Time-Dependent Failure Models

→**Probability models** used to describe non-constant failure rate processes

Overview

- Weibull Distribution
  - Probability Functions
  - Characteristics of Weibull Distribution
  - MTTF, Variance, Median and Mode
  - Conditional Reliability
- Normal Distribution
  - Probability Functions
  - Mean, Variance and Moments
  - Central Limit Theorem
- Lognormal Distribution
  - Probability Functions
  - Mean, Variance

Weibull Distribution

- Weibull distribution can attain many shapes for various values of shape parameter β.

- It can model a great variety of data and life characteristics, including constant, increasing, and decreasing failure rates (CFR, IFR, and DFR).

- Therefore, it is one of the most widely used lifetime distributions in reliability engineering.

Two-Parameter Weibull Distribution:

- β = shape parameter, or slope parameter
- θ = scale parameter, or characteristic life parameter

Probability Functions

CFR, IFR, DFR

↓

Hazard Function

$$h(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}, t \geq 0, \theta > 0, \beta > 0$$

↓

Reliability Function

$$R(t) = \exp\left[-\left(\frac{t}{\theta}\right)^{\beta}\right]$$

↓

Cumulative Distribution Function

$$F(t) = 1 - \exp\left[-\left(\frac{t}{\theta}\right)^{\beta}\right]$$

↓

Probability Density Function

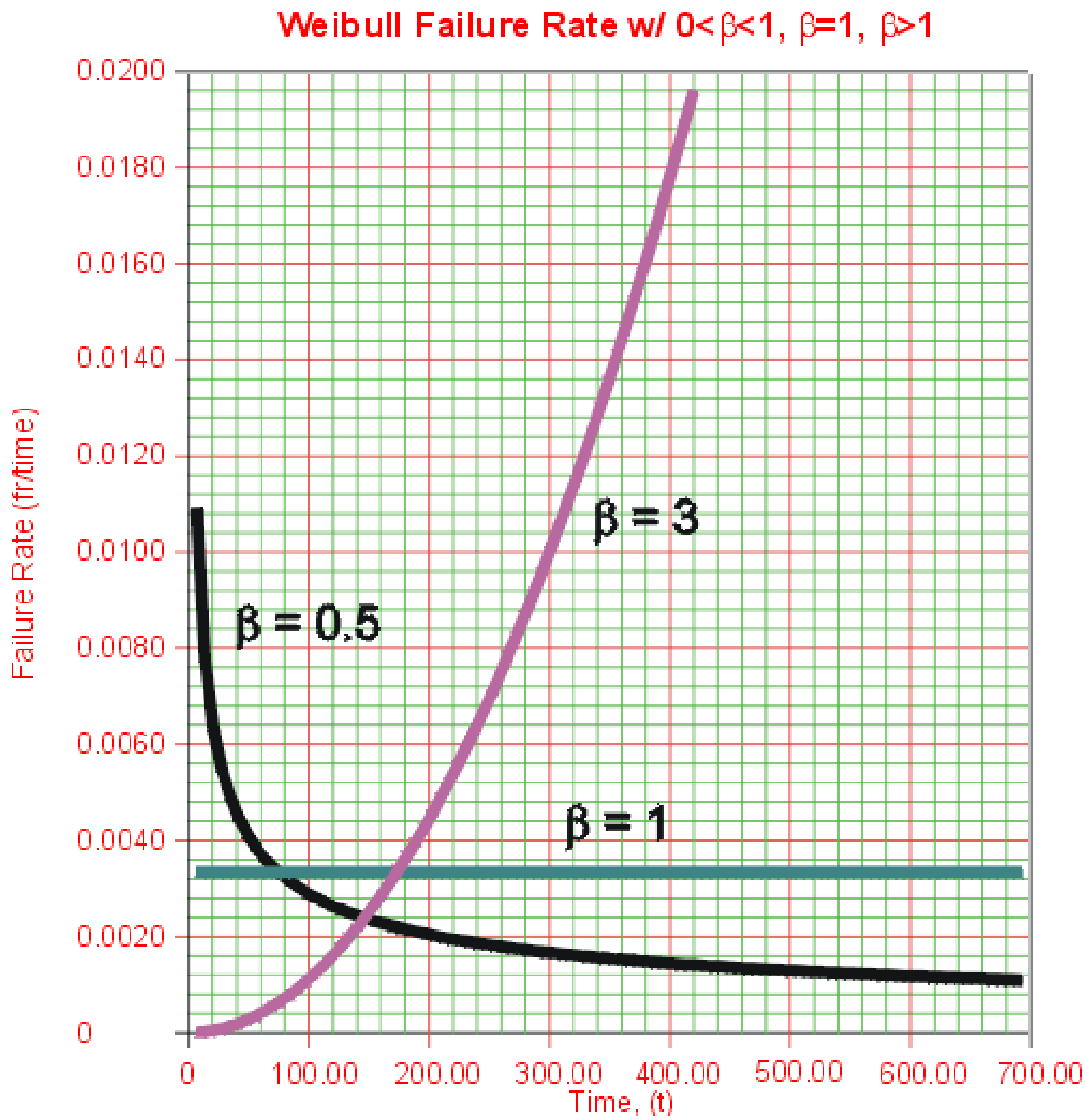
$$f(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1} \exp\left[-\left(\frac{t}{\theta}\right)^{\beta}\right]$$

Characteristics of Weibull Distribution

- Characteristic Effects of Shape (Slope) Parameter, β
  - Effects of β on Weibull Hazard Function (CFR, IFR, DFR depending on the value of β)
  - Effects of β on Weibull PDF
  - Effects of β on Weibull Reliability Function

- Characteristic Effects of Scale Parameter, θ

Effects of β on Weibull Hazard Function



$$h(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}, t \geq 0, \theta > 0, \beta > 0$$

Effects of β on Weibull Hazard Function

All three life stages of the bathtub curve can be modeled with the Weibull distribution and varying values of β:

- $\beta < 1$: DFR
- $\beta = 1$: CFR (Exponential distribution)
- $\beta > 1$: IFR
    - $1 < \beta < 2$, the $h(t)$ curve is <u>concave</u>

