General Factorial Design

- The results for the two-factor factorial design may be extended to the general case where there are a levels of factor A, b levels of factor B, c levels of factor C, and so on, arranged in a factorial experiment.
- There will be $abc \ldots n$ total observations if there are n replicates of the complete experiment.
- For a fixed effects model, test statistics for each main effect and interaction may be constructed by dividing the corresponding mean square for the effect or interaction by the mean square error.
- The number of degrees of freedom for any main effect is the number of levels of the factor minus one, and the number of degrees of freedom for an interaction is the product of the number of degrees of freedom associated with the individual components of the interaction

College of Engineering
Industrial Engineering

Factorial Design Definition

General Factorial Design

Model Adequacy Checking

Example

For the three-factor analysis of variance model

$$y_{ijkl} = \mu + \tau_i + \beta_j + \gamma_k + (\tau\beta)_{ij} + (\tau\gamma)_{ik} + (\beta\gamma)_{jk} + (\tau\beta\gamma)_{ijk} + \epsilon_{ijkl} \quad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, c \\ l = 1, 2, \ldots, n \end{cases}$$

Assuming that A, B, and C are fixed,

Factorial Design Definition

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	Expected Mean Square	F_0
A	SS_A	$a-1$	MS_A	$\sigma + \frac{bcn\sum \tau_i^2}{a-1}$	$F_0 = \frac{MS_A}{MS_E}$
B	SS_B	$b-1$	MS_B	$\sigma^2 + \frac{acn\sum \beta_j^2}{b-1}$	$F_0 = \frac{MS_B}{MS_E}$
C	SS_C	$c-1$	MS_C	$\sigma^2 + \frac{abn\sum \gamma_k^2}{c-1}$	$F_0 = \frac{MS_C}{MS_E}$
AB	SS_{AB}	$(a-1)(b-1)$	MS_{AB}	$\sigma^2 + \frac{cn\sum\sum (\tau\beta)_{ij}^2}{(a-1)(b-1)}$	$F_0 = \frac{MS_{AB}}{MS_E}$
AC	SS_{AC}	$(a-1)(c-1)$	MS_{AC}	$\sigma^2 + \frac{bn\sum\sum (\tau\gamma)_{ik}^2}{(a-1)(c-1)}$	$F_0 = \frac{MS_{AC}}{MS_E}$
BC	SS_{BC}	$(b-1)(c-1)$	MS_{BC}	$\sigma^2 + \frac{an\sum\sum (\beta\gamma)_{jk}^2}{(b-1)(c-1)}$	$F_0 = \frac{MS_{BC}}{MS_E}$
ABC	SS_{ABC}	$(a-1)(b-1)(c-1)$	MS_{ABC}	$\sigma^2 + \frac{n\sum\sum\sum (\tau\beta\gamma)_{ijk}^2}{(a-1)(b-1)(c-1)}$	$F_0 = \frac{MS_{ABC}}{MS_E}$
Error	SS_E	$abc(n-1)$	MS_E	σ^2	
Total	SS_T	$abcn-1$			

Factorial Design Definition

General Factorial Design

Model Adequacy Checking

Example

$$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\sum_{l=1}^{n} y_{ijkl}^2 - \frac{y_{....}^2}{abcn}$$

$$SS_A = \frac{1}{bcn}\sum_{i=1}^{a} y_{i...}^2 - \frac{y_{....}^2}{abcn}$$

$$SS_B = \frac{1}{acn}\sum_{j=1}^{b} y_{.j..}^2 - \frac{y_{....}^2}{abcn}$$

$$SS_C = \frac{1}{abn}\sum_{k=1}^{c} y_{..k.}^2 - \frac{y_{....}^2}{abcn}$$

$$SS_{AB} = \frac{1}{cn}\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij..}^2 - \frac{y_{....}^2}{abcn} - SS_A - SS_B$$
$$= SS_{\text{Subtotals}(AB)} - SS_A - SS_B$$

$$SS_{AC} = \frac{1}{bn}\sum_{i=1}^{a}\sum_{k=1}^{c} y_{i.k.}^2 - \frac{y_{....}^2}{abcn} - SS_A - SS_C$$
$$= SS_{\text{Subtotals}(AC)} - SS_A - SS_C$$

$$SS_{BC} = \frac{1}{an}\sum_{j=1}^{b}\sum_{k=1}^{c} y_{.jk.}^2 - \frac{y_{....}^2}{abcn} - SS_B - SS_C$$
$$= SS_{\text{Subtotals}(BC)} - SS_B - SS_C$$

