

M 104 - GENERAL MATHEMATICS -2-

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Solution of the First Mid-Term Exam

First semester 1437-1438 H

Q.1 Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 2 & 0 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 0 & 1 \\ 3 & 6 \\ 1 & 3 \end{pmatrix}$.

Compute (if possible) : \mathbf{AB} and $\mathbf{B+C}$

Solution :

$$\mathbf{AB} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+0+0 & 1+4+3 \\ 0+0+0 & 0+4+2 \\ 2+0+0 & 2+0+1 \end{pmatrix} = \begin{pmatrix} 1 & 8 \\ 0 & 6 \\ 2 & 3 \end{pmatrix}$$

$$\mathbf{B+C} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 3 & 6 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1+0 & 1+1 \\ 0+3 & 2+6 \\ 0+1 & 1+3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 8 \\ 1 & 4 \end{pmatrix}$$

Q.2 Compute The determinant $\begin{vmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ -2 & 0 & 1 \end{vmatrix}$

Solution (1) : Using Sarrus Method

$$\begin{array}{ccccc} 1 & -2 & 1 & 1 & -2 \\ 2 & 1 & -1 & 2 & 1 \\ -2 & 0 & 1 & -2 & 0 \end{array}$$

$$\begin{vmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ -2 & 0 & 1 \end{vmatrix} = (1+(-4)+0) - (-2+0+(-4)) = -3 - (-6) = -3+6 = 3$$

Solution (2) : By the definition (using third row)

$$\begin{vmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ -2 & 0 & 1 \end{vmatrix} = -2 \times \begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} - 0 \times \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} + 1 \times \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}$$
$$= -2(2-1) - 0 + 1(1-(-4)) = -2(1) + 1(5) = -2+5 = 3$$

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Q.3 Solve by Gauss-Jordan the linear system :

$$\begin{cases} x & - & 2y & + & z & = & 0 \\ 2x & - & 3y & - & z & = & -2 \\ -2x & & & + & z & = & -1 \end{cases}$$

Solution : The augmented matrix is

$$\begin{aligned} & \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & -3 & -1 & -2 \\ -2 & 0 & 1 & -1 \end{array} \right) \xrightarrow{-2R_1+R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & -2 \\ -2 & 0 & 1 & -1 \end{array} \right) \\ & \xrightarrow{2R_1+R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & -2 \\ 0 & -4 & 3 & -1 \end{array} \right) \xrightarrow{4R_2+R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & -9 & -9 \end{array} \right) \\ & \xrightarrow{\frac{1}{-9}R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 1 & 1 \end{array} \right) \xrightarrow{3R_3+R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array} \right) \\ & \xrightarrow{-R_3+R_1} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array} \right) \xrightarrow{2R_2+R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array} \right) \end{aligned}$$

The solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Q.4 Find all the elements of the conic section $y^2 - 4x^2 + 2y + 8x - 7 = 0$ and sketch it.

Solution :

$$y^2 - 4x^2 + 2y + 8x - 7 = 0$$

$$y^2 + 2y - 4x + 8x = 7$$

$$(y^2 + 2y) - 4(x^2 - 2x) = 7$$

By completing the square

$$(y^2 + 2y + 1) - 4(x^2 - 2x + 1) = 7 + 1 - 4$$

$$(y + 1)^2 - 4(x - 1)^2 = 4$$

$$\frac{(y + 1)^2}{4} + \frac{4(x - 1)^2}{4} = 1$$

$$\frac{(y + 1)^2}{4} + \frac{(x - 1)^2}{1} = 1$$

The conic section is a Hyperbola .

The center is $P(1, -1)$.

$$a^2 = 1 \implies a = 1$$

$$b^2 = 4 \implies b = 2$$

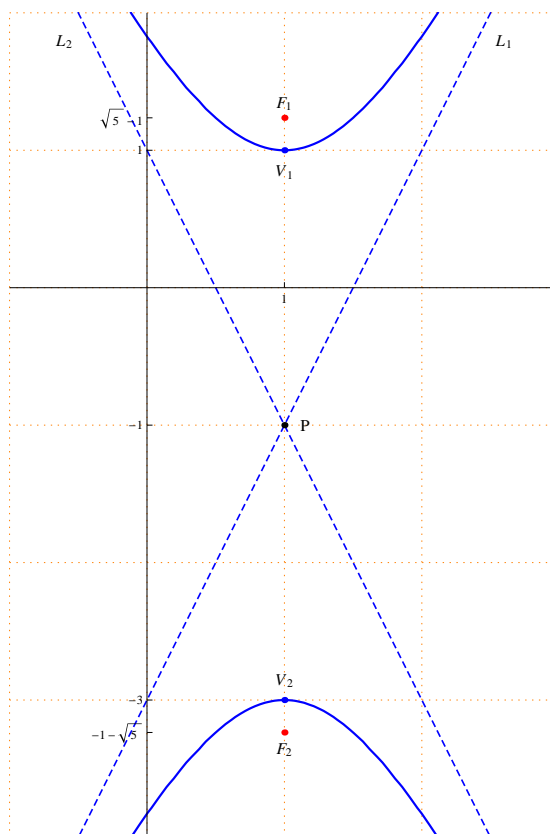
$$c^2 = a^2 + b^2 = 1 + 4 = 5 \implies c = \sqrt{5}$$

The vertices are $V_1(1, 1)$ and $V_2(1, -3)$

The foci are $F_1(1, -1 + \sqrt{5})$ and $F_2(1, -1 - \sqrt{5})$

The equations of the asymptotes are $L_1 : (y + 1) = 2(x - 1)$

and $L_2 : (y + 1) = -2(x - 1)$



Q.5 Find the standard equation of the parabola with focus $F(5, 1)$ and with directrix $x = -1$, and sketch it.

Solution :

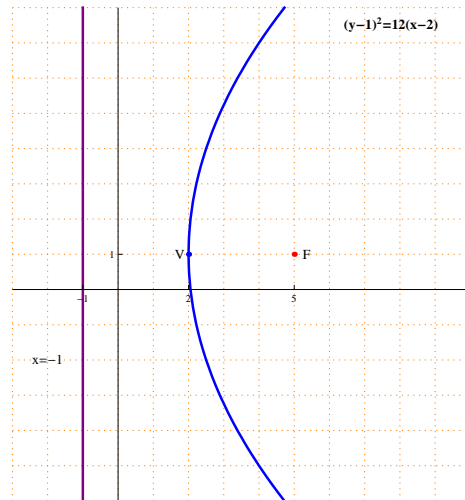
From the positions of the focus and directrix the parabola opens to the right.

