111 ريض - حساب التكامل
الفصل الدراسي الأول 1443 هـ
حل الاختبار الفصلي
د. طارق عبدالرحمن مُحَمَّد الفاضل

السؤال الأول (12 درجة):

(1) جد $F'(x)$ إذا كانت $F(x)=\int_{e^{\sin x}}^{2^{\sqrt{x}}} \frac{1}{\sqrt{2+t^2}}\,dt$

الحل :

$$F'(x)=\frac{d}{dx}\int_{e^{\sin x}}^{2^{\sqrt{x}}} \frac{1}{\sqrt{2+t^2}}\,dt$$

$$=\frac{1}{\sqrt{2+\left(2^{\sqrt{x}}\right)^2}}\,2^{\sqrt{x}}\,\frac{1}{2\sqrt{x}}\ln 2-\frac{1}{\sqrt{2+\left(e^{\sin x}\right)^2}}\,e^{\sin x}\cos x$$

$$=\frac{2^{\sqrt{x}}\ln 2}{2\sqrt{x}\,\sqrt{2+4^{\sqrt{x}}}}-\frac{e^{\sin x}\cos x}{\sqrt{2+e^{2\sin x}}}$$

احسب $\frac{dy}{dx}$ فيمايلي :

(2) $y=3^{3x}\ \sin^{-1}\left(e^x\right)$

الحل :

$$\frac{dy}{dx}=\left(3^{3x}\ (3)\ \ln 3\right)\ \sin^{-1}\left(e^x\right)+3^{3x}\ \left(\frac{1}{\sqrt{1-\left(e^x\right)^2}}\,e^x\right)$$

$$=3^{3x+1}\ln 3\ \sin^{-1}\left(e^x\right)+\frac{3^x\,e^x}{\sqrt{1-e^{2x}}}$$

(3) $y=\sqrt[4]{x}\ \log_4\left|2+\tan(2x^2)\right|$

الحل :

$$y=\sqrt[4]{x}\ \log_4\left|2+\tan(2x^2)\right|=x^{\frac{1}{4}}\ \log_4\left|2+\tan(2x^2)\right|$$

$$\frac{dy}{dx}=\left(\frac{1}{4}x^{-\frac{3}{4}}\right)\log_4\left|2+\tan(2x^2)\right|+x^{\frac{1}{4}}\ \left(\frac{\sec^2(2x^2)\,4x}{2+\tan(2x^2)}\,\frac{1}{\ln 4}\right)$$

$$=\frac{\log_4\left|2+\tan(2x^2)\right|}{4x^{\frac{3}{4}}}+\frac{4x^{\frac{5}{4}}\sec^2(2x^2)}{\ln 4\,[2+\tan(2x^2)]}$$

$$y = \ln \left| \frac{\sqrt[3]{x^2+1}\ \sec(2x)}{\sqrt{x+\sin x}} \right| \quad (4)$$

الحل :

$$y = \ln \left| \frac{\left(x^2+1\right)^{\frac{1}{3}}\ \sec(2x)}{(x+\sin x)^{\frac{1}{2}}} \right| = \ln \left| \left(x^2+1\right)^{\frac{1}{3}}\ \sec(2x) \right| - \ln \left| (x+\sin x)^{\frac{1}{2}} \right|$$

$$= \ln \left| \left(x^2+1\right)^{\frac{1}{3}} \right| + \ln |\sec(2x)| - \ln \left| (x+\sin x)^{\frac{1}{2}} \right|$$

$$= \frac{1}{3} \ln |x^2+1| + \ln |\sec(2x)| - \frac{1}{2} \ln |x+\sin x|$$

باشتقاق الطرفين

$$y' = \frac{1}{3} \left(\frac{2x}{x^2+1} \right) + \frac{2\sec(2x)\ \tan(2x)}{\sec(2x)} - \frac{1}{2} \left(\frac{1+\cos x}{x+\sin x} \right)$$

$$y' = \frac{2x}{3(x^2+1)} + 2\tan(2x) - \frac{1+\cos x}{2(x+\sin x)}$$

$$y = (\sec x)^{\tan x} \quad (5)$$

الحل :

$$y = (\sec x)^{\tan x} \implies \ln |y| = \ln \left| (\sec x)^{\tan x} \right| = \tan x\ \ln |\sec x|$$