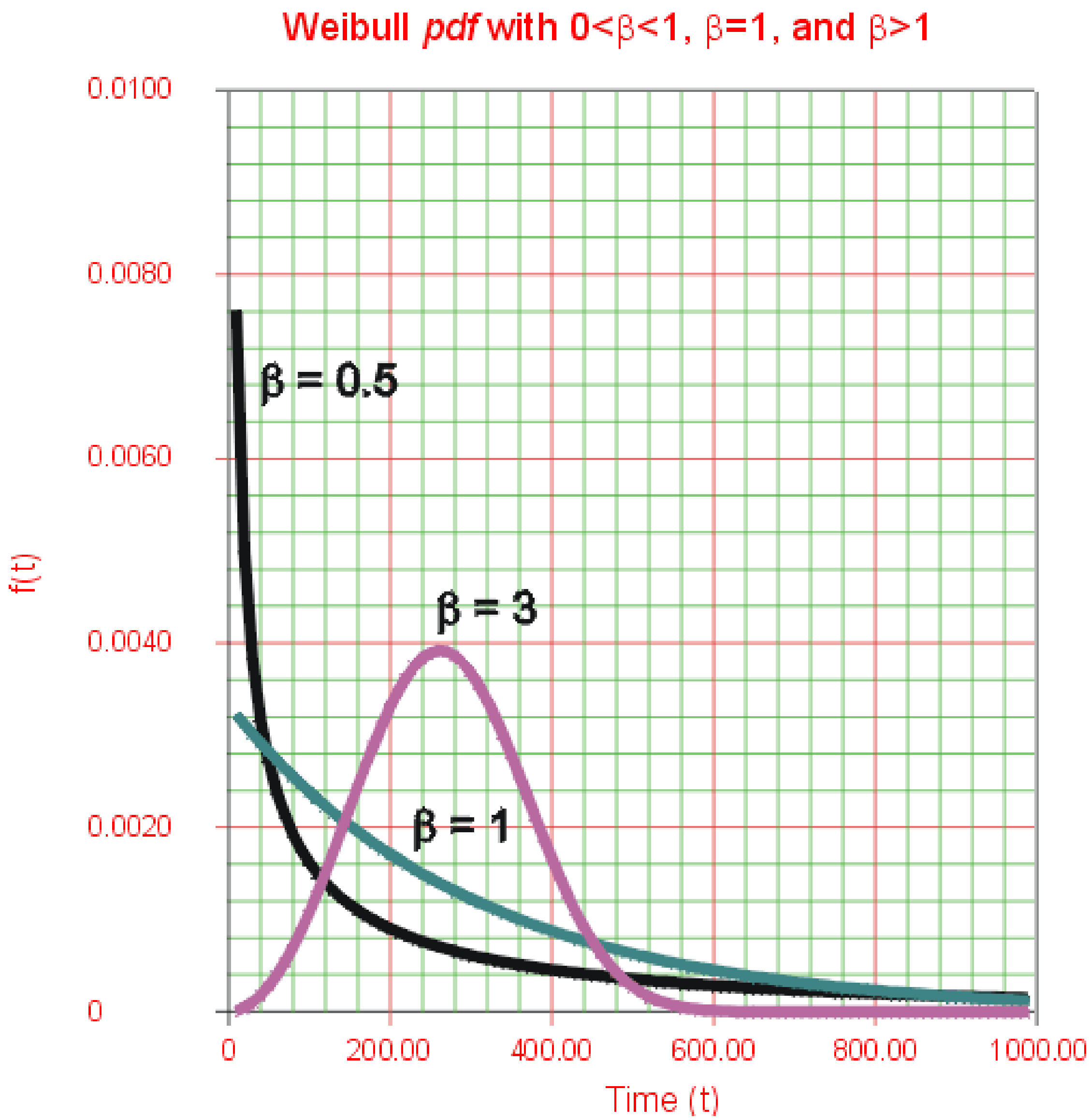
      → the failure rate increases at a decreasing rate as t increases
    - $\beta = 2$, a <u>straight line</u> relationship between $h(t)$ and t
    - $\beta > 2$, the $h(t)$ curve is <u>convex</u>

      → the failure rate increases at an increasing rate as t increases

Effects of β on Weibull Hazard Function

<table>
<tr><th colspan="2">Value</th><th>Property</th><th>Application</th></tr>
<tr><td colspan="2">$0<\beta<1$</td><td>DFR (pdf is similar to Exponential Dist.)</td><td>Model infant mortality</td></tr>
<tr><td colspan="2">$\beta=1$</td><td>CFR (pdf is Exponential Dist.)</td><td>Model useful life</td></tr>
<tr><td rowspan="4">$\beta>1$</td><td>$1<\beta<2$</td><td>IFR (hazard function is concave)</td><td rowspan="4">Model wearout</td></tr>
<tr><td>$\beta=2$</td><td>LFR (hazard function is linearly increasing) (pdf is Rayleigh distribution)</td></tr>
<tr><td>$\beta>2$</td><td>IFR (hazard function is convex)</td></tr>
<tr><td>$3\leq\beta\leq4$</td><td>IFR (pdf approaches normal, symmetrical)</td></tr>
</table>

Effects of β on Weibull PDF



$$f(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1} \exp\left[-\left(\frac{t}{\theta}\right)^{\beta}\right]$$

Effects of β on Weibull PDF

$0 < \beta < 1$:

- As $t \rightarrow 0$, $f(t) \rightarrow \infty$.
- As $t \rightarrow \infty$, $f(t) \rightarrow 0$.
- $f(t)$ decreases monotonically as t increases
- $f(t)$ is convex
- The mode is non-existent.

