M 104 - GENERAL MATHEMATICS -2-

Dr. Tariq A. AlFadhel[1]

Solution of the First Mid-Term Exam

First semester 1433-1434 H

Q.1 Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 0 \\ 2 & 0 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \\ 2 & 0 & 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 & -1 \\ 2 & 3 \\ 0 & 4 \end{pmatrix}$

Compute (if possible) : **BA** , **BAC** and **CB**

Solution :

$$\mathbf{BA} = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-6+6 & 2-6+0 & 3+0+3 \\ 4+15-12 & 8+15+0 & 12+0-6 \\ 2+0+2 & 4+0+0 & 6+0+1 \end{pmatrix} = \begin{pmatrix} 1 & -4 & 6 \\ 7 & 23 & 6 \\ 4 & 4 & 7 \end{pmatrix}$$

$$\mathbf{BAC} = (\mathbf{BA})\mathbf{C} = \begin{pmatrix} 1 & -4 & 6 \\ 7 & 23 & 6 \\ 4 & 4 & 7 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 3 \\ 0 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1-8+0 & -1-12+24 \\ 7+46+0 & -7+69+24 \\ 4+8+0 & -4+12+28 \end{pmatrix} = \begin{pmatrix} -7 & 11 \\ 53 & 86 \\ 12 & 36 \end{pmatrix}$$

CB is not possible because the number of columns of **C** (which is 2) is not equal to the number of rows of **B** (which is 3)

Q.2 Compute The determinant $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 0 \\ 1 & 2 & 3 & 5 \\ 3 & 0 & 0 & 0 \end{vmatrix}$

Solution (1) : Using the properties of the determinants

Multiply C_2 by 3 and C_3 by 2

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 0 \\ 1 & 2 & 3 & 5 \\ 3 & 0 & 0 & 0 \end{vmatrix} = \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) \begin{vmatrix} 1 & 6 & 6 & 4 \\ 0 & 6 & 6 & 0 \\ 1 & 6 & 6 & 5 \\ 3 & 0 & 0 & 0 \end{vmatrix} = 0 \text{ (since } C_2 = C_3)$$

Solution (2) : Interchange R_1 and R_4.

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$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 0 \\ 1 & 2 & 3 & 5 \\ 3 & 0 & 0 & 0 \end{vmatrix} = -\begin{vmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 1 & 2 & 3 & 5 \\ 1 & 2 & 3 & 4 \end{vmatrix} = -3\begin{vmatrix} 2 & 3 & 0 \\ 2 & 3 & 5 \\ 2 & 3 & 4 \end{vmatrix}$$

Using Sarrus method

$$\begin{matrix} 2 & 3 & 0 & 2 & 3 \\ 2 & 3 & 5 & 2 & 3 \\ 2 & 3 & 4 & 2 & 3 \end{matrix}$$

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 0 \\ 1 & 2 & 3 & 5 \\ 3 & 0 & 0 & 0 \end{vmatrix} = -3[(24+30+0)-(0+30+24)] = -3[54-54] = 0$$

Q.3 Find all the elements of the conic section $9x^2 + 16y^2 + 96y - 36x + 36 = 0$ and sketch it.

Solution :

$9x^2 + 16y^2 + 96y - 36x + 36 = 0 \;\Rightarrow\; 9x^2 - 36x + 16y^2 + 96y = -36$

$\Rightarrow\; 9(x^2 - 4x) + 16(y^2 + 6y) = -36$

$\Rightarrow\; 9(x^2 - 4x + 4) + 16(y^2 + 6y + 9) = -36 + 36 + 144$

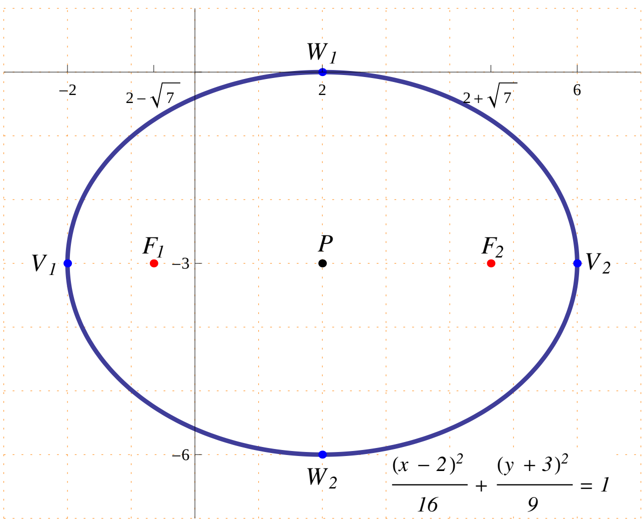
$\Rightarrow\; 9(x-2)^2 + 16(y+3)^2 = 144 \;\Rightarrow\; \dfrac{9(x-2)^2}{144} + \dfrac{16(y+3)^2}{144} = 1$

$\dfrac{(x-2)^2}{16} + \dfrac{(y+3)^2}{9} = 1$

The conic section is an Ellipse

$a^2 = 16 \;\Rightarrow\; a = 4$ and $b^2 = 9 \;\Rightarrow\; b = 3$

$c^2 = a^2 - b^2 = 16 - 9 = 7 \;\Rightarrow\; b = \sqrt{7}$



The center is $P = (2, -3)$

The vertices are $V_1 = (-2, -3)$ and $V_2 = (6, -3)$, and the length of the major axis is $2a = 6$.

The foci are $F_1 = \left(2 - \sqrt{7}, -3\right)$ and $F_2 = \left(2 + \sqrt{7}, -3\right)$.

The end points of the minor axis are $W_1 = (2, 0)$ and $W_2 = (2, -6)$, and its length is $2b = 4$

Q.4 Find the standard equation of the hyperbola with foci $(4, 3)$ and $(-2, 3)$ and with vertex $(3, 3)$, then sketch it.

Solution :

Since the line passing through the two foci is parallel to the x-axis then the standard equation has the form $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$.

The center of the hyperbola is the middle point between $(-2, 3)$ and $(4, 3)$, hence $(h, k) = (1, 3)$

c is the distance between the center and one of the foci , hence $c = 3$

a is the distance between the center and the vertex $(3, 3)$, hence $a = 2$

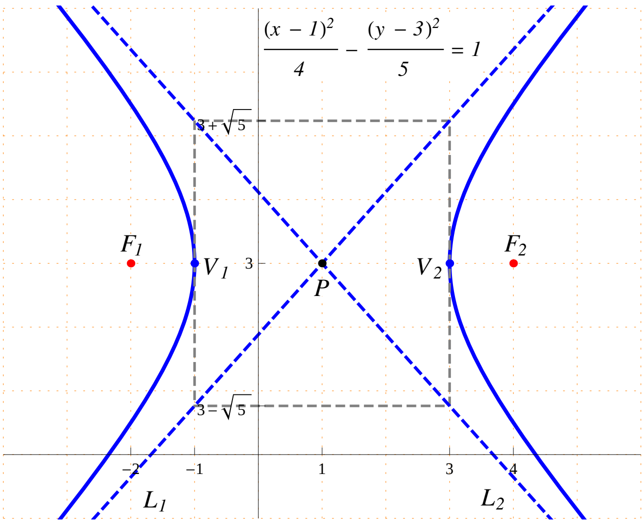
$c^2 = a^2 + b^2 \;\Rightarrow\; 9 = 4 + b^2 \;\Rightarrow\; b^2 = 9 - 4 = 5 \;\Rightarrow\; b = \sqrt{5}$

The standard equation of the hyperbola is $\dfrac{(x-1)^2}{4} - \dfrac{(y-3)^2}{5} = 1$.

The other vertex is $(-1, 3)$ and the length of the transverse axis is $2a = 4$.