The equation of the parabola has the form $(y - k)^2 = 4a(x - h)$.

The vertex lies between the focus and the directrix , hence the vertex is $V(2, 1)$

a is the distance between the focus and the vertex , hence $a = 3$

The standard equation of the parabola is $(y - 1)^2 = 12(x - 2)$



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Solution of the Second Mid-Term Exam

First semester 1437-1438 H

Q.1 Compute the integrals :

(a) $\int_0^1 16x(x^2 - 1)^7 dx$

(b) $\int \frac{2x - 1}{(x - 2)(x - 3)} dx$

(c) $\int (x + 1) \sin x dx$

(d) $\int x^3 \ln |x| dx$

(e) $\int \frac{2x}{(x + 1)^2} dx$

Solution :

$$\begin{aligned} \text{(a)} \int_0^1 16x(x^2 - 1)^7 dx &= 8 \int_0^1 (x^2 - 1)^7 2x dx = 8 \left[\frac{(x^2 - 1)^8}{8} \right]_0^1 \\ &= 8 \left[\frac{(1 - 1)^8}{8} - \frac{(0 - 1)^8}{8} \right] = 8 \left(0 - \frac{1}{8} \right) = 0 - 1 = -1 \end{aligned}$$

Using the formula $\int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + c$, where $n \neq -1$

(b) $\int \frac{2x - 1}{(x - 2)(x - 3)} dx$

Using the method of partial fractions

$$\frac{2x - 1}{(x - 2)(x - 3)} = \frac{A_1}{x - 2} + \frac{A_2}{x - 3}$$

$$2x - 1 = A_1(x - 3) + A_2(x - 2)$$

Put $x = 2$:

$$2(2) - 1 = A_1(2 - 3) \implies A_1 = -3$$

Put $x = 3$:

$$2(3) - 1 = A_2(3 - 2) \implies A_2 = 5$$

$$\int \frac{2x - 1}{(x - 2)(x - 3)} dx = \int \left(\frac{-3}{x - 2} + \frac{5}{x - 3} \right) dx$$

$$= -3 \int \frac{1}{x - 2} dx + 5 \int \frac{1}{x - 3} dx = -3 \ln |x - 2| + 5 \ln |x - 3| + c$$

$$(c) \int (x+1) \sin x \, dx$$

Using integration by parts

$$\begin{aligned} u &= x+1 & dv &= \sin x \, dx \\ du &= dx & v &= -\cos x \end{aligned}$$

$$\begin{aligned} \int (x+1) \sin x \, dx &= (x+1)(-\cos x) - \int -\cos x \, dx \\ &= -(x+1) \cos x + \int \cos x \, dx = -(x+1) \cos x + \sin x + c \end{aligned}$$

$$(d) \int x^3 \ln |x| \, dx$$

Using integration by parts :

$$\begin{aligned} u &= \ln |x| & dv &= x^3 \, dx \\ du &= \frac{1}{x} \, dx & v &= \frac{x^4}{4} \end{aligned}$$

$$\begin{aligned} \int \ln |x| \, dx &= \frac{x^4}{4} \ln |x| - \int \frac{x^4}{4} \frac{1}{x} \, dx \\ &= \frac{x^4}{4} \ln |x| - \frac{1}{4} \int x^3 \, dx = \frac{x^4}{4} \ln |x| - \frac{1}{4} \frac{x^4}{4} + c \end{aligned}$$

$$(e) \int \frac{2x}{(x+1)^2} \, dx$$

Using the method of partial fractions

$$\frac{2x}{(x+1)^2} = \frac{A_1}{x+1} + \frac{A_2}{(x+1)^2}$$

$$\frac{2x}{(x+1)^2} = \frac{A_1(x+1)}{(x+1)^2} + \frac{A_2}{(x+1)^2}$$

$$2x = A_1(x+1) + A_2 = A_1x + (A_1 + A_2)$$

By comparing the coefficients of both sides :

$$\begin{aligned} A_1 &= 2 & \longrightarrow & (1) \\ A_1 + A_2 &= 0 & \longrightarrow & (2) \end{aligned}$$

From equations (1) and (2) : $A_1 = 2$ and $A_2 = -2$

$$\begin{aligned} \int \frac{2x}{(x+1)^2} \, dx &= \int \left(\frac{2}{x+1} + \frac{-2}{(x+1)^2} \right) \, dx \\ &= 2 \int \frac{1}{x+1} \, dx - 2 \int (x+1)^{-2} \, dx = 2 \ln |x+1| - 2 \frac{(x+1)^{-1}}{-1} + c \end{aligned}$$

$$= 2 \ln |x + 1| + \frac{2}{x + 1} + c$$

Q.2 Find the area of the region bounded by the curves :

$$y = x^2 + 1 \text{ and } y = 2$$

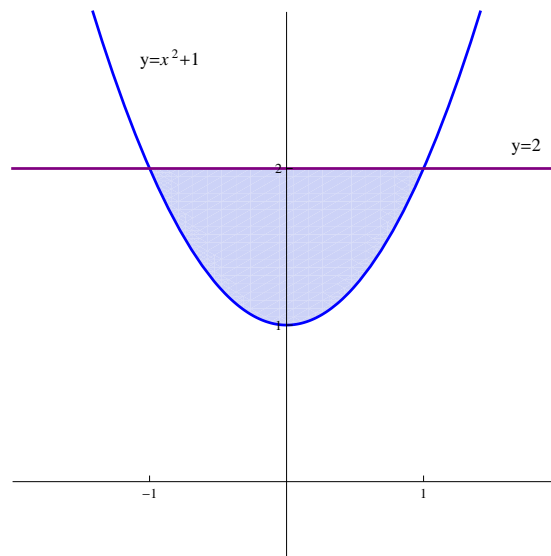
Solution :

$y = 2$ is a straight line parallel to the x -axis and passes through $(0, 2)$

$y = x^2 + 1$ is a parabola opens upwards with vertex $(0, 1)$

Points of intersection of $y = x^2 + 1$ and $y = 2$:

$$x^2 + 1 = 2 \implies x^2 = 1 \implies x = \pm 1$$



$$\begin{aligned} \text{Area} &= \int_{-1}^1 [2 - (x^2 + 1)] dx = \int_{-1}^1 (2 - x^2 - 1) dx = \int_{-1}^1 (1 - x^2) dx \\ &= \left[x - \frac{x^3}{3} \right]_{-1}^1 = \left(1 - \frac{1}{3} \right) - \left(-1 - \frac{-1}{3} \right) = 1 - \frac{1}{3} + 1 - \frac{1}{3} \\ &= 2 - \frac{2}{3} = \frac{6 - 2}{3} = \frac{4}{3} \end{aligned}$$