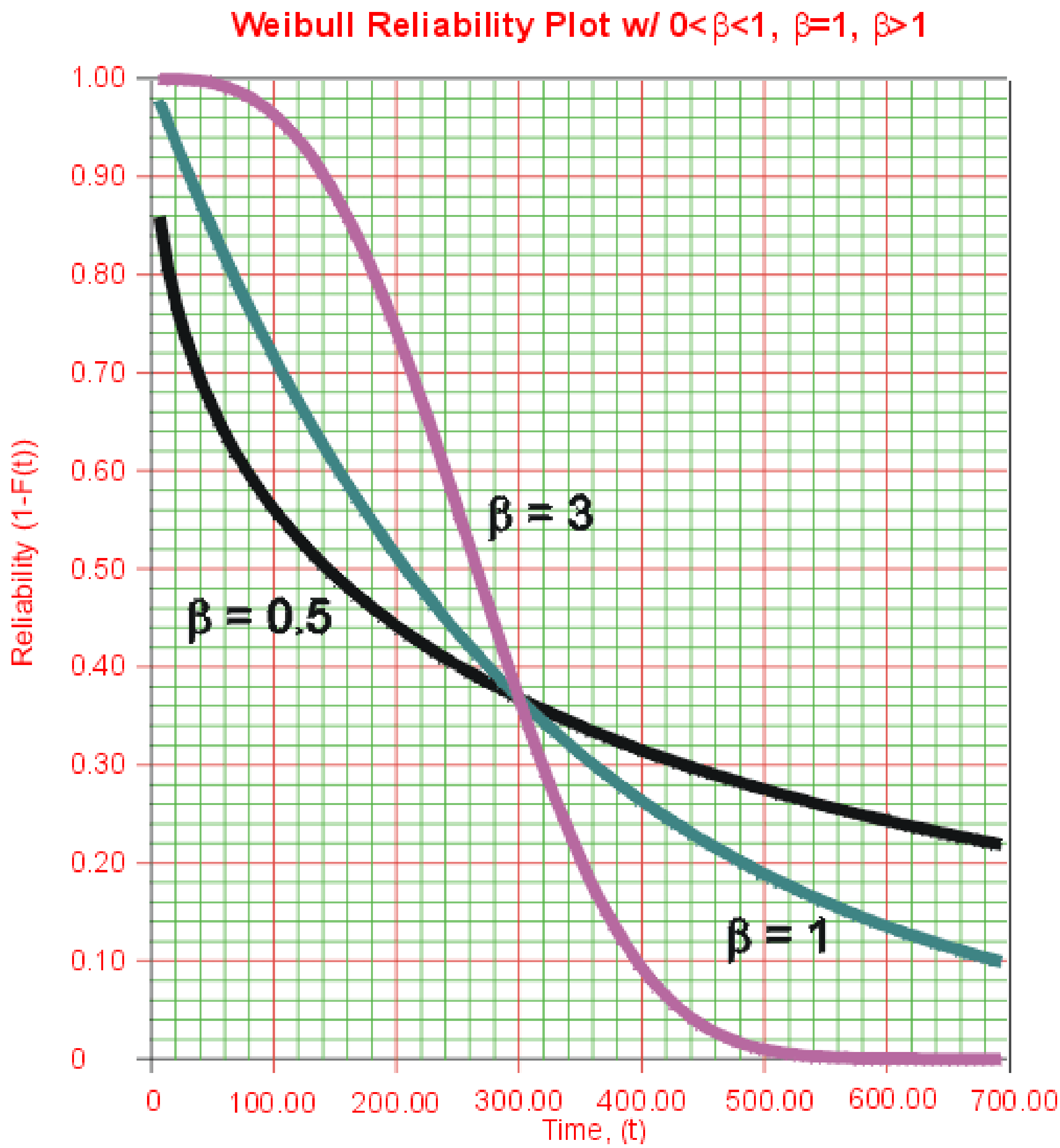
$\beta = 1$:

- exponential distribution

$\beta > 1$:

- $f(t) = 0$ at $t = 0$
- $f(t)$ increases as t approaches to the mode and decreases thereafter
- $1 < \beta < 2.6$, $f(t)$ is positively skewed → right tail
- $2.6 < \beta < 3.7$, its coefficient of skewness approaches zero →no tail, approximate the normal *pdf*,
- $\beta > 3.7$, $f(t)$ is negatively skewed → left tail

Effects of β on Weibull Reliability Function



$$R(t) = \exp\left[-\left(\frac{t}{\theta}\right)^{\beta}\right]$$

Effects of β on Weibull Reliability Function

$$R(t) = \exp\left[-\left(\frac{t}{\theta}\right)^{\beta}\right]$$

- $0 < \beta < 1$, R(t) decreases sharply and monotonically, and is convex.
- $\beta = 1$, R(t) decreases monotonically but less sharply than for $0 < \beta < 1$, and is convex.
- $\beta > 1$, R(t) decreases as *t* increases. As wear-out sets in, the curve goes through an inflection point and decreases sharply.

Scale Parameter, *θ*

- The scale parameter influences both the mean and the spread of the Weibull distribution.

- All Weibull reliability functions pass through the point (*θ, 0.368*).

$$R(\theta) = \exp\left[-\left(\frac{\theta}{\theta}\right)^{\beta}\right] = e^{-1} = 0.368$$