The equations of the assymptotes are $L_1 \;:\; y - 3 = \dfrac{\sqrt{5}}{2}(x - 1)$

and $L_2 \;:\; y - 3 = -\dfrac{\sqrt{5}}{2}(x - 1)$



Q.5 Find all the elements of the conic section $y^2 - 4y - 4x = 0$

Solution :

$y^2 - 4y - 4x = 0 \;\Rightarrow\; y^2 - 4y = 4x \;\Rightarrow\; y^2 - 4y + 4 = 4x + 4$

$\Rightarrow\; (y-2)^2 = 4(x+1)$

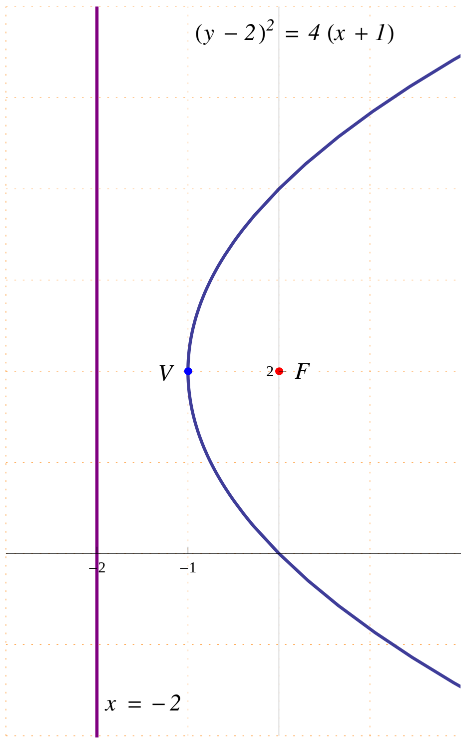
The conic section is parabola opens to the right.

The vertex is $V = (-1, 2)$

$4a = 4 \;\Rightarrow\; a = 1$

The focus is $F = (0, 2)$

The equation of the directrix is $x = -2$



M 104 - GENERAL MATHEMATICS -2-

Dr. Tariq A. AlFadhel[2]

Solution of the Second Mid-Term Exam

First semester 1433-1434 H

Q.1 Solve by Gauss elimination method :

$$\begin{array}{ccccccc} x & + & y & + & z & = & 4 \\ x & - & y & - & z & = & 6 \\ 2x & + & y & - & z & = & 3 \end{array}$$

Solution : The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & -1 & -1 & 6 \\ 2 & 1 & -1 & 3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & -1 & -1 & 6 \\ 2 & 1 & -1 & 3 \end{array}\right) \xrightarrow{-R_1+R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & -2 & 2 \\ 2 & 1 & -1 & 3 \end{array}\right)$$

$$\xrightarrow{-2R_1+R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & -2 & 2 \\ 0 & -1 & -3 & -5 \end{array}\right) \xrightarrow{-\frac{1}{2}R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & -3 & -5 \end{array}\right)$$

$$\xrightarrow{R_2+R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -2 & -6 \end{array}\right) \xrightarrow{-\frac{1}{2}R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$z = 3$

$y + z = -1 \implies y + 3 = -1 \implies y = -4$

$x + y + z = 4 \implies x - 4 + 3 = 4 \implies x = 5$

Q.2 Solve by Cramer's rule :

$$\begin{array}{ccccc} x & - & 2y & = & 3 \\ 3x & + & 5y & = & 20 \end{array}$$

Solution :

$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 3 & 5 \end{pmatrix}$, $\mathbf{A}_1 = \begin{pmatrix} 3 & -2 \\ 20 & 5 \end{pmatrix}$ and $\mathbf{A}_2 = \begin{pmatrix} 1 & 3 \\ 3 & 20 \end{pmatrix}$

$\det(\mathbf{A}) = \begin{vmatrix} 1 & -2 \\ 3 & 5 \end{vmatrix} = (1 \times 5) - (3 \times -2) = 5 - (-6) = 5 + 6 = 11$.

$\det(\mathbf{A}_1) = \begin{vmatrix} 3 & -2 \\ 20 & 5 \end{vmatrix} = (3 \times 5) - (20 \times -2) = 15 - (-40) = 15 + 40 = 55$.

$\det(\mathbf{A}_2) = \begin{vmatrix} 1 & 3 \\ 3 & 20 \end{vmatrix} = (1 \times 20) - (3 \times 3) = 20 - 9 = 11$.

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$$x = \frac{\det(\mathbf{A}_1)}{\det(\mathbf{A})} = \frac{55}{11} = 5$$
$$y = \frac{\det(\mathbf{A}_2)}{\det(\mathbf{A})} = \frac{11}{11} = 1$$

Q.3 Compute the integrals :

$(a)\ \int (x^2 + x)^{11}(2x + 1)\ dx \quad (b)\ \int \frac{2x - 1}{x^2 + 1}\ dx$

$(c)\ \int \frac{x + 2}{(x - 2)^2(x - 1)}\ dx \quad (d)\ \int x^{12} \ln(x)\ dx$

Solution :

(a) $\int (x^2 + x)^{11}(2x + 1)\ dx = \frac{(x^2 + x)^{12}}{12} + c$.

(b) $\int \frac{2x - 1}{x^2 + 1}\ dx = \int \frac{2x}{x^2 + 1}\ dx - \int \frac{1}{x^2 + 1}\ dx$

$= \ln(x^2 + 1) - \tan^{-1} x + c$.

(c) $\int \frac{x + 2}{(x - 2)^2(x - 1)}\ dx$

Using the method of partial fractions

$$\frac{x + 2}{(x - 1)(x - 2)^2} = \frac{A}{x - 1} + \frac{B}{x - 2} + \frac{C}{(x - 2)^2}$$
$$\frac{x + 2}{(x - 1)(x - 2)^2} = \frac{A(x - 2)^2}{(x - 1)(x - 2)^2} + \frac{B(x - 1)(x - 2)}{(x - 1)(x - 2)^2} + \frac{C(x - 1)}{(x - 1)(x - 2)^2}$$
$$x + 2 = A(x - 2)^2 + B(x - 1)(x - 2) + C(x - 1)$$
$$x + 2 = A(x^2 - 4x + 4) + B(x^2 - 3x + 2) + C(x - 1)$$
$$x + 2 = Ax^2 - 4Ax + 4A + Bx^2 - 3Bx + 2B + Cx - C$$
$$x + 2 = (A + B)x^2 + (-4A - 3b + C)x + (4A + 2B - C)$$

By comparing the coefficients :

$$\begin{array}{lllllll} A & + & B & & & = & 0 \longrightarrow (1) \\ -4A & - & 3B & + & C & = & 1 \longrightarrow (2) \\ 4A & + & 2B & - & C & = & 2 \longrightarrow (3) \end{array}$$

Addingg equation (2) and equation (3) : $-B = 3 \implies B = -3$.

From equation (1) : $A - 3 = 0 \implies A = 3$

From equation (2) : $-12 + 9 + C = 1 \implies C = 4$

$$\frac{x + 2}{(x - 1)(x - 2)^2} = \frac{3}{x - 1} + \frac{-3}{x - 2} + \frac{4}{(x - 2)^2}$$
$$\int \frac{x + 2}{(x - 2)^2(x - 1)}\ dx = \int \left(\frac{3}{x - 1} + \frac{-3}{x - 2} + \frac{4}{(x - 2)^2} \right)\ dx$$
$$= \int \frac{3}{x - 1}\ dx + \int \frac{-3}{x - 2}\ dx + \int \frac{4}{(x - 2)^2}\ dx$$
$$= 3\int \frac{1}{x - 1}\ dx - 3\int \frac{1}{x - 2}\ dx + 4\int (x - 2)^{-2}\ dx$$
$$= 3\ln|x - 1| - 3\ln|x - 2| + 4\ \frac{(x - 2)^{-1}}{-1} + c$$

$$= 3\ln|x-1| - 3\ln|x-2| - \frac{4}{x-2} + c$$

(d) $\int x^{12}\ln(x)\ dx$

Using integration by parts

$u = \ln(x) \qquad dv = x^{12}dx$

$du = \frac{1}{x}\ dx \qquad v = \frac{x^{13}}{13}$

$$\int x^{12}\ln(x)\ dx = \frac{x^{13}}{13}\ln(x) - \int \frac{1}{x}\ \frac{x^{13}}{13}\ dx$$

$$= \frac{x^{13}}{13}\ln(x) - \frac{1}{13}\int x^{12}\ dx = \frac{x^{13}}{13}\ln(x) - \frac{1}{13}\ \frac{x^{13}}{13} + c$$

M 104 - GENERAL MATHEMATICS -2-

Dr. Tariq A. AlFadhel[3]

Solution of the Final Exam

First semester 1433-1434 H

Q.1 **(a)** Compute : $2\mathbf{BA}-\mathbf{AB}$ for $\mathbf{A}=\begin{pmatrix}2&2&0\\1&7&1\\0&1&3\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}1&7&1\\2&1&2\\1&9&1\end{pmatrix}$

(b) Compute the determinant $\begin{vmatrix}11&1&7\\1&2&8\\4&-1&0\end{vmatrix}$.