باشتقاق الطرفين

$$\frac{y'}{y} = \sec^2 x\ \ln |\sec x| + \tan x\ \left(\frac{\sec x \tan x}{\sec x} \right)$$

$$y' = y\ \left[\sec^2 x\ \ln |\sec x| + \tan^2 x \right]$$

$$y' = (\sec x)^{\tan x}\ \left[\sec^2 x\ \ln |\sec x| + \tan^2 x \right]$$

$$y = \cosh^{-1} \left(\frac{1}{x} \right) \quad (6)$$

الحل :

$$\frac{dy}{dx} = \frac{1}{\sqrt{\left(\frac{1}{x}\right)^2 - 1}}\ \frac{-1}{x^2} = \frac{-1}{x^2\ \sqrt{\frac{1}{x^2} - 1}}$$

السؤال الثاني (18 درجة) : أحسب التكاملات التالية

(1) $\int \frac{\csc^2(\sqrt{x})}{\sqrt{x}}\, dx$

الحل :

$$\int \frac{\csc^2(\sqrt{x})}{\sqrt{x}}\, dx = \int \csc^2(\sqrt{x})\ \frac{1}{\sqrt{x}}\, dx = 2\int \csc^2(\sqrt{x})\ \frac{1}{2\sqrt{x}}\, dx$$

$$= 2\left(-\cot(\sqrt{x})\right) + c = -2\cot(\sqrt{x}) + c$$

باستخدام القانون

$$\int \csc^2(f(x))\ f'(x)\, dx = -\cot(f(x)) + c$$

(2) $\int (x^2-2)^5\ x\, dx$

الحل :

$$\int (x^2-2)^5\ x\, dx = \frac{1}{2}\int (x^2-2)^5\ (2x)\, dx = \frac{1}{2}\,\frac{(x^2-2)^6}{6} + c$$

باستخدام القانون $\int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} + c$ حيث $n \neq -1$.

(3) $\int_0^1 x\, 10^{1+x^2}\, dx$

الحل :

$$\int_0^1 x\, 10^{1+x^2}\, dx = \frac{1}{2}\int_0^1 10^{1+x^2}(2x)\, dx = \frac{1}{2}\left[\frac{10^{1+x^2}}{\ln 10}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{10^{1+1^2}}{\ln 10} - \frac{10^{1+0^2}}{\ln 10}\right] = \frac{1}{2}\left[\frac{10^2}{\ln 10} - \frac{10}{\ln 10}\right] = \frac{1}{2}\left(\frac{90}{\ln 10}\right) = \frac{90}{2\ln 10}$$

باستخدام القانون $\int a^{f(x)}\ f'(x)\, dx = \frac{a^{f(x)}}{\ln a} + c$.

(4) $\int \frac{x-2}{x+1}\, dx$

الحل :

$$\int \frac{x-2}{x+1}\,dx = \int \frac{(x+1)-3}{x+1}\,dx = \int \left(\frac{x+1}{x+1} - \frac{3}{x+1}\right)\,dx$$

$$= \int \left(1 - \frac{3}{x+1}\right)\,dx = \int 1\,dx - \int \frac{3}{x+1}\,dx = \int 1\,dx - 3\int \frac{1}{x+1}\,dx$$

$$= x - 3\ln|x+1| + c$$

باستخدام القانون $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$.

(5) $\int \frac{e^{3x}}{\sqrt{16+e^{6x}}}\,dx$

الحل :

$$\int \frac{e^{3x}}{\sqrt{16+e^{3x}}}\,dx = \int \frac{e^{3x}}{\sqrt{(4)^2+(e^{3x})^2}}\,dx = \frac{1}{3}\int \frac{3\,e^{3x}}{\sqrt{(4)^2+(e^{3x})^2}}\,dx$$

$$= \frac{1}{3}\sinh^{-1}\left(\frac{e^{3x}}{4}\right) + c$$

باستخدام القانون

$$\int \frac{f'(x)}{\sqrt{a^2+[f(x)]^2}}\,dx = \sinh^{-1}\left(\frac{f(x)}{a}\right) + c$$

(6) $\int \frac{\coth(\ln x)}{x}\,dx$

الحل :

$$\int \frac{\coth(\ln x)}{x}\,dx = \int \coth(\ln x)\left(\frac{1}{x}\right)\,dx = \ln|\sinh(\ln x)| + c$$

باستخدام القانون

$$\int \coth(f(x))\;f'(x)\,dx = \ln|\sinh(f(x))| + c$$

(7) $\int \frac{x}{x^4+16}\,dx$

الحل :

$$\int \frac{x}{x^4+16}\,dx = \int \frac{x}{(x^2)^2+(4)^2}\,dx = \frac{1}{2}\int \frac{2x}{(x^2)^2+(4)^2}\,dx$$

$$= \frac{1}{2}\,\frac{1}{4}\,\tan^{-1}\left(\frac{x^2}{4}\right) + c = \frac{1}{8}\,\tan^{-1}\left(\frac{x^2}{4}\right) + c$$