$$SS_{ABC} = \frac{1}{n}\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c} y_{ijk.}^2 - \frac{y_{....}^2}{abcn} - SS_A - SS_B - SS_C - SS_{AB} - SS_{AC} - SS_{BC}$$
$$= SS_{\text{Subtotals}(ABC)} - SS_A - SS_B - SS_C - SS_{AB} - SS_{AC} - SS_{BC}$$

$$SS_E = SS_T - SS_{\text{Subtotals}(ABC)}$$

Model Adequacy

The residual analysis is the primary diagnostic tool to check the model adequacy for the analysis of variance

$$e_{ijk} = y_{ijk} - \hat{y}_{ijk}$$

Because $\hat{y}_{ijk} = \bar{y}_{ij.}$

Then

$$e_{ijk} = y_{ijk} - \bar{y}_{ij.}$$

Example

A soft drink bottler is interested in obtaining more uniform fill heights in the bottles produced by his manufacturing process. The filling machine theoretically fills each bottle to the correct target height, but in practice, there is variation around this target, and the bottler would like to understand better the sources of this variability and eventually reduce it. The process engineer can control three variables during the filling process: the percent carbonation (A), the operating pressure in the filler (B), and the bottles produced per minute or the line speed (C). The pressure and speed are easy to control, but the percent carbonation is more difficult to control during actual manufacturing because it varies with product temperature. However, for purposes of an experiment, the engineer can control carbonation at three levels: 10, 12, and 14 percent. She chooses two levels for pressure (25 and 30 psi) and two levels for line speed (200 and 250 bpm). She decides to run two replicates of a factorial design in these three factors, with all 24 runs taken in random order. The response variable observed is the average deviation from the target fill height observed in a production run of bottles at each set of conditions. The data that resulted from this experiment is shown following table. Positive deviations are fill heights above the target, whereas negative deviations are fill heights below the target

Example

Percent Carbonation (A)	Operating Pressure (B): 25 psi, Line Speed (C) 200	25 psi, Line Speed (C) 250	30 psi, Line Speed (C) 200	30 psi, Line Speed (C) 250	$y_{i...}$
10	−3, −1 (−4)	−1, 0 (−1)	−1, 0 (−1)	1, 1 (2)	−4
12	0, 1 (1)	2, 1 (3)	2, 3 (5)	6, 5 (11)	20
14	5, 4 (9)	7, 6 (13)	7, 9 (16)	10, 11 (21)	59
$B \times C$ Totals $y_{.jk.}$	6	15	20	34	$75 = y_{....}$
$y_{.j..}$	21		54		

$A \times B$ Totals $y_{ij..}$

A	B = 25	30
10	−5	1
12	4	16
14	22	37

$A \times C$ Totals $y_{i.k.}$

A	C = 200	250
10	−5	1
12	6	14
14	25	34

Example

Percent Carbonation (A)	Operating Pressure (B): 25 psi, Line Speed (C): 200	25 psi, 250	30 psi, 200	30 psi, 250	$y_{i...}$
10	−3, −1 (−4)	−1, 0 (−1)	−1, 0 (−1)	1, 1 (2)	−4
12	0, 1 (1)	2, 1 (3)	2, 3 (5)	6, 5 (11)	20
14	5, 4 (9)	7, 6 (13)	7, 9 (16)	10, 11 (21)	59
$B \times C$ Totals $y_{.jk.}$	6	15	20	34	$75 = y_{....}$
$y_{.j..}$	21		54		

$A \times B$ Totals $y_{ij..}$

A	B: 25	30
10	−5	1
12	4	16
14	22	37

$A \times C$ Totals $y_{i.k.}$

A	C: 200	250
10	−5	1
12	6	14
14	25	34

Example

$$SS_{\text{Carbonation}} = \frac{1}{bcn}\sum_{i=1}^{a} y_{i...}^2 - \frac{y_{....}^2}{abcn}$$

$$= \frac{1}{8}[(-4)^2 + (20)^2 + (59)^2] - \frac{(75)^2}{24} = 252.750$$

$$SS_{\text{Pressure}} = \frac{1}{acn}\sum_{j=1}^{b} y_{.j..}^2 - \frac{y_{....}^2}{abcn}$$

$$= \frac{1}{12}[(21)^2 + (54)^2] - \frac{(75)^2}{24} = 45.375$$

$$SS_{\text{Speed}} = \frac{1}{abn}\sum_{k=1}^{c} y_{..k.}^2 - \frac{y_{....}^2}{abcn}$$