(c) Solve by Gauss Elimination Method : $\begin{array}{rcrcrcr} x&-&2y&+&z&=&8\\ -2x&+&y&-&3z&=&-13\\ 4x&-&y&+&z&=&9\end{array}$

Solution :

(a) $\mathbf{BA}=\begin{pmatrix}1&7&1\\2&1&2\\1&9&1\end{pmatrix}\begin{pmatrix}2&2&0\\1&7&1\\0&1&3\end{pmatrix}$

$$=\begin{pmatrix}2+7+0&2+49+1&0+7+3\\4+1+0&4+7+2&0+1+6\\2+9+0&2+63+1&0+9+3\end{pmatrix}=\begin{pmatrix}9&52&10\\5&13&7\\11&66&12\end{pmatrix}$$

$$2\mathbf{BA}=2\begin{pmatrix}9&52&10\\5&13&7\\11&66&12\end{pmatrix}=\begin{pmatrix}2\times9&2\times52&2\times10\\2\times5&2\times13&2\times7\\2\times11&2\times66&2\times12\end{pmatrix}=\begin{pmatrix}18&104&20\\10&26&14\\22&132&24\end{pmatrix}$$

$$\mathbf{AB}=\begin{pmatrix}2&2&0\\1&7&1\\0&1&3\end{pmatrix}\begin{pmatrix}1&7&1\\2&1&2\\1&9&1\end{pmatrix}$$

$$=\begin{pmatrix}2+4+0&14+2+0&2+4+0\\1+14+1&7+7+9&1+14+1\\0+2+3&0+1+27&0+2+3\end{pmatrix}=\begin{pmatrix}6&16&6\\16&23&16\\5&28&5\end{pmatrix}$$

$$2\mathbf{BA}-\mathbf{AB}=\begin{pmatrix}18&104&20\\10&26&14\\22&132&24\end{pmatrix}-\begin{pmatrix}6&16&6\\16&23&16\\5&28&5\end{pmatrix}$$

$$=\begin{pmatrix}18-6&104-16&20-6\\10-16&26-23&14-16\\22-5&132-28&24-5\end{pmatrix}=\begin{pmatrix}12&88&14\\-6&3&-2\\17&104&19\end{pmatrix}$$

(b) Using Sarrus method $\begin{array}{ccccc}11&1&7&11&1\\1&2&8&1&2\\4&-1&0&4&-1\end{array}$

$$\begin{vmatrix}11&1&7\\1&2&8\\4&-1&0\end{vmatrix}=(0+32-7)-(56-88+0)=25-(-32)=25+32=57$$

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(c) The augmented matrix is $\left(\begin{array}{ccc|c} 1 & -2 & 1 & 8 \\ -2 & 1 & -3 & -13 \\ 4 & -1 & 1 & 9 \end{array}\right)$

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 8 \\ -2 & 1 & -3 & -13 \\ 4 & -1 & 1 & 9 \end{array}\right) \xrightarrow{2R_1+R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 8 \\ 0 & -3 & -1 & 3 \\ 4 & -1 & 1 & 9 \end{array}\right)$$

$$\xrightarrow{-4R_1+R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 8 \\ 0 & -3 & -1 & 3 \\ 0 & 7 & -3 & -23 \end{array}\right) \xrightarrow{3R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 8 \\ 0 & -3 & -1 & 3 \\ 0 & 21 & -9 & -69 \end{array}\right)$$

$$\xrightarrow{7R_2+R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 8 \\ 0 & -3 & -1 & 3 \\ 0 & 0 & -16 & -48 \end{array}\right) \xrightarrow{-\frac{1}{16}R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 8 \\ 0 & -3 & -1 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$z = 3$

$-3y - z = 3 \implies -3y - 3 = 3 \implies -3y = 6 \implies y = -2$

$x - 2y + z = 8 \implies x - 2(-2) + 3 = 8 \implies x + 4 + 3 = 8 \implies x = 1$

The solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

Q.2 Find all the elements of the two following conic sections and sketch their graphs:

(a) $x^2 + 9y^2 - 72y - 4x + 139 = 0$

(b) $9x^2 - 4y^2 - 16y - 18x - 43 = 0$

Solution :

(a) $x^2 + 9y^2 - 72y - 4x + 139 = 0$

$(x^2 - 4x) + (9y^2 - 72y) = -139$

$(x^2 - 4x) + 9(y^2 - 8y) = -139$

$(x^2 - 4x + 4) + 9(y^2 - 8y + 16) = -139 + 4 + 144$

$(x - 2)^2 + 9(y - 4)^2 = 9$

$\dfrac{(x-2)^2}{9} + \dfrac{9(y-4)^2}{9} = 1$

$\dfrac{(x-2)^2}{9} + (y-4)^2 = 1$

The conic section is an ellipse

The center is $P(2, 4)$

$a^2 = 9 \quad \Rightarrow \quad a = 3$

$b^2 = 1 \quad \Rightarrow \quad b = 1$

$c^2 = a^2 - b^2 = 9 - 1 = 8 \quad \Rightarrow \quad c = \sqrt{8} = 2\sqrt{2}$

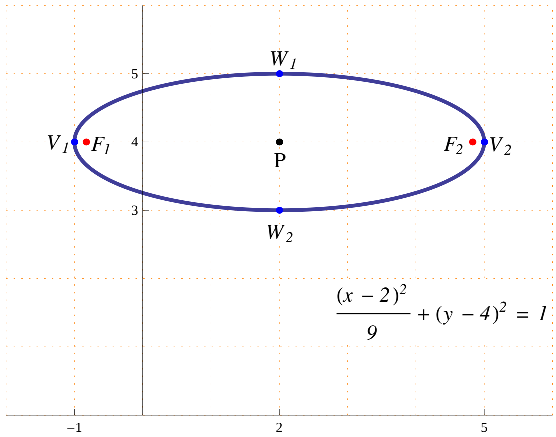
The vertices are $V_1(-1, 4)$ and $V_2(5, 4)$.

The major axis is parallel to the x-axis and its length is $2a = 6$

The foci are $F_1\left(2 - 2\sqrt{2}, 4\right)$ and $F_2\left(2 + 2\sqrt{2}, 4\right)$

The end-points of the minor axis are $W_1(2,5)$ and $W_2(2,3)$

The minor axis is parallel to the y-axis and its length is $2b = 2$



(b) $9x^2 - 4y^2 - 16y - 18x - 43 = 0$

$(9x^2 - 18x) - (4y^2 + 16y) = 43$

$9(x^2 - 2x) - 4(y^2 + 4y) = 43$

$9(x^2 - 2x + 1) - 4(y^2 + 4y + 4) = 43 + 9 - 16$

$9(x-1)^2 - 4(y+2)^2 = 36$

$\dfrac{9(x-1)^2}{36} - \dfrac{4(y+2)^2}{36} = 1$

$\dfrac{(x-1)^2}{4} - \dfrac{(y+2)^2}{9} = 1$

The conic section is a hyperbola

The center is $P(1,-2)$

$a^2 = 4 \quad \Rightarrow \quad a = 2$

$b^2 = 9 \quad \Rightarrow \quad b = 3$

$c^2 = a^2 + b^2 = 4 + 9 = 13 \quad \Rightarrow \quad c = \sqrt{13}$

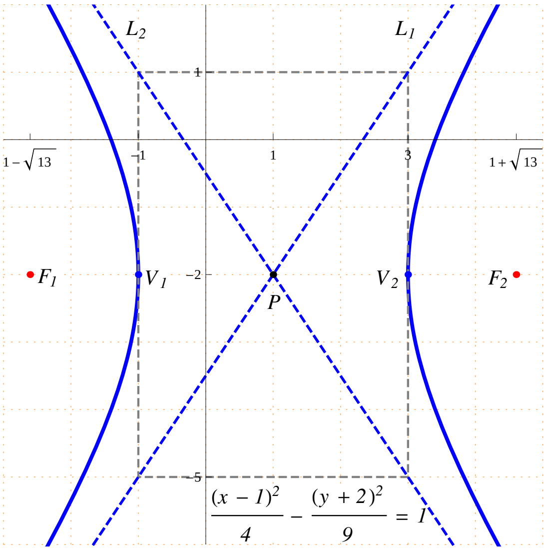
The transverse axis is parallel to the x-axis and its length is $2a = 4$

The vertices are $v_1(-1,-2)$ and $V_2(3,-2)$

The focis are $F_1\left(1-\sqrt{13},-2\right)$ and $F_2\left(1+\sqrt{13},-2\right)$.