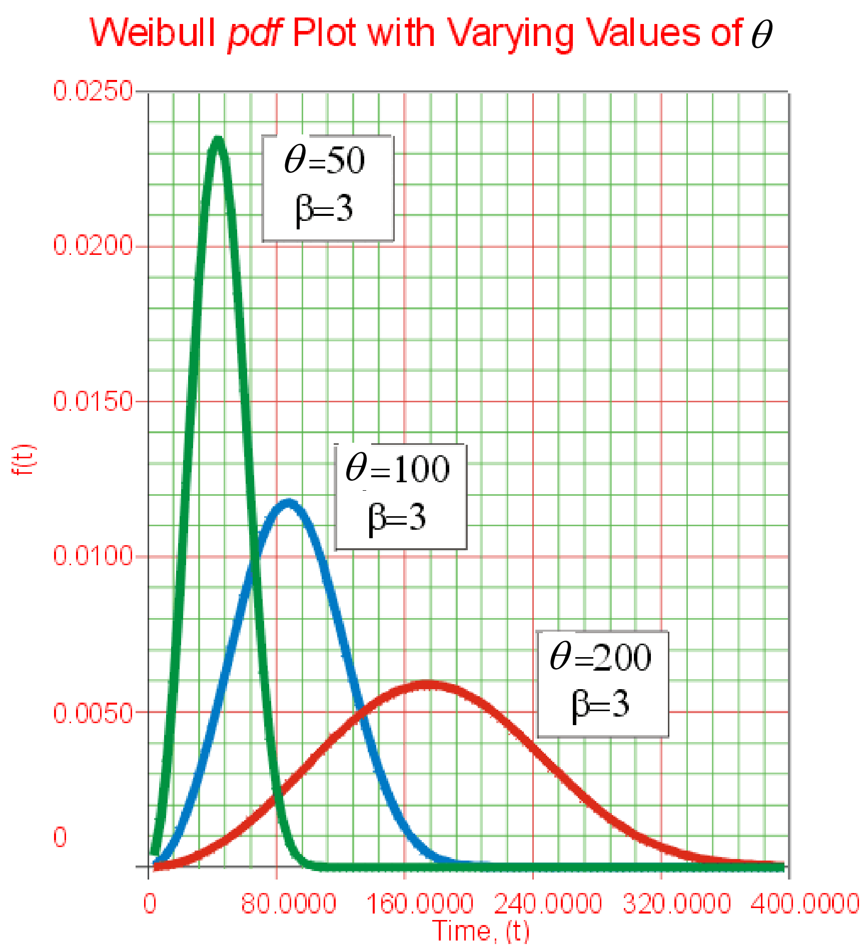
- θ is also called characteristic life.
- θ has the same units as *T*, such as hours, miles, cycles, actuations, etc.

Effects of θ on Weibull PDF

$$f(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1} \exp\left[-\left(\frac{t}{\theta}\right)^{\beta}\right]$$

- Increasing the value of θ has the effect of stretching out the *pdf*

  → influences both mean and spread
- Since the area under a *pdf* curve is a constant value of one, the "peak" of the *pdf* curve will also decrease with the increase of θ



Examples (Weibull Dist.)

Example 1

A two-speed synchronized transfer case used in a large industrial dump truck experiences failures that seem to be well approximated by a two parameter Weibull distribution with

$$\theta = 18,000 \text{ km and } \beta = 2.7$$

(1) Characterized the failure process based on the values of θ and β.

(2) What is the 10,000 km reliability?

$$R(10000) = e^{-(\frac{10000}{18000})^{2.7}} = 0.815$$

(3) What is the 24,000 km reliability?

$$\mathrm{R}(24000) = \mathrm{e}^{-(\frac{24000}{18000})^{2.7}} = 0.1137$$

Examples (Weibull Dist.)

Example 1 (Cont.)

A two-speed synchronized transfer case used in a large industrial dump truck experiences failures that seem to be well approximated by a two parameter Weibull distribution with

$$\theta = 18,000 \text{ km and } \beta = 2.7$$

(4) What is the B_{10} life? (km at which 10% of the population will fail, or 90% reliability is desired.)

$$R(\beta_\alpha) = 1 - \alpha/100$$

$$R(\beta_{10}) = 0.9 = e^{-(\frac{t}{18000})^{2.7}} \Rightarrow \beta_{10} = 7821.7$$

(5) If the Weibull slope can be changed to $\beta = 1.7$ by changing the synchronizer gear tooth design, how will this affect the above answers?

$$R(10000) = 0.6920 < 0.815$$

$$R(24000) = 0.1957 > 0.1137$$

Examples (Weibull Dist.)

Example 2

The characteristic life for a highly turbocharged diesel engine in a military application is 1,800 miles with a Weibull slope of 1.97. What is the B_{10} life?

Examples (Weibull Dist.)

Example 3

A device that shows running-in failure pattern, has threshold time to failure of 150 days and characteristic life of 100days. What is the probability that this device will fail before 200 days of running times? What is the probability that this device will survive for 180days? What is the age-specific failure rate at 180 days? (Hint: the shape factor in this case could be one of the following; 1.2 ,0.5, 4.5)

MTTF and Variance

$$\text{MTTF} = \theta \Gamma\left(1 + \frac{1}{\beta}\right)$$

$$\sigma^2 = \theta^2 \left\{ \Gamma\left(1 + \frac{2}{\beta}\right) - \left[\Gamma\left(1 + \frac{1}{\beta}\right)\right]^2 \right\}$$

where $\Gamma(x)$ is the gamma function:

$$\Gamma(x) = \int_0^\infty y^{x-1} e^{-y} dy$$

To obtain the gamma function, use Standard Table in the textbook, and

$$\Gamma(x) = (x-1)\Gamma(x-1), \quad x > 0$$

$$\Gamma(x) = (x-1)!, \quad x = \text{integer} > 0$$

Median and Mode

- **Design Life and B_α Life**

$$R(t_R) = e^{-(t_R/\theta)^\beta} = R \quad \Rightarrow \quad t_R = \theta(-\ln R)^{1/\beta}$$

$$R(B_\alpha) = e^{-(B_\alpha/\theta)^\beta} = 1 - \alpha/100$$

$$\Rightarrow \quad B_\alpha = \theta\left(-\ln(1-\alpha/100)\right)^{1/\beta}$$

- **Median**

$$R(B_{50}) = 0.5 = e^{-(B_{50}/\theta)^\beta}$$

$$\Downarrow$$

$$B_{50} = t_{\text{med}} = \theta(\ln 2)^{1/\beta}$$

- **Mode**

$$t_{\text{mode}} = \begin{cases} \theta(1-1/\beta)^{1/\beta} & \text{for } \beta > 1 \\ 0 & \text{for } \beta \le 1 \end{cases}$$

Conditional Reliability

- Conditional reliability is useful in describing the reliability of an item following a burn-in period or after a warranty period, T_0.
- For Weibull distribution, the conditional reliability is

$$R(t \mid T_0) = \frac{P[T > t + T_0]}{P[T > T_0]}$$

$$= \frac{e^{-[(t+T_0)/\theta]^\beta}}{e^{-(T_0/\theta)^\beta}}$$

$$= \exp\left[(T_0/\theta)^\beta - [(t+T_0)/\theta]^\beta \right]$$

- $R(t \mid T_0) = R(t)$ for all t if and only if $R(t)$ is exponential.