باستخدام القانون

$$\int \frac{f'(x)}{a^2 + [f(x)]^2}\,dx = \frac{1}{a}\,\tan^{-1}\left(\frac{f(x)}{a}\right) + c \,.$$

$$\int (x-1)\,\sqrt{x+1}\,dx \quad (8)$$

الحل : بوضع $\sqrt{x+1} = u \implies x+1 = u^2 \implies x = u^2 - 1 \implies x - 1 = u^2 - 2$

$$dx = 2u\,du$$

$$\int (x-1)\,\sqrt{x+1}\,dx = \int \left(u^2-2\right)\,u\,2u\,du = 2\int u^2\left(u^2-2\right)\,du$$

$$= 2\int \left(u^4 - 2u^2\right)\,du = 2\left(\frac{u^5}{5} - 2\,\frac{u^3}{3}\right) + c$$

$$= \frac{2}{5}\left(\sqrt{x+1}\right)^5 - \frac{4}{3}\left(\sqrt{x+1}\right)^3 + c = \frac{2}{5}\,(x+1)^{\frac{5}{2}} - \frac{4}{3}\,(x+1)^{\frac{3}{2}} + c$$

$$\int e^{2x}\sinh x\,dx \quad (9)$$

الحل :

$$\int e^{2x}\sinh x\,dx = \int e^{2x}\left(\frac{e^x - e^{-x}}{2}\right)\,dx = \int \left(\frac{e^{3x} - e^x}{2}\right)\,dx$$

$$= \int \left(\frac{e^{3x}}{2} - \frac{e^x}{2}\right)\,dx = \frac{1}{2}\,\frac{1}{3}\int e^{3x}\,(3)\,dx - \frac{1}{2}\int e^x\,dx$$

$$= \frac{1}{6}\,e^{3x} - \frac{1}{2}\,e^x + c = \frac{e^{3x}}{6} - \frac{e^x}{2} + c$$

باستخدام القانون $\int e^{f(x)}\,f'(x)\,dx = e^{f(x)} + c$

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الجزء الأول (7 درجات):

(1) أوجد قيمة c التي تحقق نظرية القيمة المتوسطة للدالة $f(x) = \sqrt[3]{x-1}$ على الفترة $[1,9]$

الحل : باستخدام العلاقة $(b-a)f(c) = \int_a^b f(x)\,dx$

حيث $f(x) = \sqrt[3]{x-1}$ و $[a,b] = [1,9]$

$$(9-1)\ \sqrt[3]{x-1} = \int_1^9 \sqrt[3]{x-1}\,dx$$

$$8\ \sqrt[3]{c-1} = \int_1^9 (x-1)^{\frac{1}{3}}\ dx = \left[\frac{(x-1)^{\frac{4}{3}}}{\frac{4}{3}}\right]_1^9 = \left[\frac{3}{4}\ (x-1)^{\frac{4}{3}}\right]_1^9$$

$$8\ \sqrt[3]{c-1} = \frac{3}{4}\ (9-1)^{\frac{4}{3}} - \frac{3}{4}\ (1-1)^{\frac{4}{3}} = \frac{3}{4}\ (8)^{\frac{4}{3}} - 0 = \frac{3}{4}\,(2)^4 = \frac{3}{4}\,(16) = 12$$

$$\sqrt[3]{c-1} = \frac{12}{8} = \frac{3}{2} \implies c-1 = \left(\frac{3}{2}\right)^3 = \frac{27}{8} \implies c = \frac{27}{8} + 1 = \frac{35}{8}$$

لاحظ أن $c = \frac{35}{8} \in (1,9)$.

(2) جد $F'(x)$ إذا كانت $F(x) = \int_{\sin 3x}^{e^{5x}} \sqrt{4+t^2}\,dt$.

الحل : $F'(x) = \frac{d}{dx}\int_{\sin 3x}^{e^{5x}} \sqrt{4+t^2}\,dt$

$$F'(x) = \sqrt{4+\left(e^{5x}\right)^2}\ \left(e^{5x}\,5\right) - \sqrt{4+(\sin 3x)^2}\ (\cos 3x\,(3))$$

$$= 5e^{5x}\sqrt{4+e^{10x}} - 3\cos 3x\sqrt{4+\sin^2 3x}$$

(3) جد $f'(x)$ إذا كانت $f(x) = \sinh^{-1}\left(4^{x+1}\right) + \log_2\left|\cosh(x^2)\right|$.

