# CHAPTER SIX

THREE MOMENT EQUATION

Three moment equation is a relationship among the bending moments at three consecutive supports of a horizontal beam. It first proposed by Clapeyron in 1857. In this method, two consecutive spans of the continuous beam are considered at one time, each span is treated individually as a simply supported beam with external loads and two end support moments.

In a continuous beam, consider any two spans with three supports as shown in Figure below:
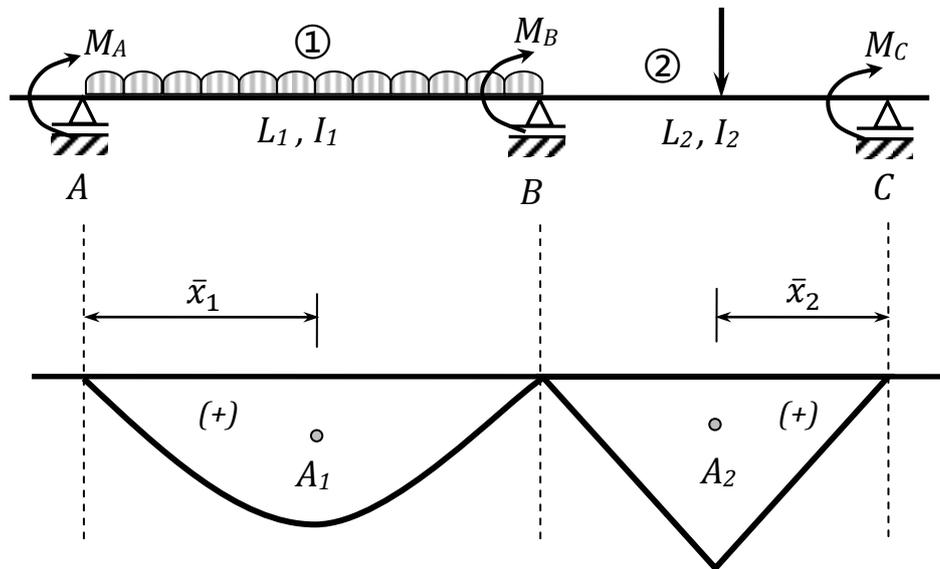
The three moment Equation is given by:

$$M_A\left(\frac{L_1}{I_1}\right) + 2M_B\left(\frac{L_1}{I_1} + \frac{L_2}{I_2}\right) + M_C\left(\frac{L_2}{I_2}\right) = -6\sum\left(\frac{A\,\bar{x}}{L\,I}\right)_{for\ two\ spans}$$

Where:
   L_1 is the length of span 1 (AB)
   L_2 is the length of span 1 (BC)
   I_1 is moment of inertia of span 1
   I_2 is moment of inertia of span 2
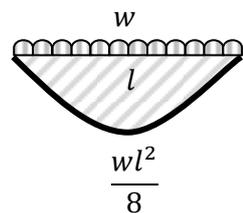   *M_A is the moment at joint A*
   M_B is the moment at joint B
   *M_C is the moment at joint C*

A_1 is area of bending moment due to loads at span 1
A_2 is area of bending moment due to loads at span 2
$\bar{x}_1$ is the distance from the centre of A_1 to left support
$\bar{x}_2$ is the distance from the centre of A_2 to left support
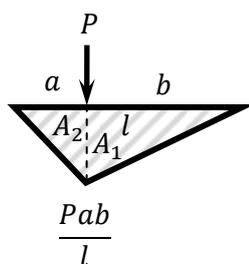
Notes:
- *The Equation is derived by using moment area method, depends on the compatibility of rotation.*
- *The sagging moment is taken positive.*
- *The area of moment diagrams for different loads and distance from centre of moment diagram to support are given below:*

$$A = \frac{2}{3}\left(\frac{wl^2}{8}\right)(l) \qquad\qquad \bar{x} = \frac{l}{2}$$
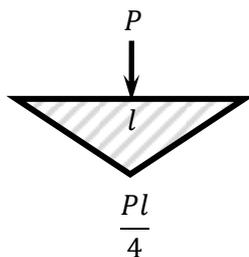
$$A = \frac{1}{2}\left(\frac{Pl}{4}\right)(l) \qquad\qquad \bar{x} = \frac{l}{2}$$

