C.*
2. *the displacements at mid beam.*

Solution:

- *Draw the conjugate beam:*

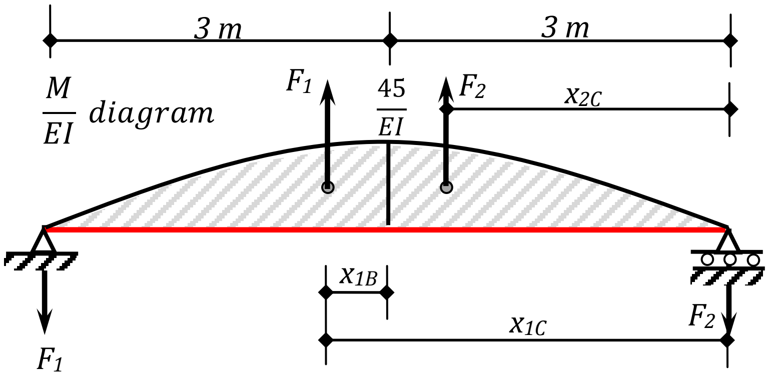


- *Draw the bending moment diagram for actual beam:*



$$M = \frac{10\,(6)^2}{8} = 45$$

- *Draw the M/EI diagram for conjugate beam:*



$$F_1 = \frac{2}{3}\left(\frac{45}{EI}\right)(3) = \frac{90}{EI},\ F_2 = F_1 = \frac{90}{EI}$$

$$x_{1B} = \frac{3}{8}(3) = \frac{9}{8}, x_{1C} = \frac{3}{8}(3) + 3 = \frac{33}{8}, x_{2C} = \frac{5}{8}(3) = \frac{15}{8}$$

- *For rotation at A, find the shear force at conjugate beam at A:*

$$V_A = F_1 = \frac{90}{EI},\ downward \downarrow$$

So:

$$\theta_A = \frac{90}{EI}$$

- *For rotation at C, find the shear force at conjugate beam at C:*

$$V_A = -F_1 + F_1 + F_2 = F_2 = \frac{90}{EI},\ upward \uparrow$$

So:

$$\theta_C = \frac{90}{EI}$$

- *For deflection at B, find the moment at conjugate beam at B:*

$$M_B = -F_1(3) + F_1(x_{1B}) = -\frac{90}{EI}(3) + \frac{90}{EI}\left(\frac{9}{8}\right)$$

$$= -\frac{675}{4EI}, (hog)$$

So:

$$\delta_B = \frac{674}{4EI},\ downward \downarrow$$