Q.3 Find the volume of the solid of revolution generated by rotation about the x -axis of the region \mathbf{R} limited by the following curves :

$$y = 0, x = -1, x = 1 \text{ and } y = 2x^2$$

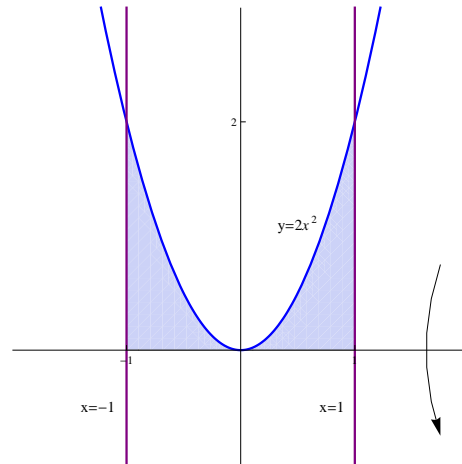
Solution :

$y = 0$ is the x -axis

$x = -1$ is a straight line parallel to the y -axis and passes through $(-1, 0)$

$x = 1$ is a straight line parallel to the y -axis and passes through $(1, 0)$

$y = 2x^2$ is a parabola opens upwards with vertex $(0, 0)$



Using Disk method :

$$\begin{aligned} \text{Volume} &= \pi \int_{-1}^1 (2x^2)^2 dx = \pi \int_{-1}^1 4x^4 dx = 4\pi \int_{-1}^1 x^4 dx \\ &= 4\pi \left[\frac{x^5}{5} \right]_{-1}^1 = 4\pi \left[\frac{1}{5} - \left(\frac{-1}{5} \right) \right] = 4\pi \left(\frac{2}{5} \right) = \frac{8\pi}{5} \end{aligned}$$

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Solution of the Final Exam

First semester 1437-1438 H

Q.1 (a) Compute (if possible) \mathbf{AB} and \mathbf{BA} for $\mathbf{A} = \begin{pmatrix} 0 & 3 & 1 \\ 2 & 3 & 2 \\ 2 & 3 & 1 \end{pmatrix}$

and $\mathbf{B} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \\ 2 & 4 \end{pmatrix}$

(b) Compute the determinant $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 2 & 1 & 1 \end{vmatrix}$.

(c) Solve by Gauss Method : $\begin{cases} x - y + z = 2 \\ x - 2y + z = 0 \\ 2x - y + z = 3 \end{cases}$

Solution :

(a) $\mathbf{AB} = \begin{pmatrix} 0 & 3 & 1 \\ 2 & 3 & 2 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \\ 2 & 4 \end{pmatrix}$
 $= \begin{pmatrix} 0+0+2 & 0+3+4 \\ 2+0+4 & 6+3+8 \\ 2+0+2 & 6+3+4 \end{pmatrix} = \begin{pmatrix} 2 & 7 \\ 6 & 17 \\ 4 & 13 \end{pmatrix}$

\mathbf{BA} is impossible , because the number of columns of \mathbf{B} does not equal the number of rows of \mathbf{A}

(b) Solution (1): Using Sarrus Method

$$\begin{array}{cccccc} 1 & 2 & 3 & 1 & 2 & \\ & 1 & 3 & 1 & 1 & 3 \\ & 2 & 1 & 1 & 2 & 1 \end{array}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 2 & 1 & 1 \end{vmatrix} = (3 + 4 + 3) - (18 + 1 + 2) = 10 - 21 = -11$$

Solution (2) : Using the definition (using the first row) :

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 2 & 1 & 1 \end{vmatrix} = 1 \times \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} - 2 \times \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + 3 \times \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}$$
$$= (3 - 1) - 2(1 - 2) + 3(1 - 6) = 2 + 2 - 15 = -11$$

(c) Using Gauss Method :

$$\begin{pmatrix} 1 & -1 & 1 & | & 2 \\ 1 & -2 & 1 & | & 0 \\ 2 & -1 & 1 & | & 3 \end{pmatrix} \xrightarrow{-R_1+R_2} \begin{pmatrix} 1 & -1 & 1 & | & 2 \\ 0 & -1 & 0 & | & -2 \\ 2 & -1 & 1 & | & 3 \end{pmatrix}$$

$$\xrightarrow{-2R_1+R_3} \begin{pmatrix} 1 & -1 & 1 & | & 2 \\ 0 & -1 & 0 & | & -2 \\ 0 & 1 & -1 & | & -1 \end{pmatrix} \xrightarrow{R_2+R_3} \begin{pmatrix} 1 & -1 & 1 & | & 2 \\ 0 & -1 & 0 & | & -2 \\ 0 & 0 & -1 & | & -3 \end{pmatrix}$$

$$-z = -3 \implies z = 3$$

$$-y = -2 \implies y = 2$$

$$x - y + z = 2 \implies x - 2 + 3 = 2 \implies x = 1$$

The solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Q.2 (a) Find the standard equation of the hyperbola with foci $F_1(2, 0)$, $F_2(-2, 0)$ and a vertex $V_1(1, 0)$ then sketch it.

(b) Find the elements of the conic section $x^2 + 4y^2 - 8y + 4x + 4 = 0$ and sketch it.

Solution :

(a) The two foci and the vertex are located on the x -axis.