The equations of the asymptotes are $L_1 \ : \ y = \dfrac{3}{2}(x-1) - 2$

and $L_2 \ : \ y = -\dfrac{3}{2}(x-1) - 2$



Q.3 **(a)** Compute the integrals :

(i) $\displaystyle\int \frac{1}{x(x+1)}\ dx$ (ii) $\displaystyle\int (x-1)\cos x\ dx$ (iii) $\displaystyle\int \frac{2x-2}{x^2-4x+5}\ dx$

(b) Find the area of the surface delimited by the curves :

$y = 0$, $x = 0$, $x = 4$ and $y = \sqrt{x} + 5$.

(c) The region R between the curves $y = 2$, $y = x^2+2$ and $y = -x+4$ is rotated about the y-axis to form a solid of revolution S . Find the volume of S.

Solution :

(a) (i) $\displaystyle\int \frac{1}{x(x+1)}\ dx$

Using the method of partial fractions

$$\frac{1}{x(x+1)} = \frac{A_1}{x} + \frac{A_2}{x+1}$$

$$\frac{1}{x(x+1)} = \frac{A_1(x+1)}{x(x+1)} + \frac{A_2x}{x(x+1)}$$

$$1 = A_1(x+1) + A_2x = A_1x + A_1 + A_2x$$

$$1 = (A_1 + A_2)x + A_1$$

By comparing the coefficients :

$$\begin{array}{lllll} A_1 & + & A_2 & = & 0 & \rightarrow (1) \\ A_1 & & & = & 1 & \rightarrow (2) \end{array}$$

From equation (1) : $A_1 + A_2 = 0 \implies 1 + A_2 = 0 \implies A_2 = -1$

$$\frac{1}{x(x+1)} = \frac{1}{x} + \frac{-1}{x+1}$$

$$\int \frac{1}{x(x+1)}\ dx = \int \left(\frac{1}{x} - \frac{1}{x+1}\right)\ dx$$

$$= \int \frac{1}{x}\, dx - \int \frac{1}{x+1}\, dx = \ln|x| - \ln|x+1| + c$$

(ii) $\int (x-1)\cos x \; dx$

Using integration by parts

$u = x - 1 \qquad dv = \cos x \; dx$

$du = dx \qquad v = \sin x$

$$\int (x-1)\cos x \; dx = (x-1)\sin x - \int \sin x \; dx$$

$$= (x-1)\sin x - (-\cos x) + c = (x-1)\sin x + \cos x + c$$

(iii) $\int \frac{2x-2}{x^2-4x+5}\, dx$

$$\int \frac{2x-2}{x^2-4x+5}\, dx = \int \frac{(2x-4)+2}{x^2-4x+5}\, dx$$

$$= \int \frac{2x-4}{x^2-4x+5}\, dx + \int \frac{2}{x^2-4x+5}\, dx$$

$$= \int \frac{2x-4}{x^2-4x+5}\, dx + 2\int \frac{1}{(x^2-4x+4)+1}\, dx$$

$$= \int \frac{2x-4}{x^2-4x+5}\, dx + 2\int \frac{1}{(x-2)^2+(1)^2}\, dx$$

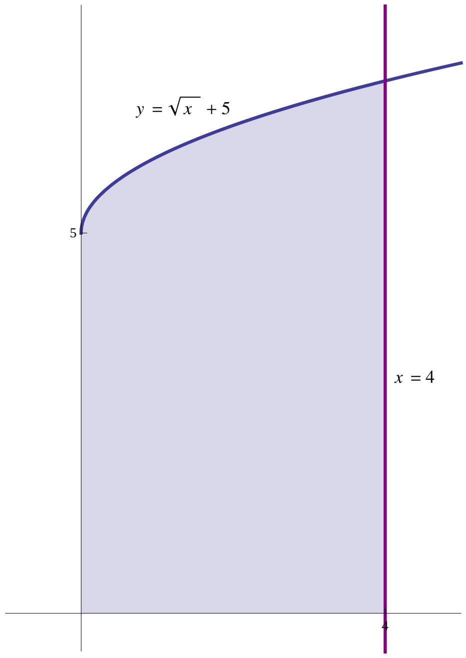
$$= \ln|x^2-4x+5| + 2\tan^{-1}(x-2) + c$$

(b) $y = 0$ is the x-axis and $x = 0$ is the y-axis .

$x = 4$ is a straight line parallel to the y-axis and passing through $(4, 0)$.

$y = \sqrt{x} + 5 \implies y - 5 = \sqrt{x} \implies (y-5)^2 = x$.

$y = \sqrt{x} + 5$ is the upper half of the parabola $(y-5)^2 = x$ which opens to the right and with vertex $(0, 5)$.



$$\text{Area} = \int_0^4 \left(\sqrt{x} + 5\right)\, dx = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 5x\right]_0^4 = \left[\frac{2}{3}x^{\frac{3}{2}} + 5x\right]_0^4$$

$$\text{Area} = \left(\frac{2}{3}(4)^{\frac{3}{2}} + 20\right) - (0+0) = \frac{2}{3} \times 8 + 20 = \frac{16}{3} + \frac{60}{3} = \frac{76}{3}$$

(c) $y = 2$ is a straight line parallel to the x-axis and passing through $(0, 2)$.

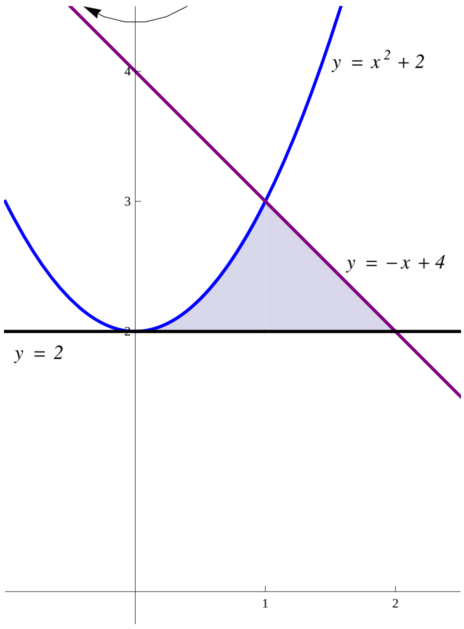
$y = -x+4$ is a straight line with slope -1 and passing through $(0, 4)$.

$y = x^2 + 2 \;\Rightarrow\; x^2 = (y-2)$ is a parabola with vertex $(0, 2)$ and opens upwards

Points of intersection of $y = -x + 4$ and $y = x^2 + 2$

$x^2 + 2 = -x + 4 \quad\Rightarrow\quad x^2 + x - 2 = 0 \quad\Rightarrow\quad (x-1)(x+2) = 0$

$\Rightarrow \quad x = 1\ ,\ x = -2 \quad\Rightarrow\quad y = 3\ ,\ y = 6$



$y = -x + 4 \quad \Rightarrow \quad x = -y + 4$

$y = x^2 + 2 \quad \Rightarrow \quad x = \sqrt{y-2}$

Using Washer Method :

$$\text{Volume} = \pi \int_2^3 \left[(-y+4)^2 - \left(\sqrt{y-2} \right)^2 \right] \, dy$$

$$= \pi \int_2^3 \left[(y^2 - 8y + 16) - (y-2) \right] \, dy = \pi \int_2^3 (y^2 - 8y + 16 - y + 2) \, dy$$

$$= \pi \int_2^3 (y^2 - 9y + 18) \, dy = \pi \left[\frac{y^3}{3} - 9\frac{y^2}{2} + 18y \right]_2^3$$

$$= \pi \left[\left(\frac{27}{3} - 9 \times \frac{9}{2} + 18 \times 3 \right) - \left(\frac{8}{3} - 9 \times \frac{4}{2} + 18 \times 2 \right) \right]$$

$$= \pi \left(\frac{19}{3} - \frac{45}{2} + 18 \right) = \pi \left(\frac{38 - 135 + 108}{6} \right) = \frac{11}{6}\pi$$

Q.4 **(a)** Find f_x and f_y for the function $f(x,y) = x^2y^2 + xye^y + \dfrac{x+y}{x^2+y^2}$

(b) Solve the differential equation $x\dfrac{dy}{dx} - y = 2013x^2e^x$

Solution :

(a) $f_x = \dfrac{\partial f}{\partial x} = 2xy^2 + ye^y + \dfrac{(1)(x^2+y^2) - (x+y)(2x)}{(x^2+y^2)^2}$

$f_y = \dfrac{\partial f}{\partial y} = 2yx^2 + x(e^y + ye^y) + \dfrac{(1)(x^2+y^2) - (x+y)(2y)}{(x^2+y^2)^2}$

(b) $x\dfrac{dy}{dx} - y = 2013x^2e^x$

$xy' - y = 2013x^2e^x$

$y' - \frac{1}{x}y = 2013xe^x$

$P(x) = -\frac{1}{x}$ and $Q(x) = 2013xe^x$.