$$= \frac{1}{12}[(26)^2 + (49)^2] - \frac{(75)^2}{24} = 22.042$$

$$SS_{AB} = \frac{1}{cn}\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij..}^2 - \frac{y_{....}^2}{abcn} - SS_A - SS_B$$

$$= \frac{1}{4}[(-5)^2 + (1)^2 + (4)^2 + (16)^2 + (22)^2 + (37)^2] - \frac{(75)^2}{24}$$

$$- 252.750 - 45.375$$

$$= 5.250$$

Example

$$SS_{AC} = \frac{1}{bn}\sum_{i=1}^{a}\sum_{k=1}^{c} y_{i.k.}^2 - \frac{y_{....}^2}{abcn} - SS_A - SS_C$$

$$= \frac{1}{4}[(-5)^2 + (1)^2 + (6)^2 + (14)^2 + (25)^2 + (34)^2] - \frac{(75)^2}{24}$$

$$- 252.750 - 22.042$$

$$= 0.583$$

$$SS_{BC} = \frac{1}{an}\sum_{j=1}^{b}\sum_{k=1}^{c} y_{.jk.}^2 - \frac{y_{....}^2}{abcn} - SS_B - SS_C$$

$$= \frac{1}{6}[(6)^2 + (15)^2 + (20)^2 + (34)^2] - \frac{(75)^2}{24} - 45.375 - 22.042$$

$$= 1.042$$

$$SS_{ABC} = \frac{1}{n}\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c} y_{ijk.}^2 - \frac{y_{....}^2}{abcn} - SS_A - SS_B - SS_C - SS_{AB} - SS_{AC} - SS_{BC}$$

$$= \frac{1}{2}[(-4)^2 + (-1)^2 + (-1)^2 + \cdots + (16)^2 + (21)^2] - \frac{(75)^2}{24}$$

$$- 252.750 - 45.375 - 22.042 - 5.250 - 0.583 - 1.042$$

$$= 1.083$$

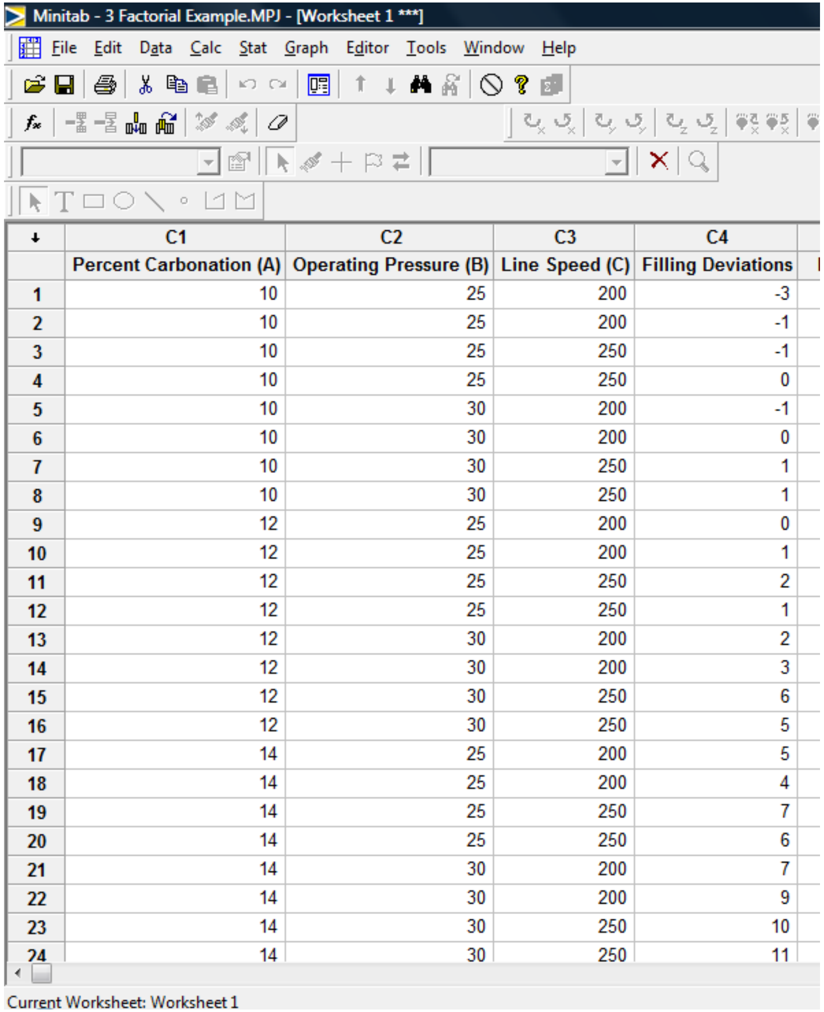
Example

$$SS_{\text{Subtotals}(ABC)} = \frac{1}{n} \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{c} y_{ijk.}^2 - \frac{y_{....}^2}{abcn} = 328.125$$