The standard equation of the hyperbola is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

$$P(h, k) = \left(\frac{-2+2}{2}, \frac{0+0}{2} \right) = (0, 0) , \text{ hence } h = 0 \text{ and } k = 0$$

a is the distance between V_1 and P , hence $a = 1$

c is the distance between F_1 and P , hence $c = 2$

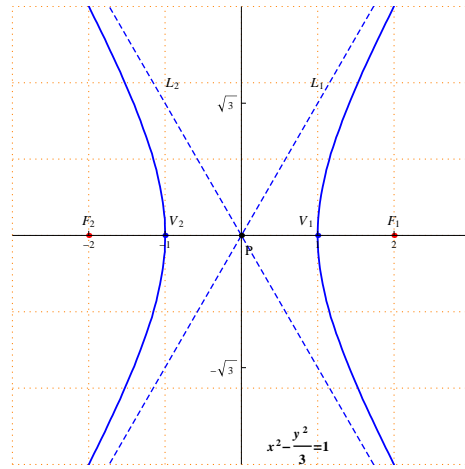
$$c^2 = a^2 + b^2 \implies 4 = 1 + b^2 \implies b^2 = 3 \implies b = \sqrt{3}$$

The standard equation of the hyperbola is $\frac{x^2}{1} - \frac{y^2}{3} = 1$

The other vertex is $V_2 = (0 - 1, 0) = (-1, 0)$

The equations of the asymptotes are :

$$L_1 : y = \sqrt{3}x \text{ and } L_2 : y = -\sqrt{3}x$$



(b) $x^2 + 4y^2 - 8y + 4x + 4 = 0$

$$x^2 - 4x + 4y^2 - 8y = -4$$

$$x^2 - 4x + 4(y^2 - 2y) = -4$$

By completing the square

$$(x^2 - 4x + 4) + 4(y^2 - 2y + 1) = -4 + 4 + 4$$

$$(x - 2)^2 + 4(y - 1)^2 = 4$$

$$\frac{(x - 2)^2}{4} + \frac{(y - 1)^2}{1} = 1$$

The conic section is an ellipse.

The center is $P(2, 1)$.

$$a^2 = 4 \implies a = 2.$$

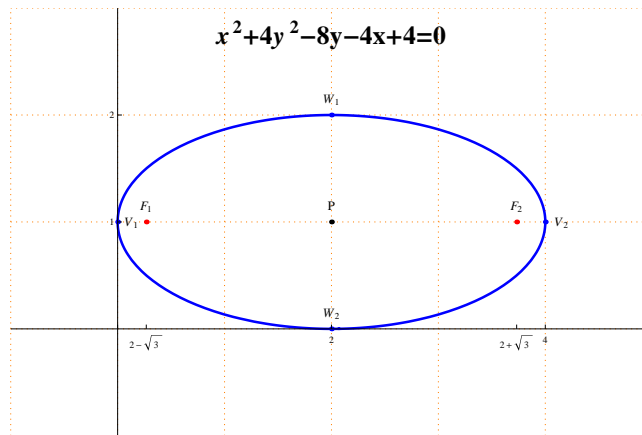
$$b^2 = 1 \implies b = 1.$$

$$c^2 = a^2 - b^2 = 4 - 1 = 3 \implies c = \sqrt{3}.$$

The vertices are $V_1(0, 1)$ and $V_2(4, 1)$

The foci are $F_1(2 - \sqrt{3}, 1)$ and $F_2(2 + \sqrt{3}, 1)$.

The end-points of the minor axis are $W_1(2, 2)$ and $W_2(2, 0)$.



Q.3 (a) Compute the integrals :

(i) $\int 6x(3x^2 + 9)^{15} dx$ (ii) $\int x^{10} \ln x dx$ (iii) $\int \frac{1}{(x-1)(x-2)} dx$

(b) Find the area of the region bounded by the graphs :

$y = 5$ and $y = x^2 + 1$.

(c) The region R between the curves $y = x^2$ and $y = x$ is rotated about the y -axis to form a solid of revolution S . Find the volume of S .

(d) Using polar coordinates find the area of the region lying in the first quadrant bounded by the circles with polar equations $r = 1$ and $r = 2$.

Solution :

(a) (i) $\int 6x(3x^2 + 9)^{15} dx = \frac{(3x^2 + 9)^{16}}{16} + c$

Using the formula $\int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + c$ where $n \neq -1$.

(ii) $\int x^{10} \ln x dx$

Using integration by parts

$$\begin{aligned} u &= \ln x & dv &= x^{10} dx \\ du &= \frac{1}{x} dx & v &= \frac{x^{11}}{11} \end{aligned}$$

$$\int x^{10} \ln x dx = \frac{x^{11}}{11} \ln x - \int \frac{x^{11}}{11} \frac{1}{x} dx$$

$$= \frac{x^{11}}{11} \ln x - \frac{1}{11} \int x^{10} dx = \frac{x^{11}}{11} \ln x - \frac{1}{11} \frac{x^{11}}{11} + c$$

(iii) $\int \frac{1}{(x-1)(x-2)} dx$

Using the method of partial fractions

$$\frac{1}{(x-1)(x-2)} = \frac{A_1}{x-1} + \frac{A_2}{x-2}$$

$$1 = A_1(x-2) + A_2(x-1)$$

$$\text{Put } x = 1 \text{ then } 1 = A_1(1-2) \implies A_1 = -1$$

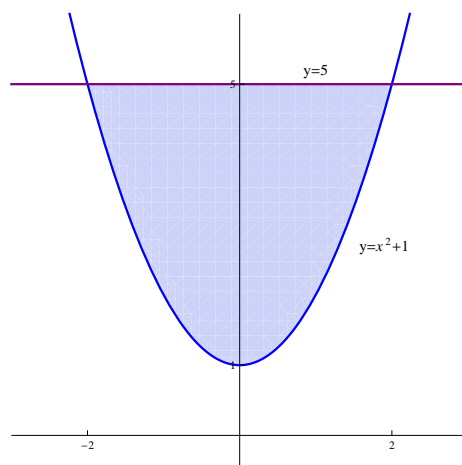
$$\text{Put } x = 2 \text{ then } 1 = A_2(2-1) \implies A_2 = 1$$

$$\int \frac{1}{(x-1)(x-2)} dx = \int \left(\frac{-1}{x-1} + \frac{1}{x-2} \right) dx$$

$$= - \int \frac{1}{x-1} dx + \int \frac{1}{x-2} dx = -\ln|x-1| + \ln|x-2| + c$$

(b) $y = 5$ is a straight line parallel to the x -axis and passes through $(0, 5)$.

$y = x^2 + 1 \implies y - 1 = x^2$ is a parabola with vertex $(1, 0)$ and opens upwards.