The integratin factor is

$u(x) = e^{\int -\frac{1}{x}\ dx} = e^{-\int \frac{1}{x}\ dx} = e^{-\ln|x|} = e^{\ln|x^{-1}|} = x^{-1} = \frac{1}{x}$

The solution of the differential equation is $y = \frac{1}{x^{-1}} \int \frac{1}{x}(2013xe^x)\ dx$

$y = x \int 2013e^x\ dx = x\,(2013e^x + c) = 2013xe^x + cx$

M 104 - GENERAL MATHEMATICS -2-

Dr. Tariq A. AlFadhel[4]

Solution of the First Mid-Term Exam

Second semester 1433-1434 H

Q.1 Let $\mathbf{A} = \begin{pmatrix} 1 & -2 & 3 \\ -1 & 3 & 4 \\ 2 & 0 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & -3 \\ 1 & 0 & 2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 5 & 0 \end{pmatrix}$

Compute (if possible) : **BC** , **ABC** and **CB**

Solution :

$$\mathbf{BC} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & -3 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} 1+4+15 & 1+6+0 \\ 2+10-15 & 2+15+0 \\ 1+0+10 & 1+0+0 \end{pmatrix} = \begin{pmatrix} 20 & 7 \\ -3 & 17 \\ 11 & 1 \end{pmatrix}$$

$$\mathbf{ABC} = \mathbf{A}(\mathbf{BC}) = \begin{pmatrix} 1 & -2 & 3 \\ -1 & 3 & 4 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 20 & 7 \\ -3 & 17 \\ 11 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 20+6+33 & 7-34+3 \\ -20-9+44 & -7+51+4 \\ 40+0+11 & 14+0+1 \end{pmatrix} = \begin{pmatrix} 59 & -24 \\ 15 & 48 \\ 51 & 15 \end{pmatrix}$$

CB is not possible because the number of columns of **C** (which is 2) is not equal to the number of rows of **B** (which is 3)

Q.2 Compute the determinant $\begin{vmatrix} 1 & -2 & 4 \\ 3 & 1 & 2 \\ -3 & 6 & -12 \end{vmatrix}$

Solution 1 : Using the properties of determinants

$$\begin{vmatrix} 1 & -2 & 4 \\ 3 & 1 & 2 \\ -3 & 6 & -12 \end{vmatrix} = 0 \text{ (because } R_3 = -3R_1)$$

Solution 2 : Using Sarrus Method

$$\begin{matrix} 1 & -2 & 4 & 1 & -2 \\ 3 & 1 & 2 & 3 & 1 \\ -3 & 6 & -12 & -3 & 6 \end{matrix}$$

$$\begin{vmatrix} 1 & -2 & -4 \\ 3 & 1 & 2 \\ -3 & 6 & -12 \end{vmatrix} = (-12+12+72) - (-12+12+72) = 0$$

Solution 3 : Using the definition of the determinant

$$\begin{vmatrix} 1 & -2 & -4 \\ 3 & 1 & 2 \\ -3 & 6 & -12 \end{vmatrix} = 1 \times \begin{vmatrix} 1 & 2 \\ 6 & -12 \end{vmatrix} - (-2) \times \begin{vmatrix} 3 & 2 \\ -3 & -12 \end{vmatrix} + 4 \times \begin{vmatrix} 3 & 1 \\ -3 & 6 \end{vmatrix}$$

$$= 1 \times (-12-12) + 2 \times (-36+6) + 4 \times (18+3) = -24 - 60 + 84 = 0$$

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Q.3 Find all the elements of the conic section $9x^2-4y^2-16y-54x+29=0$ and sketch it.

Solution :

$9x^2-4y^2-16y-54x+29=0 \Rightarrow 9x^2-54x-4y^2-16y=-29$

$\Rightarrow 9(x^2-6x)-4(y^2+4y)=-29$

$\Rightarrow 9(x^2-6x+9)-4(y^2+4y+4)=-29+81-16$

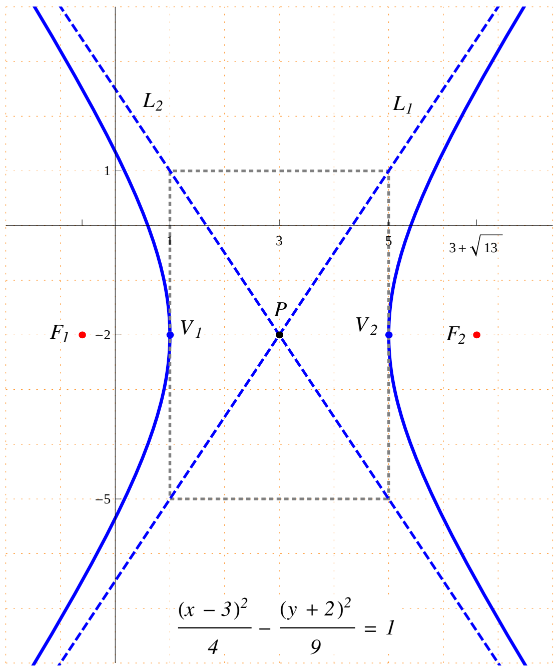
$\Rightarrow 9(x-3)^2-4(y+2)^2=36 \Rightarrow \dfrac{9(x-3)^2}{36}-\dfrac{4(y+2)^2}{36}=1$

$\Rightarrow \dfrac{(x-3)^2}{4}-\dfrac{(y+2)^2}{9}=1$

The conic section is Hypaerbola.

$a^2=4 \Rightarrow a=2$ and $b^2=9 \Rightarrow b=3$

$c^2=a^2+b^2=4+9=13 \Rightarrow c=\sqrt{13}.$



The center is $P=(3,-2)$

The vertices are $V_1=(1,-2)$ and $V_2=(5,-2)$

The Foci are $F_1=\left(3-\sqrt{13},-2\right)$ and $F_2=\left(3+\sqrt{13},-2\right)$

The transverse axis is parallel to the x-axis and its length is $2a=4$.

The equations of the asymptotes are $L_1 \ : \ y+2=\dfrac{3}{2}\,(x-3)$

and $L_2 \ : \ y+2=-\dfrac{3}{2}\,(x-3)$

Q.4 Find the standard equation of the parabola with vertex $(-4,2)$ and with directrix $y=5$ then sketch it.

Solution :

From the given information the parabola opens downwards.