Burn-In

- Early failures (infant mortality) – common "reliability" problem, esp. in electronic equipment → usually caused by manufacturing "defects"--- quality problems. Problems are typically common in "new" products and may disappear as technology matures.

- Ideal → build-in Q&R upfront and reduce such problems – but hard to do with complex technology and pressure to reduce product development cycle time.

- To achieve reliability goals and reduce field-failure, common practice to "burn-in" components and systems to screen out units that would fail early – esp. important in safety-critical applications → can be viewed as a type of 100% inspection.

- Done at use condition or low-level stress environment to avoid undue aging of components and systems.

- Burn-in is expensive → incorporate costs and benefits and decide on optimal trade-off

Examples (Weibull Dist.)

Example 4

Given a Weibull distribution with a characteristic life of 127,000 hr and a slope of 3.74, find the mean and standard deviation. Also find the probability of surviving the mean life.

Solution

$$\text{MTTF} = \theta\,\Gamma\,(1+\frac{1}{\beta})$$

$$A = \Gamma\,(1+\frac{1}{\beta}) = \Gamma(1+\frac{1}{3.74}) = \Gamma(1.27) = 0.9025$$

$$B = \Gamma\,(1+\frac{2}{\beta}) = \Gamma(1+\frac{2}{3.74}) = \Gamma(1.53) = 0.88757$$

$$\therefore MTTF = 127000(0.9025) = 114617.5$$

$$\sigma^2 = \theta^2(B-A^2) = \qquad \Rightarrow \sigma = 34382.5$$

Examples (Weibull Dist.)

Example 4

For the two-speed synchronized transfer case problem in Example 1, we have two designs. Find the mean and standard deviation of the life.

A: $\theta = 18,000 \text{ km and } \beta = 2.7$

Solution

$$\text{MTTF} = 16007.57$$

$$\sigma = 6388.294$$

B: $\theta = 18,000 \text{ km and } \beta = 1.7$

$$\text{MTTF} = 16063.74$$

$$\sigma = 9752.76$$

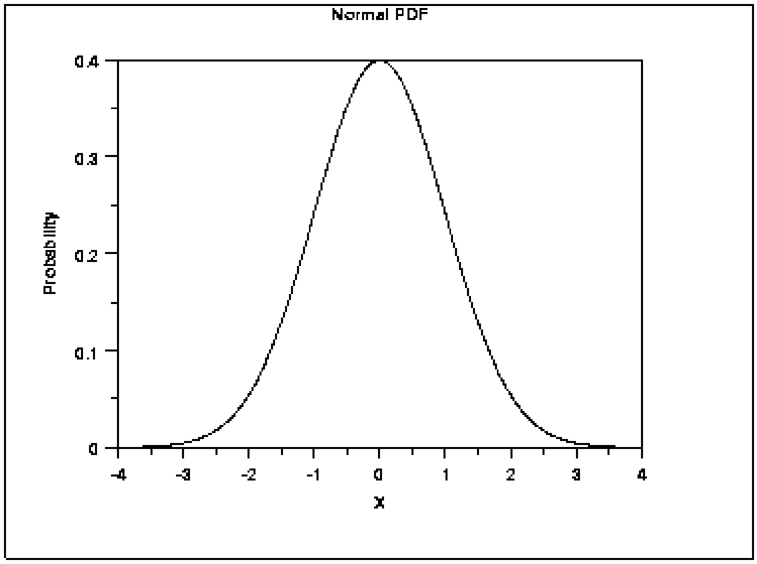
Normal (Gaussian) Distribution

- Normal distribution is not a true reliability distribution since the random variable ranges from minus infinity to plus infinity.

- When the probability that the random variable takes on negative values is negligible, the normal distribution can be used successfully to model fatigue and wear out phenomena.

- It will be used to analyze lognormal distribution.

Probability Density Function



For $T \sim N(\mu, \sigma^2)$, the PDF is:

$$f(t) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2} \frac{(t-\mu)^2}{\sigma^2} \right], \quad -\infty \leq t \leq \infty$$

- μ is the location parameter

  → shifts the graph left or right on the horizontal axis

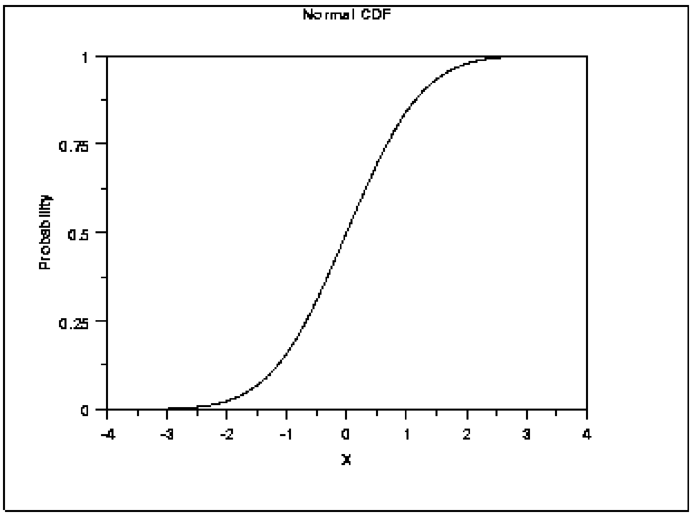
- σ^2 is the scale parameter

  → stretches or compresses the graph

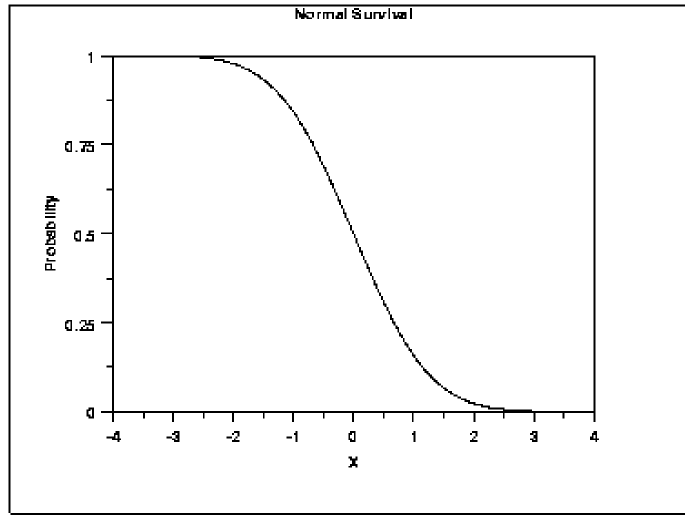
$$f(t) = \frac{1}{\sigma} \phi\left(\frac{t-\mu}{\sigma} \right)$$