Points of intersection of $y = 5$ and $y = x^2 + 1$:

$$x^2 + 1 = 5 \implies x^2 - 4 = 0 \implies (x-2)(x+2) = 0 \implies x = -2, x = 2$$

$$\text{Area} = \int_{-2}^2 [5 - (x^2 + 1)] dx = \int_{-2}^2 (4 - x^2) dx = \left[4x - \frac{x^3}{3} \right]_{-2}^2$$

$$= \left(4 \times 2 - \frac{2^3}{3} \right) - \left(4 \times -2 - \frac{(-2)^3}{3} \right) = \left(8 - \frac{8}{3} \right) - \left(-8 + \frac{8}{3} \right)$$

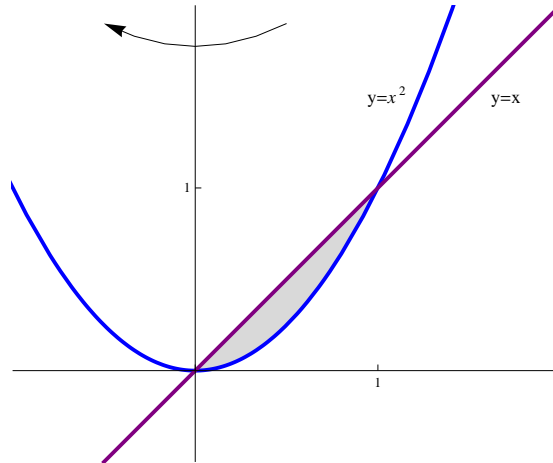
$$= 8 - \frac{8}{3} + 8 - \frac{8}{3} = 16 - \frac{16}{3} = \frac{48 - 16}{3} = \frac{32}{3}$$

(c) $y = x$ is a straight line passing through $(0, 0)$ with slope equals 1.

$y = x^2$ is a parabola opens upwards with vertex $(0, 0)$.

Points of intersection of $y = x^2$ and $y = x$:

$$x^2 = x \implies x^2 - x = 0 \implies x(x - 1) = 0 \implies x = 0, x = 1$$

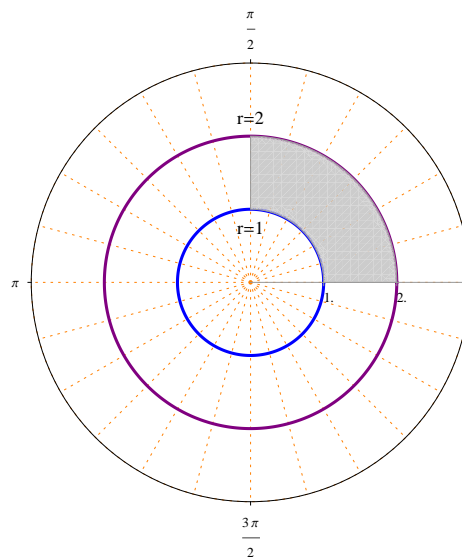


Using Cylindrical shells Method :

$$\begin{aligned} \text{Volume} &= 2\pi \int_0^1 x(x - x^2) dx = 2\pi \int_0^1 (x^2 - x^3) dx = 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 \\ &= 2\pi \left[\left(\frac{1}{3} - \frac{1}{4} \right) - (0 - 0) \right] = 2\pi \left(\frac{4 - 3}{12} \right) = \frac{\pi}{6} \end{aligned}$$

(d) $r = 1$ is a circle with center $(0, 0)$ and radius equals 1.

$r = 2$ is a circle with center $(0, 0)$ and radius equals 2.



$$\begin{aligned} \text{Area} &= \frac{1}{2} \int_0^{\frac{\pi}{2}} [(2)^2 - (1)^2] d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} (4 - 1) d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} 3 d\theta \\ &= \frac{3}{2} \int_0^{\frac{\pi}{2}} d\theta = \frac{3}{2} [\theta]_0^{\frac{\pi}{2}} = \frac{3}{2} \left(\frac{\pi}{2} - 0 \right) = \frac{3\pi}{4} \end{aligned}$$

Q.4 (a) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the implicit function z defined by the equation $x^3 z^2 \sin(y) + (x + y)e^z = 0$.

(b) Solve the differential equation $\frac{dy}{dx} - 2xy^2 = 0$.

Solution :

(a) Put $F(x, y, z) = x^3 z^2 \sin(y) + (x + y)e^z$

$$F_x = (3x^2)z^2 \sin(y) + (1 + 0)e^z = 3x^2 z^2 \sin(y) + e^z$$

$$F_y = x^3 z^2 \cos(y) + (0 + 1)e^z = x^3 z^2 \cos(y) + e^z$$

$$F_z = x^3 (2z) \sin(y) + (x + y)e^z = 2x^3 z \sin(y) + (x + y)e^z$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 z^2 \sin(y) + e^z}{2x^3 z \sin(y) + (x + y)e^z}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{x^3 z^2 \cos(y) + e^z}{2x^3 z \sin(y) + (x + y)e^z}$$

(b) $\frac{dy}{dx} - 2xy^2 = 0$

$$\frac{dy}{dx} = 2xy^2$$

$$\frac{1}{y^2} dy = 2x dx$$

It is a separable differential equation.

$$\int \frac{1}{y^2} dy = \int 2x dx$$

$$\int y^{-2} dy = \int 2x dx$$

$$\frac{y^{-1}}{-1} = x^2 + c$$

$$\frac{-1}{y} = x^2 + c$$

$$\frac{1}{y} = -x^2 - c$$

$$y = \frac{1}{-x^2 - c}$$

M 104 - GENERAL MATHEMATICS -2-

Dr. Tariq A. AlFadhel

Solution of the First Mid-Term Exam

Second semester 1437-1438 H

Q.1 Let $\mathbf{A} = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 3 & 2 \\ 2 & 0 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 2 & 1 \\ 3 & -3 \\ 1 & 7 \end{pmatrix}$.

Compute (if possible) : \mathbf{AB} and \mathbf{BC}

Solution :

$$\begin{aligned} \mathbf{AB} &= \begin{pmatrix} 2 & 2 & 1 \\ 0 & 3 & 2 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2+4+1 & 2+0+1 \\ 0+6+2 & 0+0+2 \\ 2+0+1 & 2+0+1 \end{pmatrix} = \begin{pmatrix} 7 & 3 \\ 8 & 2 \\ 3 & 3 \end{pmatrix} \end{aligned}$$

\mathbf{BC} is impossible, because the number of columns of \mathbf{B} does not equal the number of rows of \mathbf{C}

Q.2 Compute The determinant $\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix}$

Solution (1) : Using Sarrus Method

$$\begin{array}{cccccc} 1 & 2 & 1 & 1 & 2 & \\ & 2 & 1 & 1 & 2 & 1 \\ & 2 & 0 & 1 & 2 & 0 \end{array}$$

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = (1 + 4 + 0) - (2 + 0 + 4) = 5 - 6 = -1$$

Solution (2) : By the definition (using third row)

$$\begin{aligned} \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} &= 2 \times \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} - 0 \times \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + 1 \times \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \\ &= 2(2 - 1) - 0 + 1(1 - 4) = 2(1) + 1(-3) = 2 - 3 = -1 \end{aligned}$$