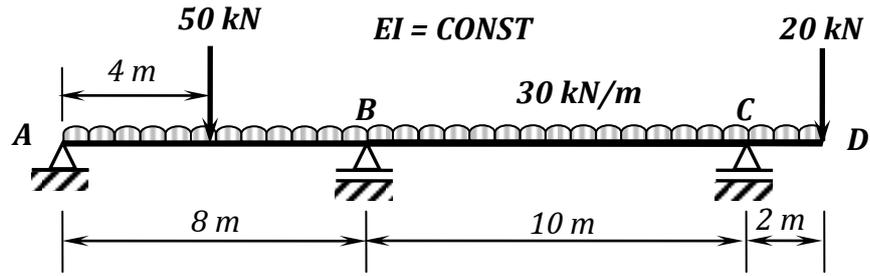$$A_1 = \frac{1}{2}\left(\frac{Pab}{l}\right)(b) \qquad \bar{x}_1 = \left(\frac{b}{3}+a\right), to\ left \quad \bar{x}_1 = \frac{2b}{3}, to\ right$$

$$A_2 = \frac{1}{2}\left(\frac{Pab}{l}\right)(a) \qquad \bar{x}_2 = \frac{2a}{3}, to\ left \quad \bar{x}_2 = \left(\frac{a}{3}+b\right), to\ right$$
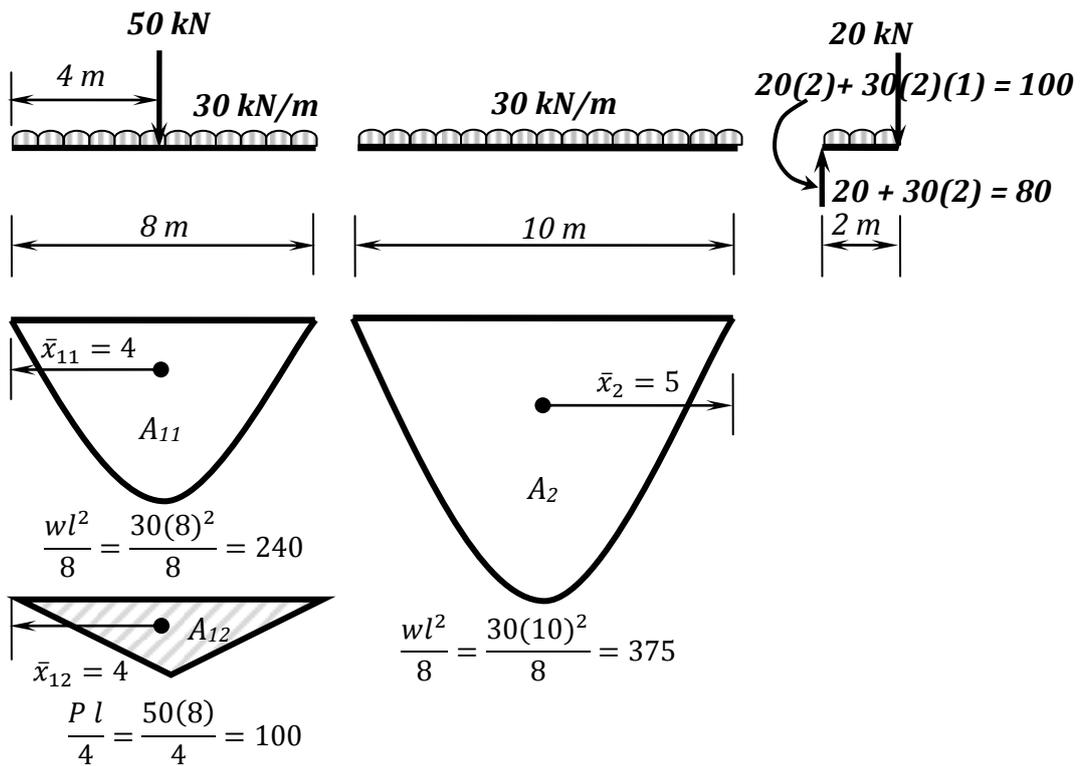
Example 1:

For the continuous beam with overhang shown in Figure below, find the support reactions.