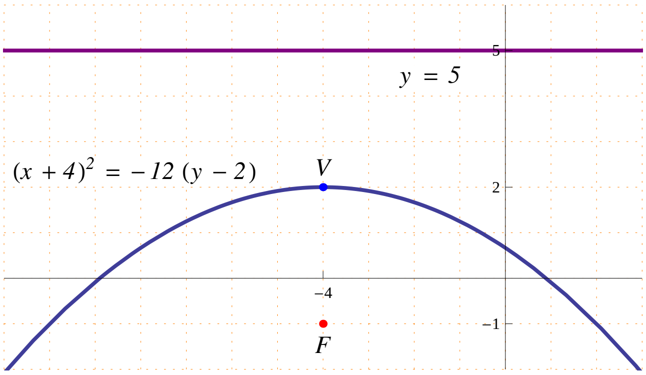
The formula for that parabola is $(x-h)^2 = -4a\ (y-k)$

Since the vertex is $(-4, 2)$ then $(h, k) = (-4, 2)$

a is the distance between the vertex and the directrix , hence $a = 3$

The standard equation of the parabola is $(x+4)^2 = -12(y-2)$

The focus is $F = (-4, -1)$



Q.5 Solve by Gauss method the linear system

$$\begin{array}{rcrcrcr} 7x & - & y & + & z & = & 4 \\ 3x & + & 2y & + & 2z & = & 17 \\ x & + & y & - & z & = & 4 \end{array}$$

Solution :

$$\left(\begin{array}{ccc|c} 7 & -1 & 1 & 4 \\ 3 & 2 & 2 & 17 \\ 1 & 1 & -1 & 4 \end{array}\right) \xrightarrow{R_1 \longleftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 3 & 2 & 2 & 17 \\ 7 & -1 & 1 & 4 \end{array}\right)$$

$$\xrightarrow{-3R_1+R_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 0 & -1 & 5 & 5 \\ 7 & -1 & 1 & 4 \end{array}\right) \xrightarrow{-7R_1+R_3} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 0 & -1 & 5 & 5 \\ 0 & -8 & 8 & -24 \end{array}\right)$$

$$\xrightarrow{-8R_2+R_3} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 0 & -1 & 5 & 5 \\ 0 & 0 & -32 & -64 \end{array}\right) \xrightarrow{-R_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 0 & 1 & -5 & -5 \\ 0 & 0 & -32 & -64 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{32}R_3} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 0 & 1 & -5 & -5 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

Therefore , $z = 2$

$y - 5z = -5 \ \Rightarrow\ y - 10 = -5 \ \Rightarrow\ y = -5 + 10 = 5$

$x + y - z = 4 \ \Rightarrow\ x + 5 - 2 = 4 \ \Rightarrow\ x + 3 = 4 \ \Rightarrow\ x = 1$

M 104 - GENERAL MATHEMATICS -2-

Dr. Tariq A. AlFadhel[5]

Solution of the Second Mid-Term Exam

Second semester 1433-1434 H

Q.1 Compute the integrals :

(a) $\displaystyle\int (3x^2+1)\sin(x^3+x+1)\ dx$

(b) $\displaystyle\int \frac{x+3}{(x-3)(x-2)}\ dx$

(c) $\displaystyle\int \frac{5}{x^2+1}\ dx$

(d) $\displaystyle\int (x^2+1)\ln x\ dx$

(e) $\displaystyle\int x^2 \sin x\ dx$

Solution :

(a) $\displaystyle\int (3x^2+1)\sin(x^3+x+1)\ dx = \int \sin(x^3+x+1)\ (3x^2+1)\ dx$

$= -\cos(x^3+x+1)+c$

(b) $\displaystyle\int \frac{x+3}{(x-3)(x-2)}\ dx$

Using the method of partial fractions

$$\frac{x+3}{(x-3)(x-2)} = \frac{A}{x-3}+\frac{B}{x-2}$$

$$\frac{x+3}{(x-3)(x-2)} = \frac{A(x-2)}{(x-3)(x-2)}+\frac{B(x-3)}{(x-2)(x-3)}$$

$x+3 = A(x-2)+B(x-3)$

Put $x=3$, then $3+3 = A(3-2)+B(3-3) \implies A=6$

Put $x=2$, then $2+3 = A(2-2)+B(2-3) \implies -B=5 \implies B=-5$

$$\frac{x+3}{(x-3)(x-2)} = \frac{6}{x-3}+\frac{-5}{x-2}$$

$$\int \frac{x+3}{(x-3)(x-2)}\ dx = \int \left(\frac{6}{x-3}-\frac{5}{x-2}\right)\ dx$$

$$= \int \frac{6}{x-3}\ dx - \int \frac{5}{x-2}\ dx = 6\int \frac{1}{x-3}\ dx - 5\int \frac{1}{x-2}\ dx$$

$= 6\ln|x-3| - 5\ln|x-2| + c$

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(c) $\displaystyle\int \frac{5}{x^2+1}\ dx = 5\int \frac{1}{x^2+1}\ dx = 5\tan^{-1}x + c$

(d) $\displaystyle\int (x^2+1)\ln x\ dx$

Using integration by parts

$$u = \ln x \qquad dv = (x^2+1)\ dx$$
$$du = \frac{1}{x}\ dx \qquad v = \frac{x^3}{3} + x$$

$$\int (x^2+1)\ln x\ dx = \left(\frac{x^3}{3}+x\right)\ln x - \int \left(\frac{x^3}{3}+x\right)\frac{1}{x}\ dx$$

$$= \left(\frac{x^3}{x}+x\right)\ln x - \int \left(\frac{x^2}{3}+1\right)\ dx$$

$$= \left(\frac{x^3}{x}+x\right)\ln x - \frac{1}{3}\int x^2\ dx - \int 1\ dx = \left(\frac{x^3}{3}+x\right)\ln x - \frac{1}{3}\,\frac{x^3}{3} - x + c$$

(e) $\displaystyle\int x^2\sin x\ dx$

Using integration by parts

$$u = x^2 \qquad dv = \sin x\ dx$$
$$du = 2x\ dx \qquad v = -\cos x$$

$$\int x^2\sin x\ dx = -x^2\cos x - \int 2x(-\cos x)\ dx = -x^2\cos x + 2\int x\cos x\ dx$$

Using integration by parts again

$$u = x \qquad dv = \cos x\ dx$$
$$du = dx \qquad v = \sin x$$

$$\int x^2\sin x\ dx = -x^2\cos x + 2\left(x\sin x - \int \sin x\ dx\right)$$

$$= -x^2\cos x + 2x\sin x + 2\cos x + c$$

Q.2 Find the area of the plane region bounded by the curves $x+y=2$, $y=2$ and $y=2x-4$.

Solution :

$y=2$, $y=2x-4$ and $y=-x+2$ are three straight lines.

Point of intersection of $y=2$ and $y=-x+2$:

$-x+2=2 \;\Rightarrow\; x=0$

$y=2$ and $y=-x+2$ intersect at the point $(0,2)$.

Point of intersection of $y=2$ and $y=2x-4$:

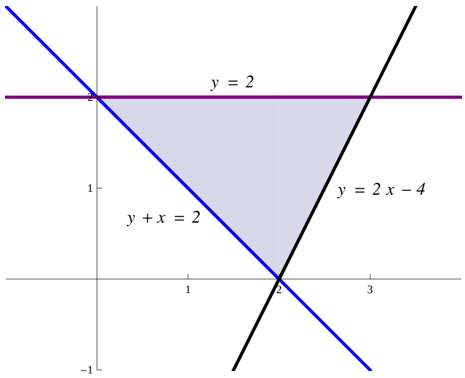
$2x - 4 = 2 \;\Rightarrow\; x = 3$

$y = 2$ and $y = 2x - 4$ intersect at the point $(3, 2)$

Point of intersection of $y = -x + 2$ and $y = 2x - 4$:

$2x - 4 = -x + 2 \;\Rightarrow\; 3x = 6 \;\Rightarrow\; x = 2$

$y = -x + 2$ and $y = 2x - 4$ intersect at the point $(2, 0)$.



$y + x = 2 \;\Rightarrow\; x = -y + 2$ and $y = 2x - 4 \;\Rightarrow\; 2x = y + 4 \;\Rightarrow\; x = \dfrac{1}{2}y + 2$

$$\text{Area} = \int_0^2 \left[\left(\frac{1}{2}y + 2 \right) - (-y + 2) \right] \, dy$$

$$\text{Area} = \int_0^2 \left(\frac{1}{2}y + y \right) \, dy = \int_0^2 \frac{3}{2}y \; dy$$

$$\text{Area} = \frac{3}{2} \left[\frac{y^2}{2} \right]_0^2 = \frac{3}{2} \left[\frac{2^2}{2} - \frac{0^2}{2} \right] = \frac{3}{2} \times 2 = 3$$

Q.3 The region R between the lines $x + y = 1$, $x = 1$ and $y = 2x + 1$ is rotated about the y-axis to form a solid of revolution S. Find the volume of S.