Q.3 Solve by Gauss method the linear system :

$$\begin{cases} x - 2y + z = 0 \\ 2x - 3y - z = 3 \\ -x + y + z = -2 \end{cases}$$

Solution : The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & -3 & -1 & 3 \\ -1 & 1 & 1 & -2 \end{array} \right) \xrightarrow{-2R_1+R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & 3 \\ -1 & 1 & 1 & -2 \end{array} \right)$$

$$\xrightarrow{R_1+R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & 3 \\ 0 & -1 & 2 & -2 \end{array} \right) \xrightarrow{R_2+R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & -1 & 1 \end{array} \right)$$

$$-z = 1 \implies z = -1$$

$$y - 3z = 3 \implies y + 3 = 3 \implies y = 0$$

$$x - 2y + z = 0 \implies x - 2(0) + (-1) = 0 \implies x = 1$$

The solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Q.4 Find the elements of the conic section $5y^2 - 4x^2 + 10y + 8x - 19 = 0$ and sketch it.

Solution :

$$5y^2 - 4x^2 + 10y + 8x - 19 = 0$$

$$5y^2 + 10y - 4x^2 + 8x = 19$$

$$5(y^2 + 2y) - 4(x^2 - 2x) = 19$$

By completing the square

$$5(y^2 + 2y + 1) - 4(x^2 - 2x + 1) = 19 + 5 - 4$$

$$5(y + 1)^2 - 4(x - 1)^2 = 20$$

$$\frac{5(y + 1)^2}{20} + \frac{4(x - 1)^2}{20} = 1$$

$$\frac{(y + 1)^2}{4} + \frac{(x - 1)^2}{5} = 1$$

The conic section is a Hyperbola .

The center is $P(1, -1)$.

$$a^2 = 5 \implies a = \sqrt{5}$$

$$b^2 = 4 \implies b = 2$$

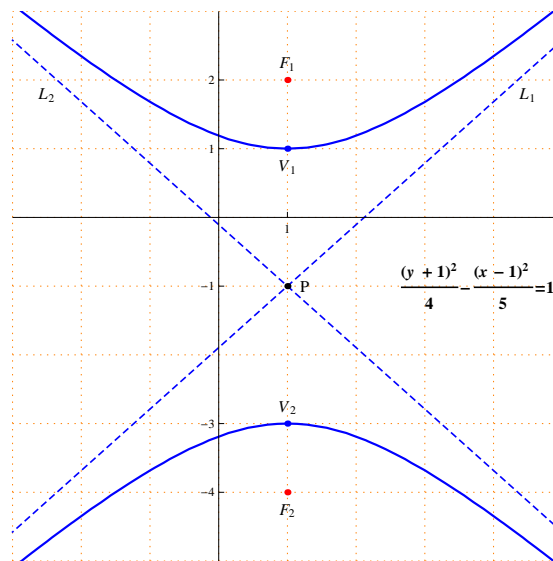
$$c^2 = a^2 + b^2 = 5 + 4 = 9 \implies c = 3$$

The vertices are $V_1(1, 1)$ and $V_2(1, -3)$

The foci are $F_1(1, 2)$ and $F_2(1, -4)$

The equations of the asymptotes are $L_1 : (y + 1) = \frac{2}{\sqrt{5}}(x - 1)$

and $L_2 : (y + 1) = -\frac{2}{\sqrt{5}}(x - 1)$



Q.5 Find the standard equation of the ellipse with foci $F_1(5, 1)$, $F_2(-5, 1)$ and with vertex $V_1(6, 1)$, and sketch it.

Solution :

The general formula of the ellipse is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

From the positions of F_1 and F_2 the major axis is parallel to the x -axis, hence $a > b$.

$$P(h, k) = \left(\frac{5 + (-5)}{2}, \frac{1 + 1}{2} \right) = (0, 1)$$

c is the distance between P and one of the foci, hence $c = 5$

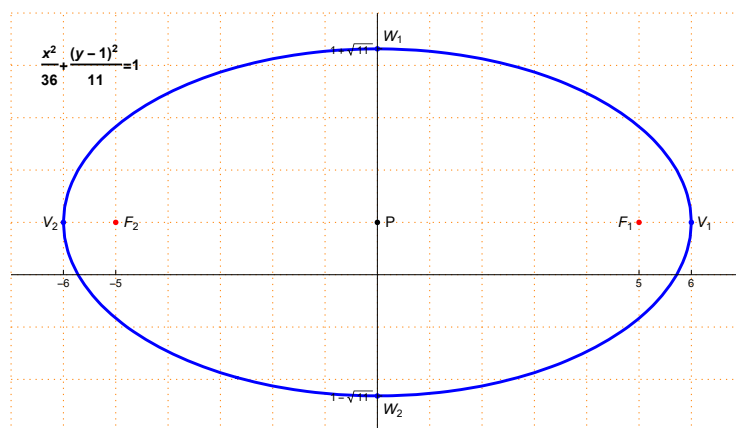
a is the distance between P and the vertex $V_1(6, 1)$, hence $a = 6$

$$c^2 = a^2 - b^2 \implies 25 = 36 - b^2 \implies b^2 = 36 - 25 = 11 \implies b = \sqrt{11}$$

The standard equation of the ellipse is $\frac{x^2}{36} + \frac{(y-1)^2}{11} = 1$

The other vertex is $V_2(-6, 1)$

The end-points of the minor axis are $W_1(0, 1 + \sqrt{11})$ and $W_2(0, 1 - \sqrt{11})$



M 104 - GENERAL MATHEMATICS -2-

Dr. Tariq A. AlFadhel

Solution of the Final Exam

Second semester 1437-1438 H

Q.1 (a) Compute (if possible) \mathbf{AB} for $\mathbf{A} = \begin{pmatrix} 1 & 4 & 2 \\ 3 & 0 & 3 \\ 0 & 2 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 1 & 6 \\ 0 & 1 & 1 \end{pmatrix}$

(b) Compute the determinant $\begin{vmatrix} 4 & 1 & 2 \\ 2 & 3 & 3 \\ 4 & 0 & 1 \end{vmatrix}$.