50 kN   EI = CONST   20 kN

4 m

B   30 kN/m   C   D

A

8 m   10 m   2 m

Solution:

- *Draw the bending moment for each span individually:*

50 kN

4 m

30 kN/m   30 kN/m   20(2)+ 30(2)(1) = 100

20 kN

20 + 30(2) = 80

8 m   10 m   2 m

$\bar{x}_{11} = 4$

A_{11}

$\bar{x}_2 = 5$

A_2

$$\frac{wl^2}{8} = \frac{30(8)^2}{8} = 240$$

A_{12}

$\bar{x}_{12} = 4$

$$\frac{wl^2}{8} = \frac{30(10)^2}{8} = 375$$

$$\frac{P\,l}{4} = \frac{50(8)}{4} = 100$$

- *Write out the three moment Equation:*

$$M_A\left(\frac{L_1}{I_1}\right) + 2M_B\left(\frac{L_1}{I_1} + \frac{L_2}{I_2}\right) + M_C\left(\frac{L_2}{I_2}\right) = -6\sum\left(\frac{A\,\bar{x}}{L\,I}\right)_{for\ two\ spans}$$

- *Set values:*
  $I_1 = I_2 = I$
  $L_1 = 8$ m, $L_2 = 10$ m
  $M_A = 0$, $M_B = ?$, $M_C = -100$ kN.m

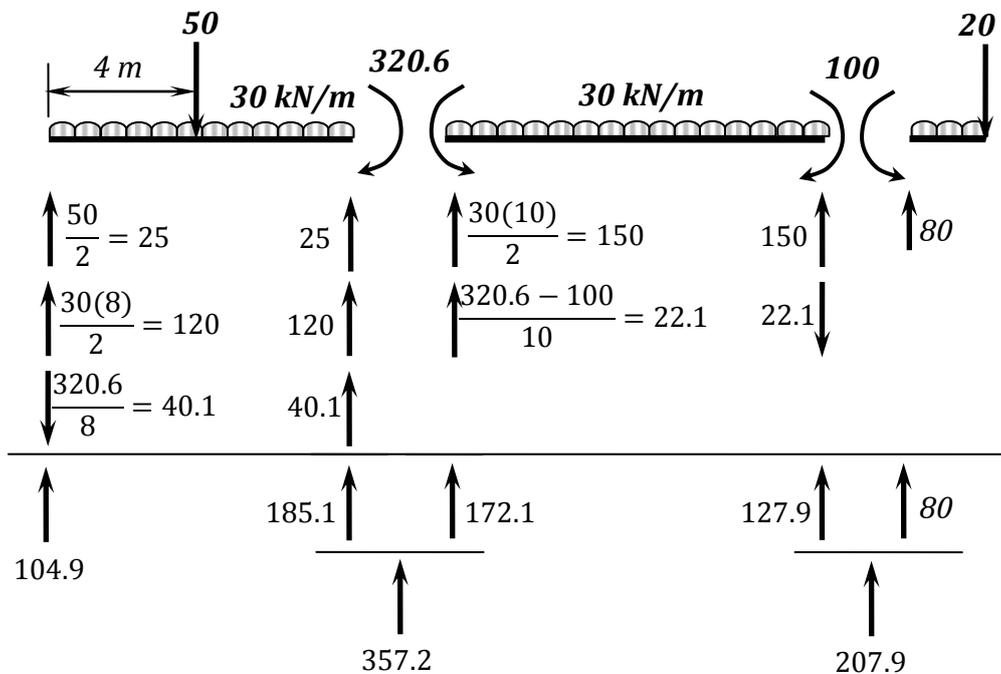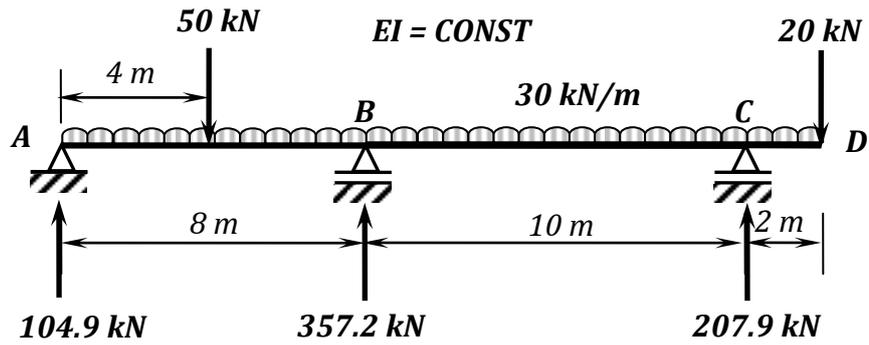| L	Area (A)	$\bar{x}$	$A\,\bar{x}$	$\dfrac{A\,\bar{x}}{L}$
8	$A_{11} = \dfrac{2}{3}(240)(8) = 1{,}280$	4	5,120	640
8	$A_{12} = \dfrac{1}{2}(100)(8) = 400$	4	1,600	200
10	$A_2 = \dfrac{2}{3}(375)(10) = 2{,}500$	5	12,500	1,250
$\sum \dfrac{A\,\bar{x}}{L}$				**2,090**