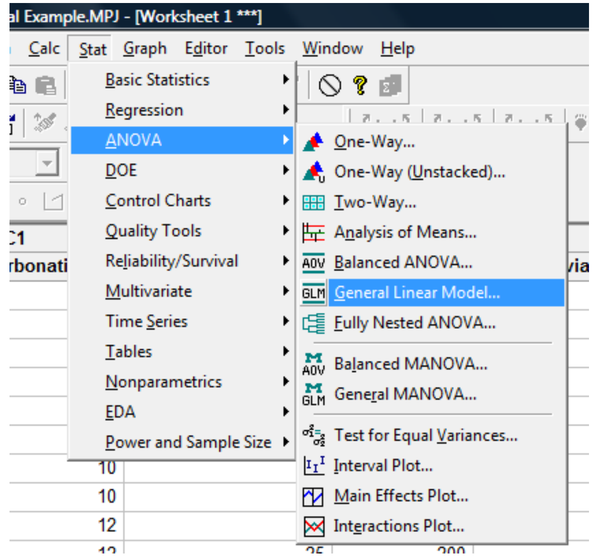
$$\begin{aligned} SS_E &= SS_T - SS_{\text{Subtotals}(ABC)} \\ &= 336.625 - 328.125 \\ &= 8.500 \end{aligned}$$

We conclude that the percentage of carbonation, operating pressure, and line speed significantly affect the fill volume.

Example



↓	C1	C2	C3	C4
	Percent Carbonation (A)	Operating Pressure (B)	Line Speed (C)	Filling Deviations
1	10	25	200	-3
2	10	25	200	-1
3	10	25	250	-1
4	10	25	250	0
5	10	30	200	-1
6	10	30	200	0
7	10	30	250	1
8	10	30	250	1
9	12	25	200	0
10	12	25	200	1
11	12	25	250	2
12	12	25	250	1
13	12	30	200	2
14	12	30	200	3
15	12	30	250	6
16	12	30	250	5
17	14	25	200	5
18	14	25	200	4
19	14	25	250	7
20	14	25	250	6
21	14	30	200	7
22	14	30	200	9
23	14	30	250	10
24	14	30	250	11