(c) Solve by Cramer's rule : $\begin{cases} x - y = 1 \\ 3x - 5y = -1 \end{cases}$

Solution :

(a) $\mathbf{AB} = \begin{pmatrix} 1 & 4 & 2 \\ 3 & 0 & 3 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 3 & 1 & 6 \\ 0 & 1 & 1 \end{pmatrix}$
 $= \begin{pmatrix} 1+12+0 & 0+4+2 & 1+24+2 \\ 3+0+0 & 0+0+3 & 3+0+3 \\ 0+6+0 & 0+2+0 & 0+12+0 \end{pmatrix} = \begin{pmatrix} 13 & 6 & 27 \\ 3 & 3 & 6 \\ 6 & 2 & 12 \end{pmatrix}$

(b) Solution (1): Using Sarrus Method

$$\begin{array}{ccccccc} 4 & 1 & 2 & 4 & 1 & & \\ 2 & 3 & 3 & 2 & 3 & & \\ 4 & 0 & 1 & 4 & 0 & & \end{array}$$

$$\begin{vmatrix} 4 & 1 & 2 \\ 2 & 3 & 3 \\ 4 & 0 & 1 \end{vmatrix} = (12 + 12 + 0) - (24 + 0 + 2) = 24 - 26 = -2$$

Solution (2) : Using the definition (using the third row) :

$$\begin{vmatrix} 4 & 1 & 2 \\ 2 & 3 & 3 \\ 4 & 0 & 1 \end{vmatrix} = 4 \times \begin{vmatrix} 1 & 2 \\ 3 & 3 \end{vmatrix} - 0 \times \begin{vmatrix} 4 & 2 \\ 2 & 3 \end{vmatrix} + 1 \times \begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix}$$
$$= 4(3 - 6) - 0 + (12 - 2) = -12 + 10 = -2$$

(c) Using Cramer's rule :

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 3 & -5 \end{pmatrix}, \mathbf{A}_x = \begin{pmatrix} 1 & -1 \\ -1 & -5 \end{pmatrix}, \mathbf{A}_y = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$$

$$|\mathbf{A}| = \begin{vmatrix} 1 & -1 \\ 3 & -5 \end{vmatrix} = (1 \times -5) - (-1 \times 3) = -5 - (-3) = -5 + 3 = -2 \neq 0$$

$$|\mathbf{A}_x| = \begin{vmatrix} 1 & -1 \\ -1 & -5 \end{vmatrix} = (1 \times -5) - (-1 \times -1) = -5 - 1 = -6$$

$$|\mathbf{A}_y| = \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} = (1 \times -1) - (1 \times 3) = -1 - 3 = -4$$

$$x = \frac{|\mathbf{A}_x|}{|\mathbf{A}|} = \frac{-6}{-2} = 3$$

$$y = \frac{|\mathbf{A}_y|}{|\mathbf{A}|} = \frac{-4}{-2} = 2$$

The solution of the linear system is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

Q.2 (a) Find the standard equation of the hyperbola with foci $(-2, 3)$, $(4, 3)$ and a vertex $(3, 3)$ then sketch it.

(b) Find the elements of the conic section $y^2 - 2y + 4x^2 + 8x + 1 = 0$ and sketch it.

Solution :

(a) The two foci and the vertex are located on a line parallel to the x -axis.

The standard equation of the hyperbola is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

$$P = (h, k) = \left(\frac{-2+4}{2}, \frac{3+3}{2} \right) = (1, 3), \text{ hence } h = 1 \text{ and } k = 3$$

a is the distance between $V_1 = (3, 3)$ and P , hence $a = 2$

c is the distance between $F_1 = (4, 3)$ and P , hence $c = 3$

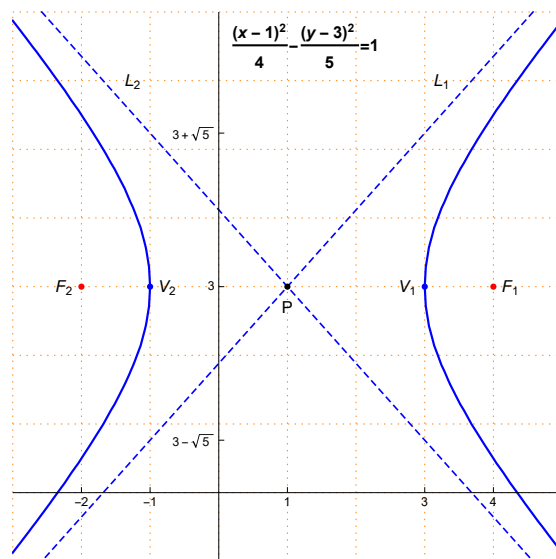
$$c^2 = a^2 + b^2 \implies 9 = 4 + b^2 \implies b^2 = 9 - 4 = 5 \implies b = \sqrt{5}$$

The standard equation of the hyperbola is $\frac{(x-1)^2}{4} - \frac{(y-3)^2}{5} = 1$

The other vertex is $V_2 = (1-2, 3) = (-1, 3)$

The equations of the asymptotes are :

$$L_1 : y - 3 = \frac{\sqrt{5}}{2}(x - 1) \text{ and } L_2 : y - 3 = -\frac{\sqrt{5}}{2}(x - 1)$$



(b) $y^2 - 2y + 4x^2 + 8x + 1 = 0$

$$4x^2 + 8x + y^2 - 2y = -1$$

$$4(x^2 + 2x) + (y^2 - 2y) = -1$$

By completing the square

$$4(x^2 + 2x + 1) + (y^2 - 2y + 1) = -1 + 4 + 1$$

$$4(x + 1)^2 + (y - 1)^2 = 4$$

$$\frac{4(x + 1)^2}{4} + \frac{(y - 1)^2}{4} = 1$$

$$(x + 1)^2 + \frac{(y - 1)^2}{4} = 1$$

The conic section is an ellipse.

The center is $P = (-1, 1)$.

$$a^2 = 1 \implies a = 1.$$

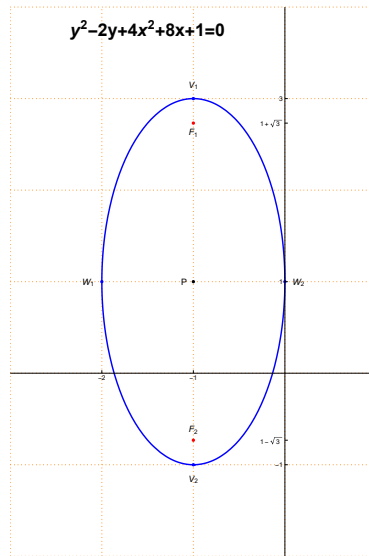
$$b^2 = 4 \implies b = 2.$$

$$c^2 = b^2 - a^2 = 4 - 1 = 3 \implies c = \sqrt{3}.$$

The vertices are $V_1 = (-1, 3)$ and $V_2 = (-1, -1)$

The foci are $F_1 = (-1, 1 + \sqrt{3})$ and $F_2 = (-1, 1 - \sqrt{3})$.