where $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ is the pdf for standard normal distribution ($\mu=0$, $\sigma^2=1$).

CDF and Reliability Function



F(t)



R(t)

- Cumulative Distribution Function, F(t)

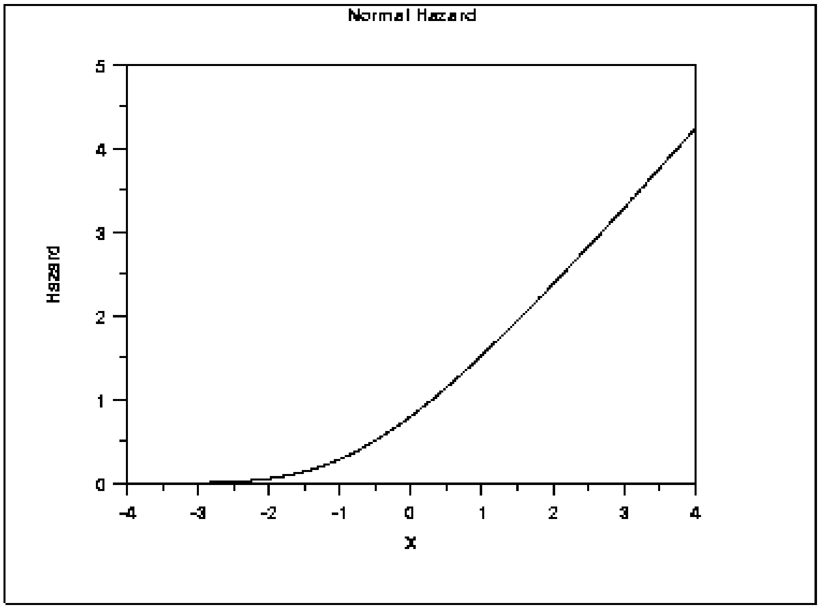
$$F(t) = \Phi\left(\frac{t-\mu}{\sigma}\right)$$

where $\Phi(z)$ is the cdf for standard normal distribution (μ=0, σ^2=1).

- Reliability Function, R(t)