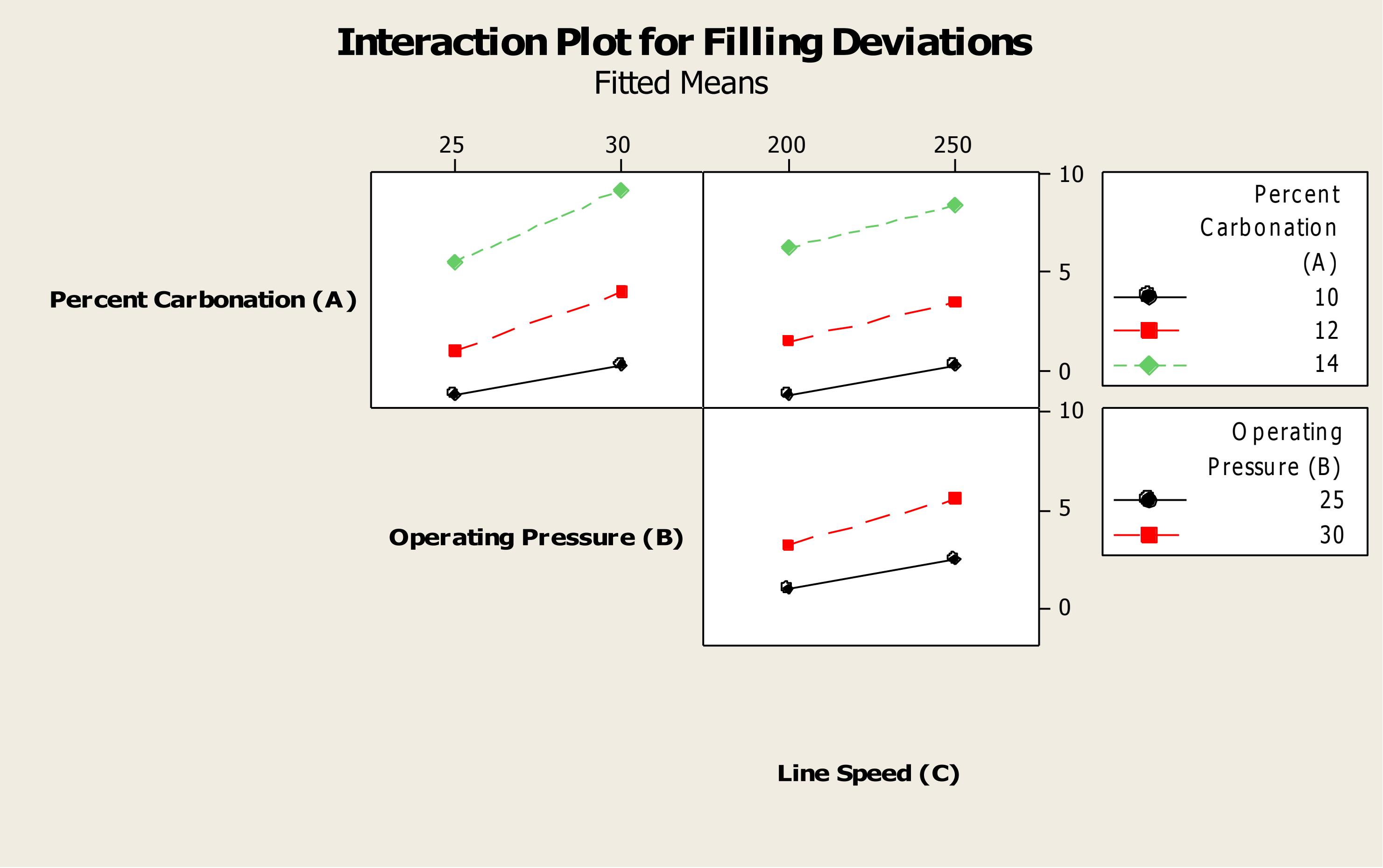


Example

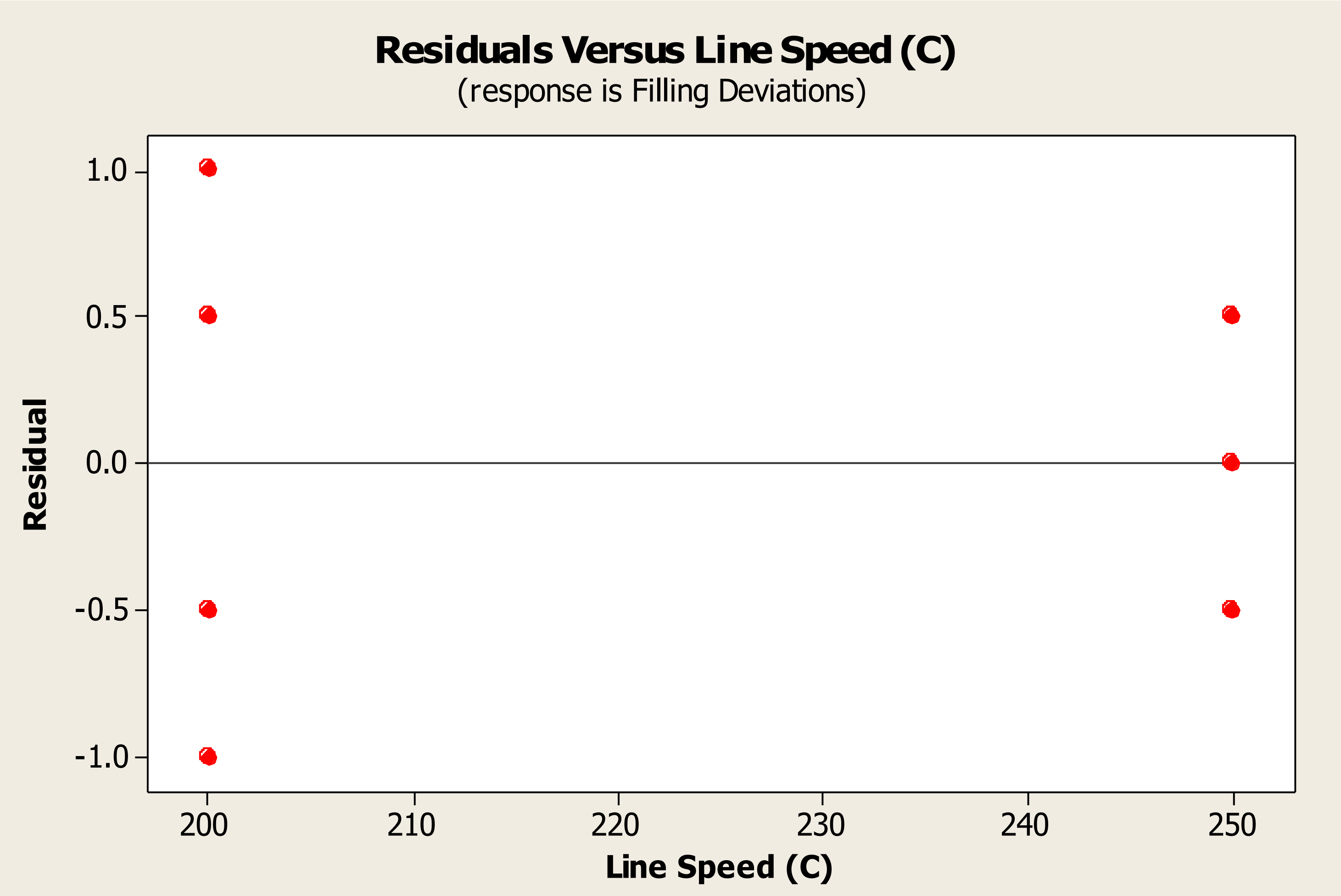
General Factorial Design