- *Substitute in Equation:*

$$0 + 2M_B \left(\frac{8}{I} + \frac{10}{I}\right) + (-100)\left(\frac{10}{I}\right) = \frac{-6}{I}(2{,}090)$$

$$2(18)M_B - 1000 = -12{,}540$$
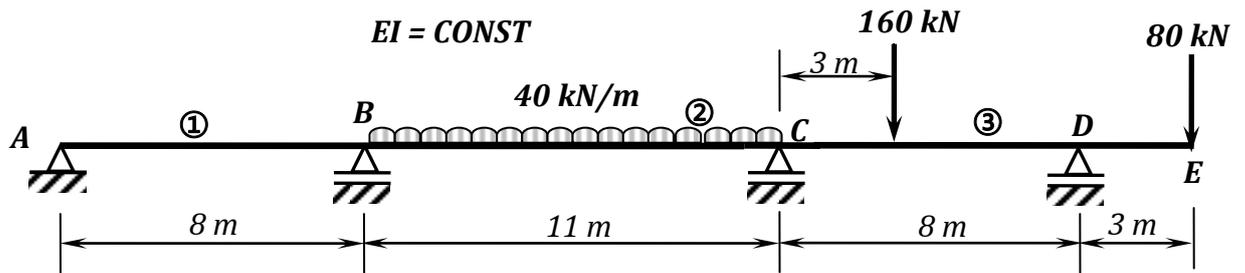
$$M_B = \frac{-11{,}540}{36} = -320.6 \; kN.m \; (hog)$$

- *Calculate reactions by using balance of elements:*

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50 kN

4 m

EI = CONST

20 kN

30 kN/m

B

C

A   *D*

8 m   **10 m**   **2 m**

104.9 kN   **357.2 kN**   **207.9 kN**

Example 2:

For the continuous beam with overhang shown in Figure below, find the moments at B and C.

EI = CONST

160 kN

80 kN

3 m

40 kN/m

B   ②   ③

A   ①   *C*   *D*   *E*

8 m   **11 m**   **8 m**   **3 m**

Solution:

- *Draw the bending moment for each span individually:*
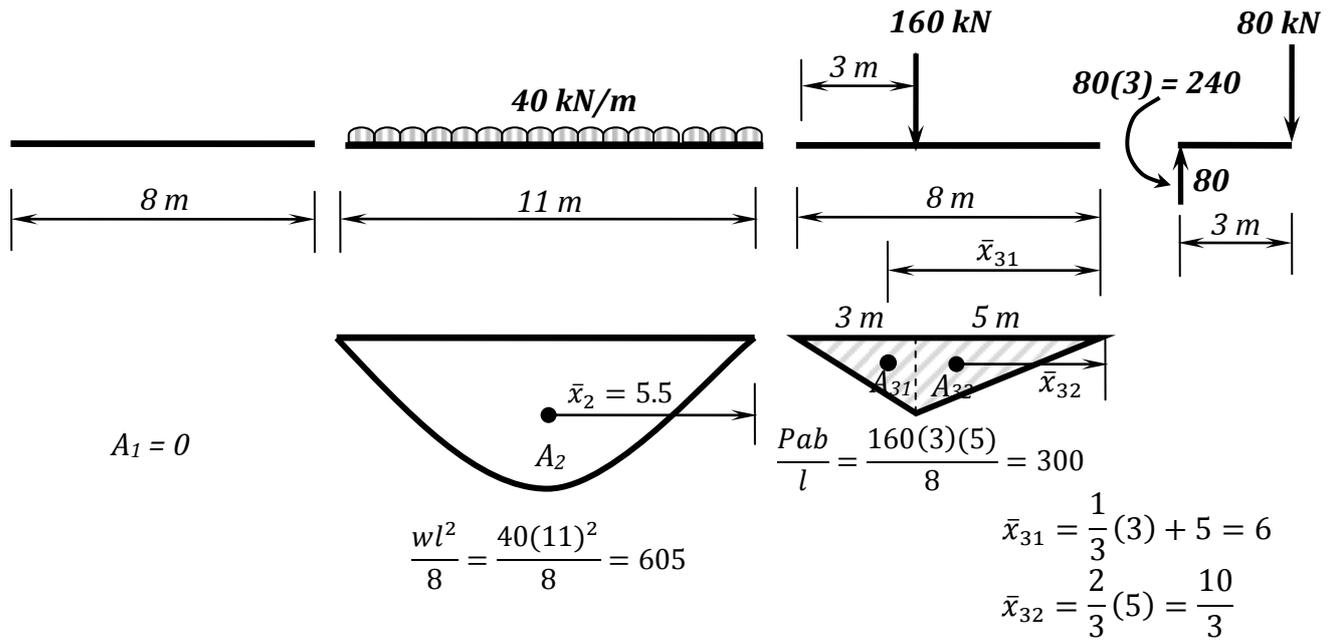
- *Write out the three moment Equation:*

$$M_A\left(\frac{L_1}{I_1}\right) + 2M_B\left(\frac{L_1}{I_1} + \frac{L_2}{I_2}\right) + M_C\left(\frac{L_2}{I_2}\right) = -6\sum\left(\frac{A\,\bar{x}}{L\,I}\right)_{for\ two\ spans}$$

- *Set values:*
  $I_1 = I_2 = I_3 = I$
  $L_1 = 8\ m,\ L_2 = 11\ m,\ L_3 = 8\ m$
  $M_A = 0,\ M_B = ?,\ M_C = ?,\ M_D = –240\ kN.m$

For span ① and ②:

L	Area (A)	$\bar{x}$	$A\,\bar{x}$	$\dfrac{A\,\bar{x}}{L}$
8	$A_1 = 0$	0	0	0
11	$A_2 = \dfrac{2}{3}(605)(11) = \dfrac{13,310}{3}$	5.5	$\dfrac{73,205}{3}$	$\dfrac{6,655}{3}$
$\sum \dfrac{A\,\bar{x}}{L}$				$\dfrac{\mathbf{6,655}}{\mathbf{3}}$

- *Substitute in Equation:*

$$0 + 2M_B\left(\frac{8}{I} + \frac{11}{I}\right) + M_C\left(\frac{11}{I}\right) = \frac{-6}{I}\left(\frac{6,655}{3}\right)$$

$$38\,M_B + 11\,M_C = -13,310 \qquad (a)$$

For span ② and ③:

L	Area (A)	$\bar{x}$	$A\,\bar{x}$	$\dfrac{A\,\bar{x}}{L}$
11	$A_2 = \dfrac{2}{3}(605)(11) = \dfrac{13{,}310}{3}$	5.5	$\dfrac{73{,}205}{3}$	$\dfrac{6{,}655}{3}$
8	$A_{31} = \dfrac{1}{2}(300)(3) = 450$	6	2,700	$\dfrac{675}{2}$
8	$A_{32} = \dfrac{1}{2}(300)(5) = 750$	$\dfrac{10}{3}$	2,500	$\dfrac{625}{2}$
	$\displaystyle\sum \dfrac{A\,\bar{x}}{L}$			$\dfrac{8{,}605}{3}$