$$R(t) = 1 - F(t) = 1 - \Phi\left(\frac{t-\mu}{\sigma}\right)$$

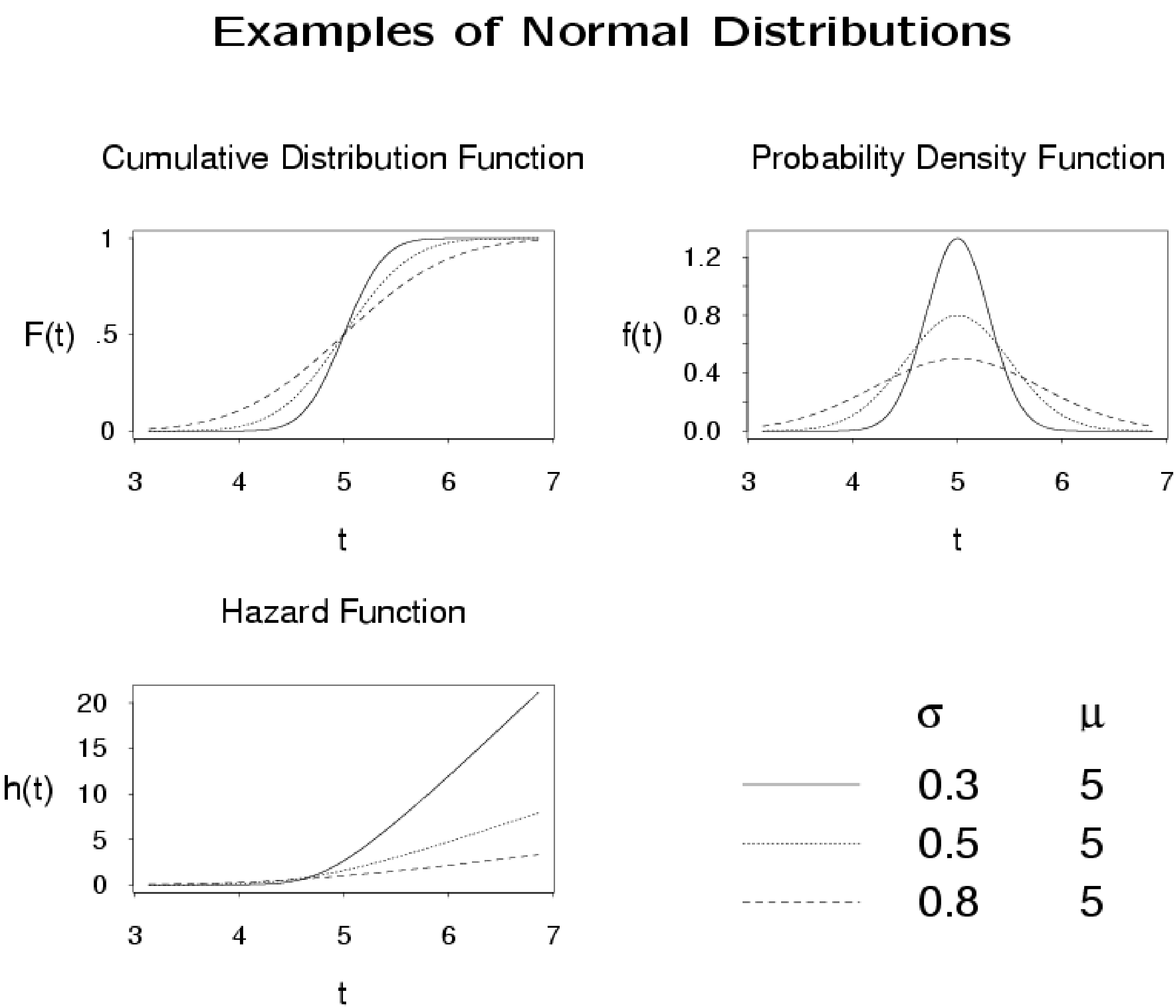
Hazard Function



$$h(t) = \frac{f(t)}{R(t)} = \frac{\frac{1}{\sigma}\phi\left(\frac{t-\mu}{\sigma}\right)}{1-\Phi\left(\frac{t-\mu}{\sigma}\right)}$$

- *h(t)* is an increasing function
  → the normal distribution can <u>only</u> be used to model wear-out (IFR) phenomena.

Examples of Normal Distributions



Mean, Variance and Moments

- **Mean and Variance**

$$E[T] = \mu$$

$$Var[T] = E\left[(T-\mu)^2\right] = \sigma^2$$

- **Moments of Normal Distribution**

$$E[(T-\mu)^m] = \begin{cases} 0 & \text{m is odd} \\ \dfrac{m!\sigma^m}{2^{m/2}(m/2)!} & \text{m is even} \end{cases}$$

Central Limit Theorem

If $X_1, X_2, \ldots, X_n$ is a random sample of size n taken from a population (either finite or infinite) with mean μ and finite variance σ^2, and if $\bar{X}$ is the sample mean, as the sample size n approaches to infinite,

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

- The central limit theorem basically states:
  - The sampling distribution of the mean becomes approximately normal regardless of the distribution of the original population.
  - The sampling distribution of the mean is centered at the population mean, μ.
  - The standard deviation of the sampling distribution of the mean approaches $\sigma / \sqrt{n}$.
- Due to the central limit theorem, the normal distribution plays a central role in classical statistics.

Examples (Normal)

Example 1

A component has a normal distribution of failure times with $\mu = 20{,}000$ cycles and $\sigma = 2{,}000$ cycles. Find the reliability of the component and the hazard function at 19,000 cycles.

Lognormal Distribution

- The lognormal distribution is used extensively in reliability applications to model failure times. (The lognormal and Weibull distributions are the most commonly used distributions in reliability applications.)

- Like the Weibull distribution, the lognormal can take a variety of shapes.

- The relationship between lognormal distribution and normal distribution is very useful in analyzing the lognormal distribution.

Lognormal Distribution

If $T \sim \text{LogNor}(\mu, \sigma^2)$, then $\text{Ln}(T) \sim \text{N}(\mu, \sigma^2)$

with **Probability Density Function**

$$f(t) = \frac{1}{\sigma t \sqrt{2\pi}} \exp\left[-\frac{1}{2} \frac{(\ln t - \mu)^2}{\sigma^2} \right], \quad t \geq 0$$

- μ is the mean of ln*(T)*, and $-\infty \leq \mu \leq \infty$
- σ^2 is the variance of ln*(T),* and $\sigma > 0$

	Lognormal ***T***	**Normal** ***Ln(T)***
Mean	$\exp\left[\mu + \frac{\sigma^2}{2}\right]$	μ
Variance	$\left(e^{2\mu+\sigma^2}\right)\left(e^{\sigma^2} - 1\right)$	σ^2
Median	e^{μ}	μ
Mode	$e^{\mu-\sigma^2}$	μ

Lognormal Distribution

- **PDF and CDF**

$$f(t) = \frac{1}{\sigma t}\phi\left(\frac{\ln t - \mu}{\sigma}\right), \quad t \geq 0$$