Model Adequacy Checking

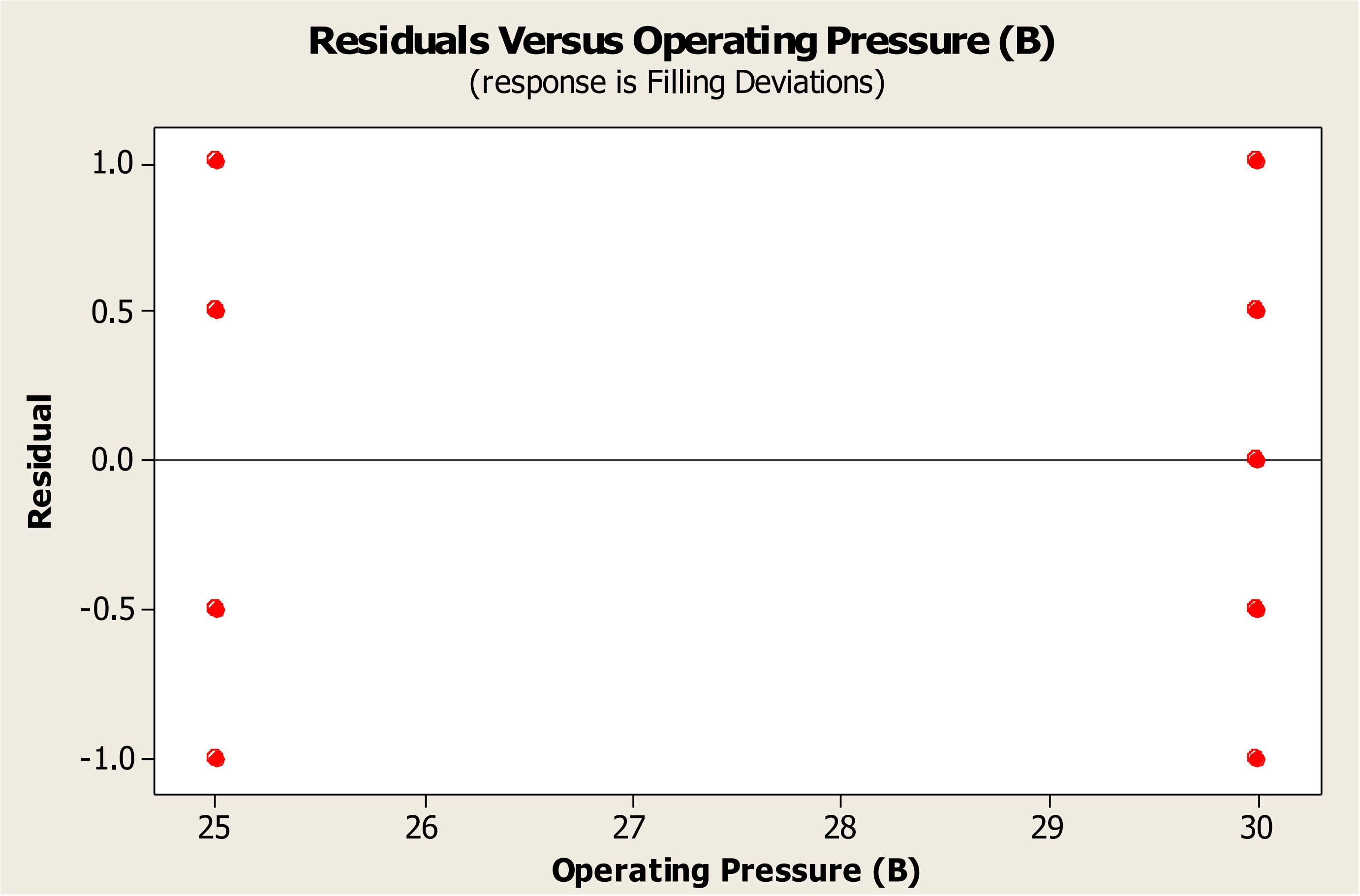
Example



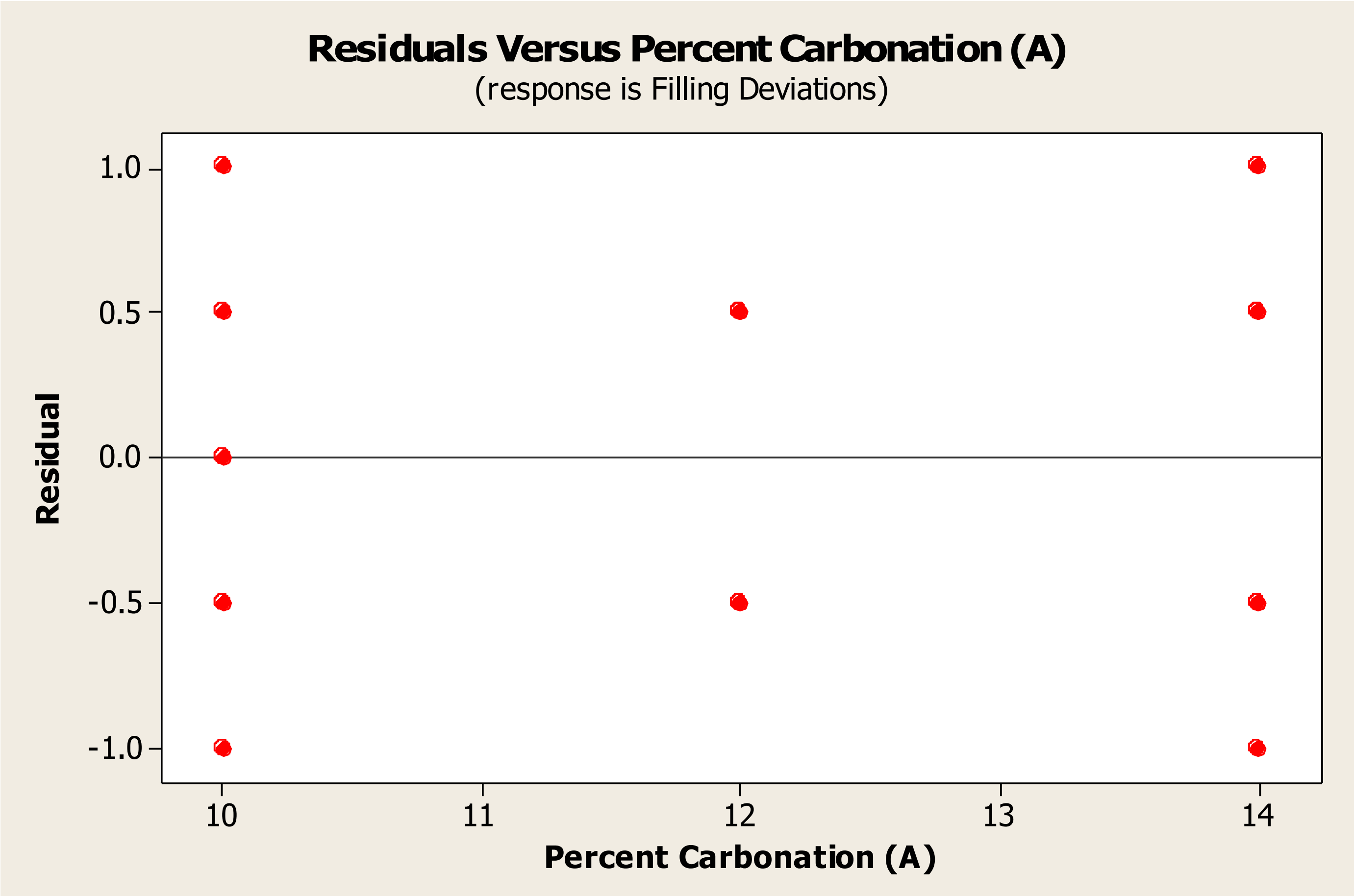
Example


Residuals Versus Line Speed (C)
(response is Filling Deviations)
Residual
1.0
0.5
0.0
-0.5
-1.0
200
210
220
230
240
250
Line Speed (C)