The end-points of the minor axis are $W_1 = (-2, 1)$ and $W_2 = (0, 1)$.



Q.3 (a) Compute the integrals :

(i) $\int \frac{x}{(x-1)(x-2)} dx$ (ii) $\int \ln x dx$ (iii) $\int (2x+1) \cos(x^2+x) dx$

(b) Find the area of the region determined by the curves :

$y = \sqrt{x}$ and $y = x$.

(c) The region R between the curves $y = 0$, $x = 2$ and $y = x^2$ is rotated about the y -axis to form a solid of revolution S . Find the volume of S .

Solution :

(a) (i) $\int \frac{x}{(x-1)(x-2)} dx$

Using the method of partial fractions

$$\frac{x}{(x-1)(x-2)} = \frac{A_1}{x-1} + \frac{A_2}{x-2}$$

$$x = A_1(x-2) + A_2(x-1)$$

Put $x = 1$ then $1 = A_1(1-2) \implies A_1 = -1$

Put $x = 2$ then $2 = A_2(2-1) \implies A_2 = 2$

$$\int \frac{x}{(x-1)(x-2)} dx = \int \left(\frac{-1}{x-1} + \frac{2}{x-2} \right) dx$$

$$= - \int \frac{1}{x-1} dx + 2 \int \frac{1}{x-2} dx = - \ln|x-1| + 2 \ln|x-2| + c$$

(ii) $\int \ln x dx$

Using integration by parts

$$\begin{aligned} u &= \ln x & dv &= 1 \, dx \\ du &= \frac{1}{x} \, dx & v &= x \end{aligned}$$

$$\int \ln x \, dx = x \ln x - \int x \frac{1}{x} \, dx$$

$$= x \ln x - \int 1 \, dx = x (\ln x) - x + c$$

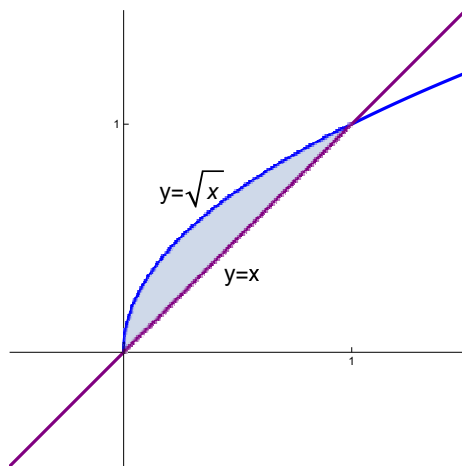
$$(iii) \int (2x + 1) \cos(x^2 + x) \, dx$$

$$\int (2x + 1) \cos(x^2 + x) \, dx = \sin(x^2 + x) + c$$

$$\text{Using the formula } \int \cos(f(x)) f'(x) \, dx = \sin(f(x)) + c$$

(b) $y = x$ is a straight line passing through the origin with slope equals 1.

$y = \sqrt{x}$ is the upper half of the parabola $x = y^2$ with vertex $(0, 0)$ and opens to the right.



Points of intersection of $y = x$ and $y = \sqrt{x}$:

$$x = \sqrt{x} \implies x^2 = x \implies x^2 - x = 0 \implies x(x - 1) = 0 \implies x = 0, x = 1$$

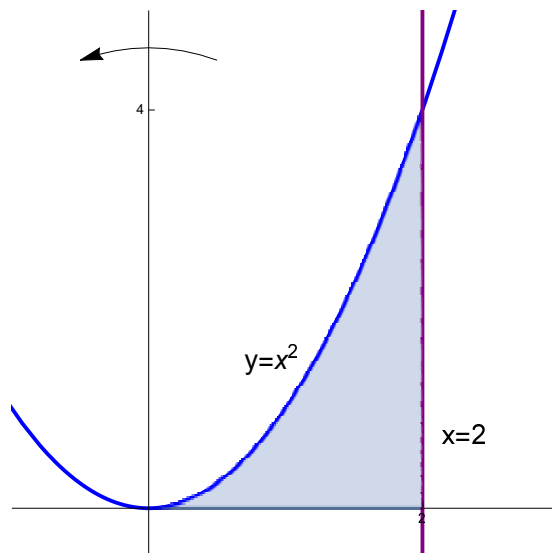
$$\text{Area} = \int_0^1 (\sqrt{x} - x) \, dx = \int_0^1 \left(x^{\frac{1}{2}} - x \right) \, dx = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^2}{2} \right]_0^1$$

$$= \left[\frac{2}{3} x^{\frac{3}{2}} - \frac{x^2}{2} \right]_0^1 = \left(\frac{2}{3} - \frac{1}{2} \right) - (0 - 0) = \frac{1}{6}$$

(c) $y = 0$ is the x -axis

$x = 2$ is a straight line parallel to the y -axis and passing through $(2, 0)$.

$y = x^2$ is a parabola opens upwards with vertex $(0, 0)$.



Using Cylindrical shells Method :

$$\begin{aligned} \text{Volume} &= 2\pi \int_0^2 x(x^2) dx = 2\pi \int_0^2 x^3 dx = 2\pi \left[\frac{x^4}{4} \right]_0^2 \\ &= 2\pi \left(\frac{2^4}{4} - \frac{0^4}{4} \right) = 2\pi(4) = 8\pi \end{aligned}$$

Q.4 Find f_x , f_y and f_z for the function $f(x, y, z) = x^2yz + xz^4 \sin(2x + y^2)$.

Solution :

$$\begin{aligned} f_x &= (2x)yz + z^4 [(1) \sin(2x + y^2) + x \cos(2x + y^2)(2 + 0)] \\ &= 2xyz + z^4 [\sin(2x + y^2) + 2x \cos(2x + y^2)] \end{aligned}$$

$$\begin{aligned} f_y &= x^2z(1) + xz^4 \cos(2x + y^2)(0 + 2y) \\ &= x^2z + 2xyz^4 \cos(2x + y^2) \end{aligned}$$

$$\begin{aligned} f_z &= x^2y(1) + (4z^3)x \sin(2x + y^2) \\ &= x^2y + 4xz^3 \sin(2x + y^2) \end{aligned}$$