- *Substitute in Equation:*

$$M_B\left(\frac{11}{I}\right) + 2M_C\left(\frac{11}{I} + \frac{8}{I}\right) + (-240)\left(\frac{8}{I}\right) = \frac{-6}{I}\left(\frac{8{,}605}{3}\right)$$

$$11\,M_B + 38\,M_C = -15{,}290 \qquad (b)$$

- *Solve for (a) and (b):*

11(a) – 38(b):

$$[(11)38\,M_B + (11)11\,M_C] - [(38)11\,M_B + (38)38\,M_C]$$
$$= [-(11)13{,}310\,] - [-(38)15{,}290]$$
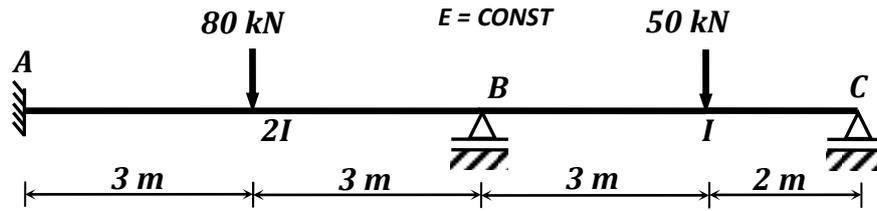
$$121\,M_C - 1{,}444\,M_C = 434{,}610$$

$$M_C = -\frac{434{,}610}{1{,}323} = -328.5 \; kN.m$$

Substitute in (a) to find MB:

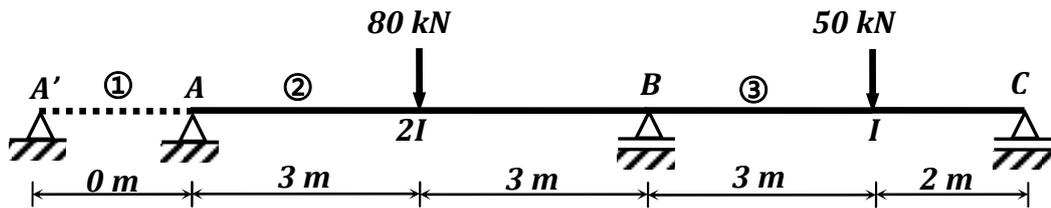$$M_B = \frac{-13{,}310 - 11(-328.5)}{38} = -255.2 \; kN.m$$

Example 2:

For the continuous beam shown in Figure below, find the moments at A and B.

80 kN   E = CONST   50 kN
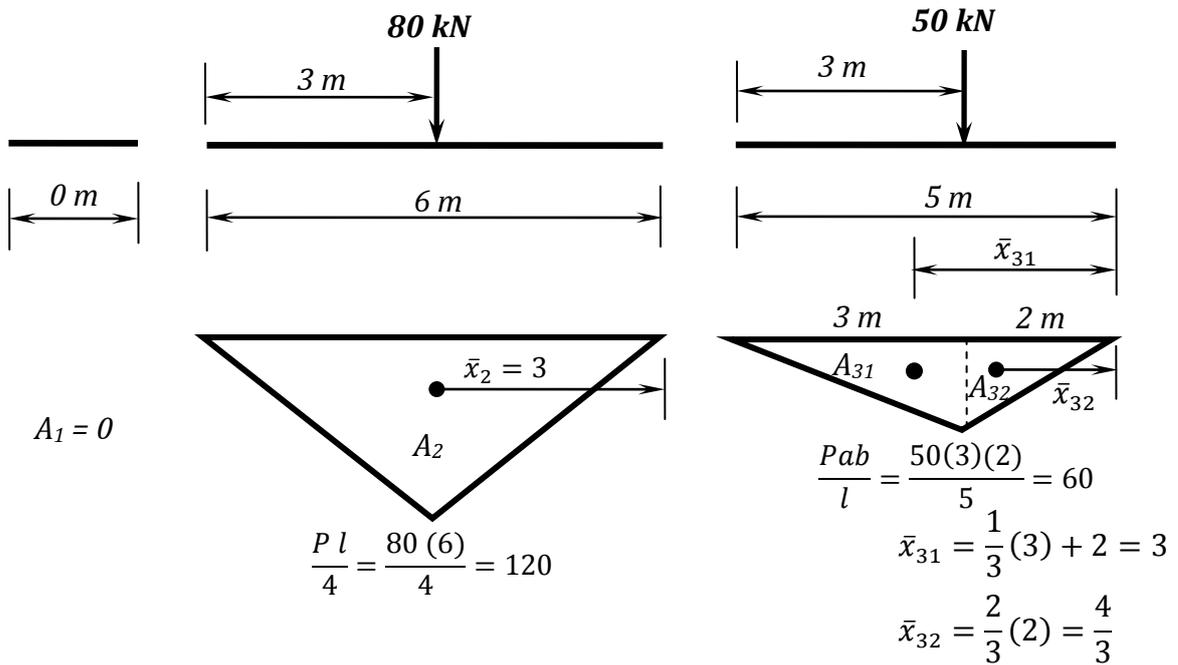
A

B

C

2I

I

3 m   3 m   3 m   2 m

Solution:

For this case, we have two unknowns moments, so we need additional Equation. Simply the Equation can be obtained by add imaginary span left to fixed support of zero length and zero moment

80 kN   50 kN

A' ① A ②   B ③   C

2I   I

0 m   3 m   3 m   3 m   2 m

- *Draw the bending moment for each span individually:*

80 kN   50 kN

3 m   3 m

0 m   6 m   5 m

$\bar{x}_{31}$

3 m   2 m

$\bar{x}_2 = 3$ A_{31} A_{32} $\bar{x}_{32}$

$A_1 = 0$ A_2

$\dfrac{P\,l}{4} = \dfrac{80\,(6)}{4} = 120$

$\dfrac{Pab}{l} = \dfrac{50(3)(2)}{5} = 60$

$\bar{x}_{31} = \dfrac{1}{3}(3) + 2 = 3$

$\bar{x}_{32} = \dfrac{2}{3}(2) = \dfrac{4}{3}$

- *Write out the three moment Equation:*

$$M_A\left(\dfrac{L_1}{I_1}\right) + 2M_B\left(\dfrac{L_1}{I_1} + \dfrac{L_2}{I_2}\right) + M_C\left(\dfrac{L_2}{I_2}\right) = -6\sum\left(\dfrac{A\,\bar{x}}{L\,I}\right)_{for\ two\ spans}$$

- *Set values:*
- $I_1 = 1,\ I_2 = 2,\ I_3 = 1$

$L_1 = 0\ m,\ L_2 = 6\ m,\ L_3 = 5\ m$

$M_{A'} = 0,\ M_A = ?,\ M_B = ?,\ M_C = 0$

For span ① and ②:

L	Area (A)	$\bar{x}$	$A\,\bar{x}$	I	$\dfrac{A\,\bar{x}}{L\,I}$
0	$A_1 = 0$	0	0	1	0
6	$A_2 = \dfrac{1}{2}(120)(6) = 360$	3	1,080	2	90
$\sum \dfrac{A\,\bar{x}}{L\,I}$					**90**

- *Substitute in Equation:*

$$0 + 2M_A\left(0 + \frac{6}{2}\right) + M_B\left(\frac{6}{2}\right) = -6(90)$$

$$6\,M_A + 3\,M_B = -540 \qquad (a)$$

For span ② and ③:

L	Area (A)	$\bar{x}$	$A\,\bar{x}$	I	$\dfrac{A\,\bar{x}}{L\,I}$
6	$A_2 = 360$	3	1,080	2	90
5	$A_{31} = \dfrac{1}{2}(60)(3) = 90$	3	270	1	54
5	$A_{32} = \dfrac{1}{2}(60)(2) = 60$	$\dfrac{4}{3}$	80	1	16
$\sum \dfrac{A\,\bar{x}}{L\,I}$					**160**

- *Substitute in Equation:*

$$M_A\left(\frac{6}{2}\right) + 2M_B\left(\frac{6}{2} + \frac{5}{1}\right) + (0)\left(\frac{5}{1}\right) = -6(160)$$

$$3\,M_A + 16\,M_B = -960 \qquad (b)$$

- *Solve for (a) and (b):*

(a) – 2(b):

$$[6\,M_A + 3\,M_B] - [(2)3\,M_A + (2)16\,M_B] = [-540\,] - (2)[-960]$$

$$3\,M_B - 32\,M_B = 1{,}380$$

$$M_B = -\frac{1{,}380}{29} = -47.6\ kN.m$$

Substitute in (a) to find MB:

$$M_A = \frac{-540 - 3(-47.6)}{6} = -66.2\ kN.m$$