$$F(t) = \Phi\left(\frac{\ln t - \mu}{\sigma}\right), \quad t \geq 0$$

- **Reliability Function**

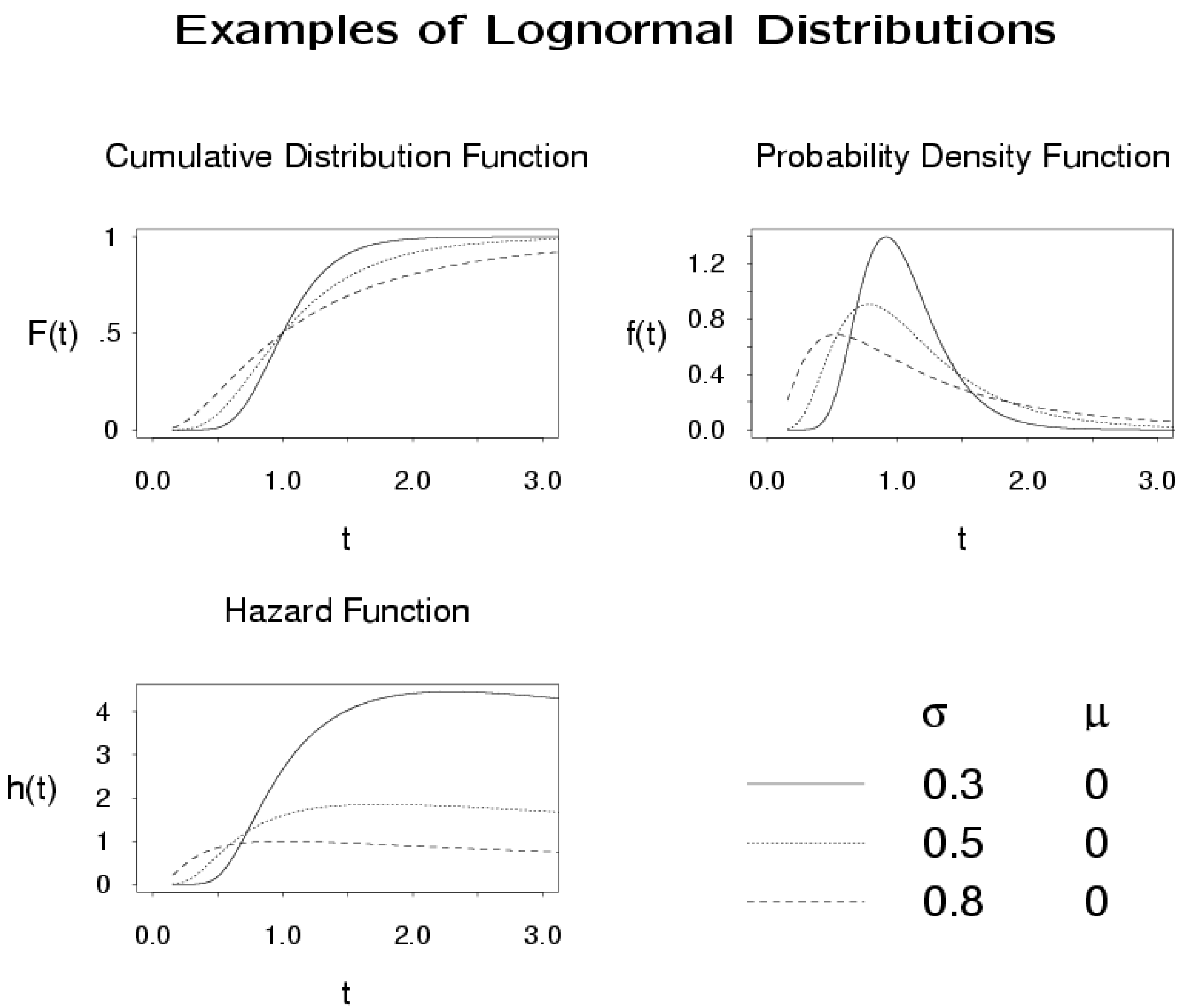
$$R(t) = 1 - F(t) = 1 - \Phi\left(\frac{\ln t - \mu}{\sigma}\right), \quad t \geq 0$$

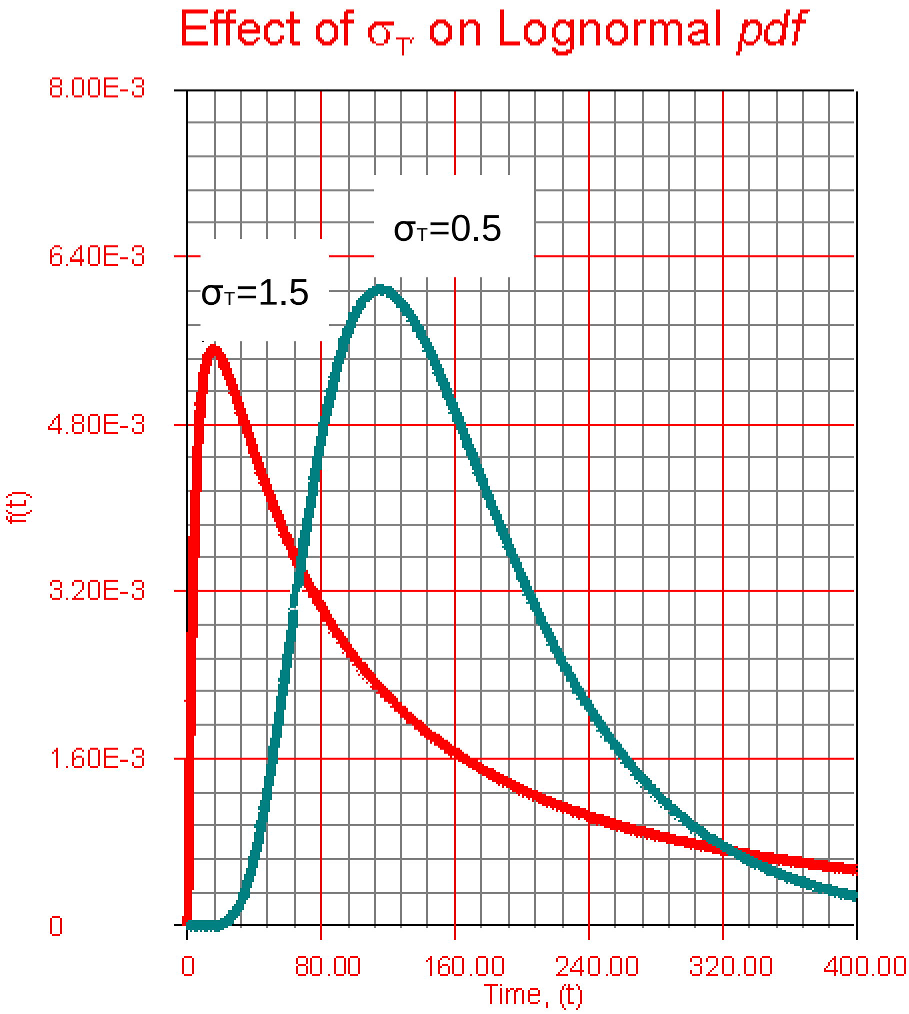
- **Hazard Function**

$$h(t) = \frac{f(t)}{R(t)} = \frac{\frac{1}{\sigma t}\phi\left(\frac{\ln t - \mu}{\sigma}\right)}{1 - \Phi\left(\frac{\ln t - \mu}{\sigma}\right)}, \quad t \geq 0$$

where $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ and $\Phi(z)$ are the pdf and cdf for standard normal distribution (μ=0, σ^2=1).

Examples of Lognormal Distribution





- The lognormal distribution is a distribution skewed to the right.
- The *pdf* starts at zero, increases to its mode, and decreases thereafter.
- The degree of skewness increases as σ_T increases.

Examples (Lognormal)

Example 1

The failure time of a certain component is log-normally distributed with $\mu = 5$ and $\sigma^2 = 1$.

1. Find the reliability of the component and the hazard rate for a life of 150 time units.
2. Find the mean and variance of the failure time of the component.