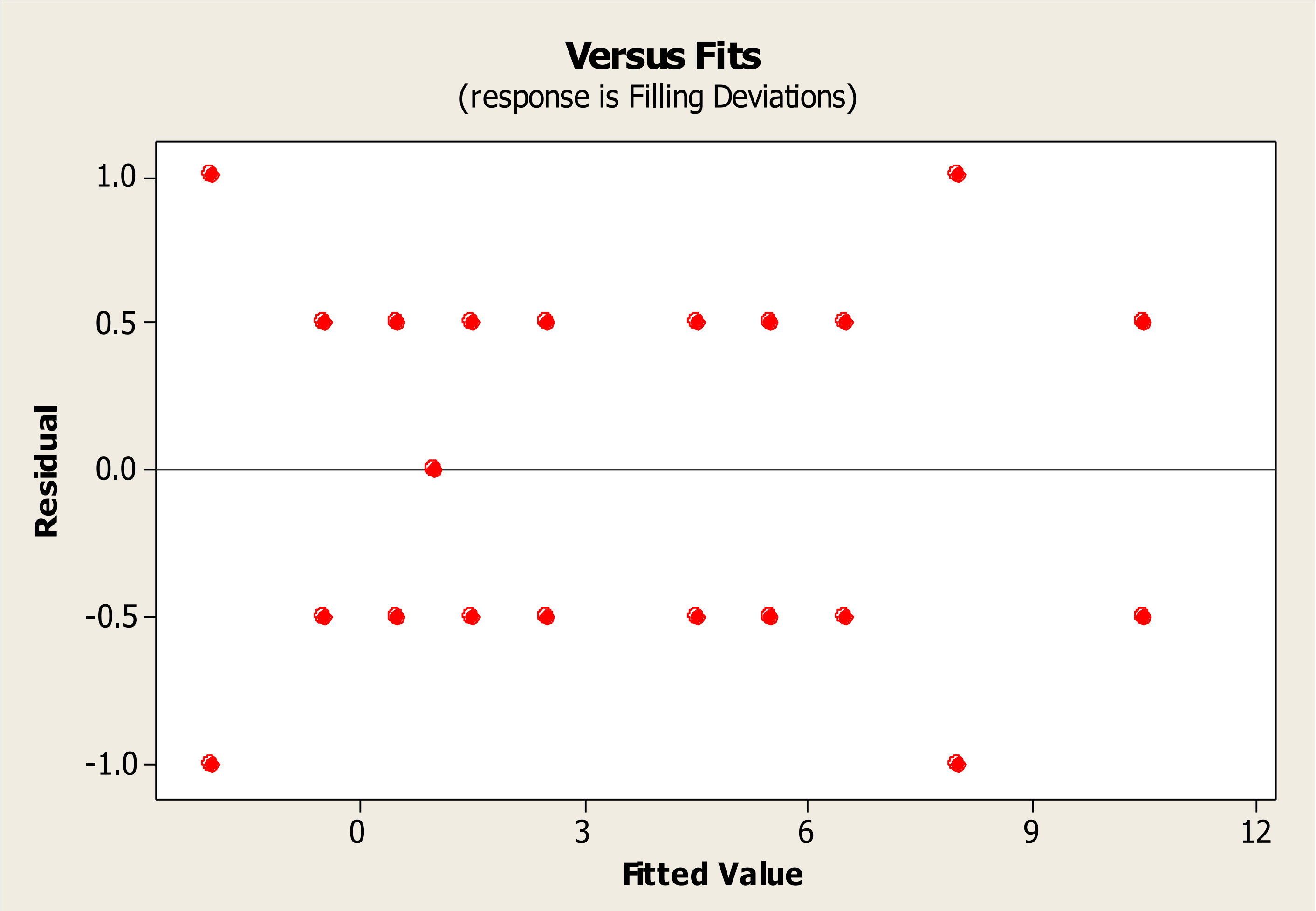
Example


Residuals Versus Operating Pressure (B)
(response is Filling Deviations)
Residual
1.0
0.5
0.0
-0.5
-1.0
25
26
27
28
29
30
Operating Pressure (B)

Example


Residuals Versus Percent Carbonation (A)
(response is Filling Deviations)
Residual
1.0
0.5
0.0
-0.5
-1.0
10
11
12
13
14
Percent Carbonation (A)

Example


Versus Fits
(response is Filling Deviations)
Residual
1.0
0.5
0.0
-0.5
-1.0
0
3
6
9
12
Fitted Value