الحل :

$$f'(x) = \frac{1}{\sqrt{1+\left(4^{x+1}\right)^2}}\left(4^{x+1}\,(1)\ln 4\right) + \frac{\sinh(x^2)\,(2x)}{\cosh(x^2)}\ \frac{1}{\ln 2}$$

$$= \frac{4^{x+1} \ln 4}{\sqrt{1+4^{2x+2}}} + \frac{2x \ \sinh(x^2)}{\cosh(x^2) \ \ln 2}$$

الجزء الثاني (14 درجة) : أحسب التكاملات التالية :

(1) $\displaystyle\int \frac{4}{x^2+6x+13}\, dx$

الحل : باستخدام الإكمال إلى مربع كامل

$$\int \frac{4}{x^2+6x+13}\, dx = 4\int \frac{1}{(x^2+6x+9)+4}\, dx = 4\int \frac{1}{(x+3)^2+(2)^2}\, dx$$

$$= 4\left(\frac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right)\right) + c = 2\ \tan^{-1}\left(\frac{x+3}{2}\right) + c$$

باستخدام القانون $\displaystyle\int \frac{f'(x)}{a^2+[f(x)]^2}\, dx = \frac{1}{a}\ \tan^{-1}\left(\frac{f(x)}{a}\right) + c$ حيث $a > 0$

(2) $\displaystyle\int x\ \sec^{-1} x\, dx$

الحل : باستخدام طريقة التكامل بالتجزئ

$$u = \sec^{-1} x \qquad dv = x\, dx$$
$$du = \frac{1}{x\sqrt{x^2-1}}\, dx \quad v = \frac{x^2}{2}$$

$$\int x\ \sec^{-1} x\, dx = \frac{x^2}{2}\sec^{-1} x - \int \frac{x^2}{2}\ \frac{1}{x\sqrt{x^2-1}}\, dx$$

$$= \frac{x^2}{2}\sec^{-1} x - \frac{1}{2}\int x\ \left(x^2-1\right)^{-\frac{1}{2}}\ dx = \frac{x^2}{2}\sec^{-1} x - \frac{1}{2}\,\frac{1}{2}\int \left(x^2-1\right)^{-\frac{1}{2}}(2x)\, dx$$

$$= \frac{x^2}{2}\sec^{-1} x - \frac{1}{4}\ \frac{\left(x^2-1\right)^{\frac{1}{2}}}{\frac{1}{2}} + c = \frac{x^2}{2}\sec^{-1} x - \frac{1}{2}\ \sqrt{x^2-1} + c$$

(3) $\displaystyle\int \frac{\sqrt{x^2-9}}{x}\, dx$

الحل : باستخدام التعويضات المثلثية

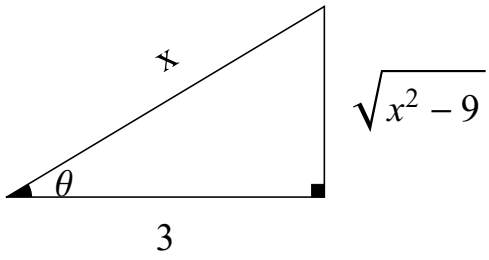
ضع $x = 3\sec\theta \implies \sec\theta = \dfrac{x}{3}$

$$dx = 3\sec\theta\ \tan\theta\, d\theta$$

$$\int \frac{\sqrt{x^2-9}}{x}\,dx = \int \frac{\sqrt{9\sec^2\theta - 9}\ (3\sec\theta\tan\theta)}{3\sec\theta}\,d\theta = \int \sqrt{9\left(\sec^2\theta - 1\right)}\ \tan\theta\,d\theta$$

$$= \int \sqrt{9\tan^2\theta}\ \tan\theta\,d\theta = \int 3\tan^2\theta\,d\theta = 3\int \left(\sec^2\theta - 1\right)\,d\theta$$

$$= 3\int \sec^2\theta\,d\theta - 3\int d\theta = 3\tan\theta - 3\theta + c$$



من المثلث نجد أن : $\tan\theta = \dfrac{\sqrt{x^2-9}}{3}$

لاحظ أن $x = 3\sec\theta \implies \sec\theta = \dfrac{x}{3} \implies \theta = \sec^{-1}\left(\dfrac{x}{3}\right)$

$$\int \frac{\sqrt{x^2-9}}{x}\,dx = 3\,\frac{\sqrt{x^2-9}}{3} - 3\sec^{-1}\left(\frac{x}{3}\right) + c = \sqrt{x^2-9} - 3\sec^{-1}\left(\frac{x}{3}\right) + c$$