Solution :

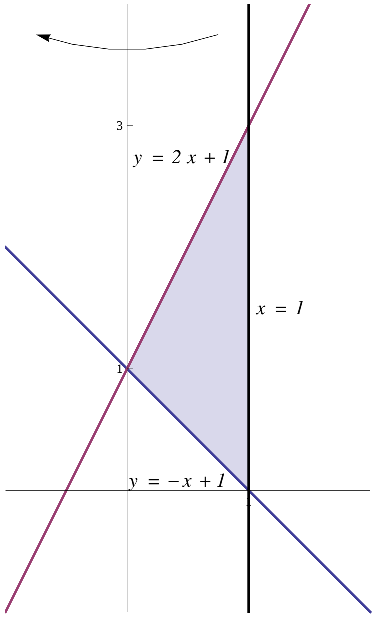
$y = -x + 1$ is a straight line with slope -1 and passing through $(0, 1)$.

$y = 2x + 1$ is a straight line with slope 2 and passing through $(0, 1)$.

$x = 1$ is a straight line parallel to the y-axis and passing through $(1, 0)$.

Point of intersection of $y = -x + 1$ and $y = -x + 2$:

$2x + 1 = -x + 1 \;\Rightarrow\; 3x = 0 \;\Rightarrow\; x = 0.$



Using Cylindrical shells method

$$\text{Volume} = 2\pi \int_0^1 x[(2x+1)-(-x+1)]\ dx = 2\pi \int_0^1 x(3x)\ dx = 2\pi \int_0^1 3x^2\ dx$$

$$\text{Volume} = 2\pi \left[x^3\right]_0^1 = 2\pi[1-0] = 2\pi$$

M 104 - GENERAL MATHEMATICS -2-

Dr. Tariq A. AlFadhel[6]

Solution of the Final Exam

Second semester 1433-1434 H

Q.1 **(a)** Compute (if possible) : $2\mathbf{AB}$ and $\mathbf{BA}$ for

$$\mathbf{A} = \begin{pmatrix} 0 & 2 & 0 \\ 3 & 2 & 3 \\ 0 & 2 & 0 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

(b) Compute the determinant $\begin{vmatrix} 1 & 0 & 3 \\ 4 & 3 & 2 \\ 5 & 6 & 0 \end{vmatrix}$.

(c) Solve : $\begin{cases} x - 2y = 5 \\ 4x - 3y = 10 \end{cases}$

Solution :

(a) $2\mathbf{AB} = (2\mathbf{A})\,\mathbf{B} = \begin{pmatrix} 2\times 0 & 2\times 2 & 2\times 0 \\ 2\times 3 & 2\times 2 & 2\times 3 \\ 2\times 0 & 2\times 2 & 2\times 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 0 & 4 & 0 \\ 6 & 4 & 6 \\ 0 & 4 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0+0+0 & 0+4+0 & 0+0+0 \\ 6+0+6 & 12+4+12 & 6+0+6 \\ 0+0+0 & 0+4+0 & 0+0+0 \end{pmatrix} = \begin{pmatrix} 0 & 4 & 0 \\ 12 & 28 & 12 \\ 0 & 4 & 0 \end{pmatrix}$$

$$\mathbf{BA} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 3 & 2 & 3 \\ 0 & 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0+6+0 & 2+4+2 & 0+6+0 \\ 0+3+0 & 0+2+0 & 0+3+0 \\ 0+6+0 & 2+4+2 & 0+6+0 \end{pmatrix} = \begin{pmatrix} 6 & 8 & 6 \\ 3 & 2 & 3 \\ 6 & 8 & 6 \end{pmatrix}$$

(b) Using Sarrus method $\begin{matrix} 1 & 0 & 3 & 1 & 0 \\ 4 & 3 & 2 & 4 & 2 \\ 5 & 6 & 0 & 5 & 6 \end{matrix}$

$$\begin{vmatrix} 1 & 0 & 3 \\ 4 & 3 & 2 \\ 5 & 6 & 0 \end{vmatrix} = (0+0+72) - (45+12+0) = 72 - 57 = 15$$

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(c) Using Gauss-Jordan method :

The augmented matrix is $\left(\begin{array}{cc|c} 1 & -2 & 5 \\ 4 & -3 & 10 \end{array}\right)$

$$\left(\begin{array}{cc|c} 1 & -2 & 5 \\ 4 & -3 & 10 \end{array}\right) \xrightarrow{-4R_1+R_2} \left(\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & 5 & -10 \end{array}\right)$$

$$\xrightarrow{\frac{1}{5}R_2} \left(\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & 1 & -2 \end{array}\right) \xrightarrow{2R_2+R_1} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -2 \end{array}\right)$$

The solution is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Using Cramer's rule :

$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 4 & -3 \end{pmatrix}$, $\mathbf{A}_1 = \begin{pmatrix} 5 & -2 \\ 10 & -3 \end{pmatrix}$ and $\mathbf{A}_2 = \begin{pmatrix} 1 & 5 \\ 4 & 10 \end{pmatrix}$

$$\det(\mathbf{A}) = |\mathbf{A}| = \begin{vmatrix} 1 & -2 \\ 4 & -3 \end{vmatrix} = (1 \times -3) - (4 \times -2) = -3 + 8 = 5$$

$$\det(\mathbf{A}_1) = |\mathbf{A}_1| = \begin{vmatrix} 5 & -2 \\ 10 & -3 \end{vmatrix} = (5 \times -3) - (10 \times -2) = -15 + 20 = 5$$

$$\det(\mathbf{A}_2) = |\mathbf{A}_2| = \begin{vmatrix} 1 & 5 \\ 4 & 10 \end{vmatrix} = (1 \times 10) - (4 \times 5) = 10 - 20 = -10$$

$$x = \frac{\det(\mathbf{A}_1)}{\det(\mathbf{A})} = \frac{5}{5} = 1$$

$$y = \frac{\det(\mathbf{A}_2)}{\det(\mathbf{A})} = \frac{-10}{5} = -2$$

The solution is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Q.2 **(a)** Find the standard equation of the ellipse with foci $(-2, 3)$ and $(6, 3)$, and the length of its major axis is 10 and then sketch it.

(b) Find the elements of the conic section $x^2 - 4y - 2x + 13 = 0$.

Solution :

(a) The standard equation of an ellipse is $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$

The center of the ellipse is the mid-point of the two foci

$$P(h, k) = \left(\frac{-2+6}{2}, \frac{3+3}{2}\right) = (2, 3).$$

The major axis passes through the two foci and it is parallel to the x-axis

The length of the major axis is 10 , means that $2a = 10 \Rightarrow a = 5$

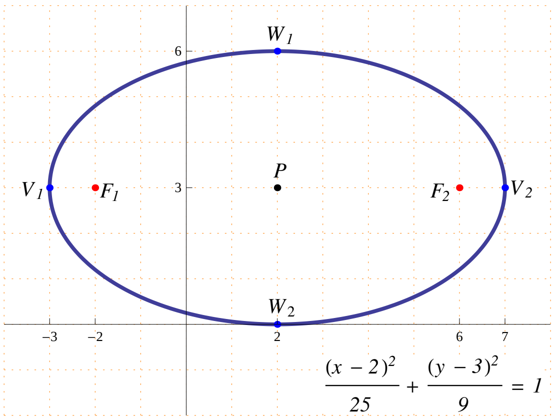
c is the distance between one of the foci and the center , hence $c = 4$

$c^2 = a^2 - b^2 \ \Rightarrow \ 16 = 25 - b^2 \ \Rightarrow \ b^2 = 9 \ \Rightarrow \ b = 3$

The standard equation of the ellipse is $\dfrac{(x-2)^2}{25} + \dfrac{(y-3)^2}{9} = 1$

The vertices of the ellipse are $V_1(-3,3)$ and $V_2(7,3)$

The end-points of the minor axis are $W_1(2,6)$ and $W_2(2,0)$



(b) $x^2 - 4y - 2x + 13 = 0 \implies x^2 - 2x = 4y - 13$

$\implies x^2 - 2x + 1 = 4y - 13 + 1 \implies (x-1)^2 = 4y - 12$

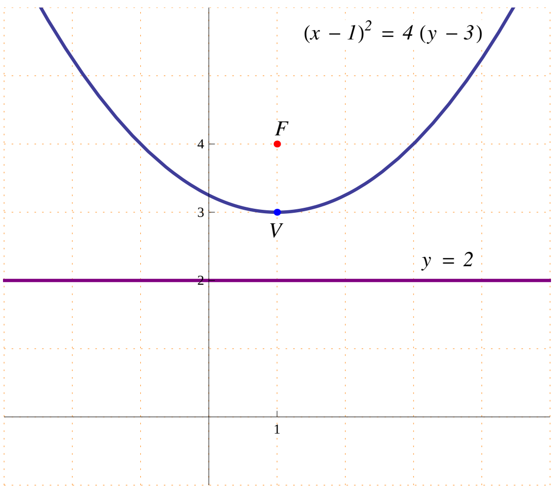
$\implies (x-1)^2 = 4(y-3)$

The conic section is a parabola with vertex $V(1,3)$ and opens upwards

$4a = 4 \ \Rightarrow \ a = 1$

The focus is $F(1,4)$

The equation of the directrix is $y = 2$



Q.3 **(a)** Compute the integrals :

(i) $\int \frac{1}{x^2+9}\,dx$ (ii) $\int x\cos x\,dx$ (iii) $\int \frac{x+1}{(x-1)^2}\,dx$

(b) Find the area of the surface determined by the curves :

$y = -x^2 + 4$ and $y = 0$.

(c) The region R between the curves $y = -x + 2$ and $y = x^2$ is rotated about the x-axis to form a solid of revolution S . Find the volume of S.

Solution :

(a) (i) $\int \frac{1}{x^2+9}\,dx = \int \frac{1}{x^2+3^2}\,dx = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + c$

(ii) $\int x\cos x\,dx$

Using integration by parts

$u = x \qquad dv = \cos x\,dx$
$du = dx \qquad v = \sin x$

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx$$

$$= x\sin x - (-\cos x) + c = x\sin x + \cos x + c$$

(iii) $\int \frac{x+1}{(x-1)^2}\,dx$

Using the method of partial fractions

$$\frac{x+1}{(x-1)^2} = \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2}$$

$$\frac{x+1}{(x-1)^2} = \frac{A_1(x-1)}{(x-1)^2} + \frac{A_2}{(x-1)^2} = \frac{A_1(x-1)+A_2}{(x-1)^2}$$

$$x + 1 = A_1(x-1) + A_2 = A_1x - A_1 + A_2$$

By comparing the coefficients of both sides :

$$\begin{array}{rcrcl} A_1 & & & = 1 & \longrightarrow (1) \\ -A_1 & + & A_2 & = 1 & \longrightarrow (2) \end{array}$$

From equation (1) : $A_1 = 1$.

From equation (2) : $-1 + A_2 = 1 \implies A_2 = 2$

$$\frac{x+1}{(x-1)^2} = \frac{1}{x-1} + \frac{2}{(x-1)^2}$$

$$\int \frac{x+1}{(x-1)^2}\,dx = \int \left(\frac{1}{x-1} + \frac{2}{(x-1)^2}\right)\,dx$$

$$= \int \frac{1}{x-1}\, dx + \int \frac{2}{(x-1)^2}\, dx = \int \frac{1}{x-1}\, dx + 2\int (x-1)^{-2}\, dx$$

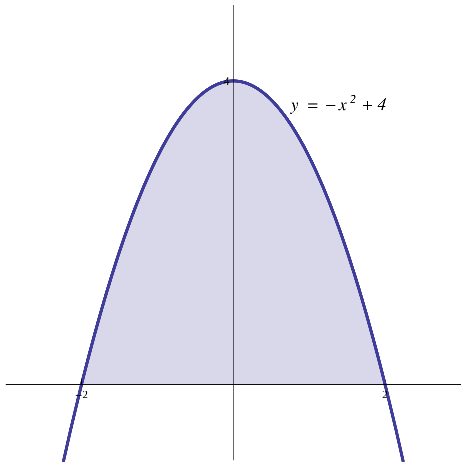
$$= \ln|x-1| + 2\frac{(x-1)^{-1}}{-1} + c = \ln|x-1| - \frac{2}{x-1} + c$$

(b) $y = 0$ is the x-axis .

$y = -x^2 + 4 \implies x^2 = -(y-4)$ is a parabola with vertex $(0, 4)$ and opens downwards.

Point of intersections of $y = -x^2 + 4$ and $y = 0$:

$-x^2 + 4 = 0 \;\Rightarrow\; x^2 - 4 = 0 \;\Rightarrow\; (x-2)(x+2) = 0 \;\Rightarrow\; x = -2\ ,\ x = 2$



$$\text{Area} = \int_{-2}^{2} (-x^2 + 4)\, dx = \left[-\frac{x^3}{3} + 4x\right]_{-2}^{2}$$

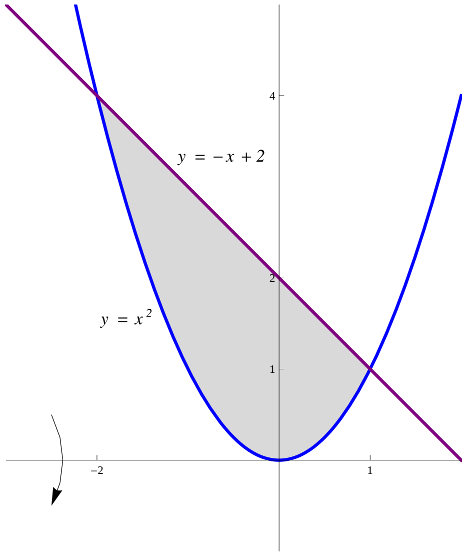
$$= \left(-\frac{8}{3} + 8\right) - \left(\frac{8}{3} - 8\right) = -\frac{8}{3} + 8 - \frac{8}{3} + 8 = 16 - \frac{16}{3} = \frac{48-16}{3} = \frac{32}{3}$$

(c) $y = -x + 2$ is a straight line with slope -1 and passing through $(0, 2)$.

$y = x^2$ is a parabola with vertex $(0, 0)$ and opens upwards .

Points of intersection of $y = -x + 2$ and $y = x^2$

$x^2 = -x + 2 \;\Rightarrow\; x^2 + x - 2 = 0 \;\Rightarrow\; (x-1)(x+2) = 0 \;\Rightarrow\; x = 1\ ,\ x = -2$



Using Washer Method :

$$\text{Volume} = \pi \int_{-2}^{1} \left[(-x+2)^2 - (x^2)^2\right] \; dx = \pi \int_{-2}^{1} (x^2 - 4x + 4 - x^4) \; dx$$

$$= \pi \int_{-2}^{1} (-x^4 + x^2 - 4x + 4) \; dx = \pi \left[-\frac{x^5}{5} + \frac{x^3}{3} - 2x^2 + 4x \right]_{-2}^{1}$$

$$= \pi \left[\left(-\frac{1}{5} + \frac{1}{3} - 2 + 4 \right) - \left(\frac{32}{5} - \frac{8}{3} - 8 - 8 \right) \right]$$

$$= \pi \left(-\frac{1}{5} + \frac{1}{3} + 2 - \frac{32}{5} + \frac{8}{3} + 16 \right) = \pi \left(-\frac{33}{5} + \frac{9}{3} + 2 + 16 \right)$$

$$= \pi \left(-\frac{33}{5} + 21 \right) = \pi \left(\frac{-33 + 105}{5} \right) = \frac{72}{5}\pi$$

Q.4 **(a)** Find f_x and f_y for the function $f(x, y) = x^2y^3 + xy\ln(y + x)$

(b) Solve the differential equation $y' + y^3e^x = 0$

Solution :

(a) $$f_x = \frac{\partial f}{\partial x} = 2xy^3 + y\ln(x+y) + xy\,\frac{1}{x+y} = 2xy^3 + y\ln(x+y) + \frac{xy}{x+y}$$

$$f_y = \frac{\partial f}{\partial y} = x^2(3y^2) + x\ln(x+y) + xy\,\frac{1}{x+y} = 3x^2y^2 + x\ln(x+y) + \frac{xy}{x+y}$$

(b) $y' + y^3e^x = 0$

$y' = -y^3e^x$

$$\frac{dy}{dx} = -y^3 e^x$$

$$-\frac{1}{y^3}\ dy = e^x\ dx$$

$$-y^{-3}\ dy = e^x\ dx$$

$$\int -y^{-3}\ dy = \int e^x\ dx$$

$$-\frac{y^{-2}}{-2} = e^x + c$$

$$\frac{1}{2y^2} = e^x + c$$

$$\frac{1}{y^2} = 2(e^x + c)$$

$$y^2 = \frac{1}{2(e^x + c)}$$

$$y = \sqrt{\frac{1}{2(e^x + c)}}$$