$$\int \frac{3x^2 - x + 8}{x^3 + 4x}\,dx \quad (4)$$

الحل : باستخدام طريقة الكسور الجزئية

$$\frac{3x^2 - x + 8}{x^3 + 4x} = \frac{3x^2 - x + 8}{x(x^2+4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

$$3x^2 - x + 8 = A(x^2+4) + (Bx + C)x$$

$$3x^2 - x + 8 = Ax^2 + 4A + Bx^2 + Cx$$

$$3x^2 - x + 8 = (A + B)x^2 + Cx + 4A$$

بمساواة معاملات كثيرتي الحدود

$$\begin{array}{lll} A + B = 3 & \longrightarrow & (1) \\ C = -1 & \longrightarrow & (2) \\ 4A = 8 & \longrightarrow & (3) \end{array}$$

من المعادلة (3) نجد أن : $A = 2$

من المعادلة (1) نجد أن : $2 + B = 3 \implies B = 1$

$$\int \frac{3x^2 - x + 8}{x^3 + 4x}\, dx = \int \left(\frac{2}{x} + \frac{x-1}{x^2+4}\right) dx$$

$$= 2\int \frac{1}{x}\, dx + \int \frac{x}{x^2+4}\, dx - \int \frac{1}{x^2+4}\, dx$$

$$= 2\int \frac{1}{x}\, dx + \frac{1}{2}\int \frac{2x}{x^2+4}\, dx - \int \frac{1}{(x)^2+(2)^2}\, dx$$

$$= 2\ln|x| + \frac{1}{2}\ln\left(x^2+4\right) - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$$

(5) $\displaystyle\int \frac{1}{\sqrt{2+\sqrt{x}}}\, dx$

الحل :

ضع $u = \sqrt{2+\sqrt{x}} \implies u^2 = 2+\sqrt{x} \implies u^2 - 2 = \sqrt{x} \implies \left(u^2-2\right)^2 = x$

$$2\left(u^2-2\right)\ 2u\, du = dx \implies 4u\left(u^2-2\right)\ du = dx$$

$$\int \frac{1}{\sqrt{2+\sqrt{x}}}\, dx = \int \frac{4u\left(u^2-2\right)}{u}\, du = 4\int \left(u^2-2\right)\ du = 4\left[4\,\frac{u^3}{3} - 2u\right] + c$$

$$= \frac{4}{3}\,u^3 - 8u + c = \frac{4}{3}\left(\sqrt{2+\sqrt{x}}\right)^3 - 8\sqrt{2+\sqrt{x}} + c$$

الجزء الثالث (19 درجة) :

(1) احسب $\lim\limits_{x\to\infty} (1+x)^{\frac{2}{x}}$

الحل :

ضع $y = (1+x)^{\frac{2}{x}} \implies \ln|y| = \ln\left|(1+x)^{\frac{2}{x}}\right| = \dfrac{2}{x}\ln|1+x| = \dfrac{2\ln|1+x|}{x}$

$$\lim_{x\to\infty} \ln|y| = \lim_{x\to\infty} \frac{2\ln|1+x|}{x} \quad \left(\frac{\infty}{\infty}\right)$$

باستخدام قاعدة لوبيتال

$$\lim_{x\to\infty} \frac{2\left(\frac{1}{1+x}\right)}{1} = \lim_{x\to\infty} \frac{2}{1+x} = 0$$

وبالتالي $\lim\limits_{x\to\infty} \ln|y| = \lim\limits_{x\to\infty} \dfrac{2\ln|1+x|}{x} = 0$

إذاً $\lim\limits_{x\to\infty} (1+x)^{\frac{2}{x}} = e^0 = 1$

(2) بين فيما إذا كان التكامل المعتل $\int_0^{\infty} \frac{e^x}{1+e^{2x}}\,dx$ متقارباً أم متباعداً .

الحل :

$$\int_0^{\infty} \frac{e^x}{1+e^{2x}}\,dx = \lim_{t\to\infty}\int_0^{t} \frac{e^x}{1+e^{2x}}\,dx = \lim_{t\to\infty}\int_0^{t} \frac{e^x}{1+(e^x)^2}\,dx$$

$$= \lim_{t\to\infty}\left[\tan^{-1}(e^x)\right]_0^t = \lim_{t\to\infty}\left[\tan^{-1}(e^t) - \tan^{-1}(1)\right] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

التكامل المعتل متقارب وقيمته $\frac{\pi}{4}$.

(3) أرسم المنطقة المحصورة بين المنحنيات $y = -x^2$ و $y = x^2 + 1$ و $x = 2$ و $x = -1$ وجد مساحتها .

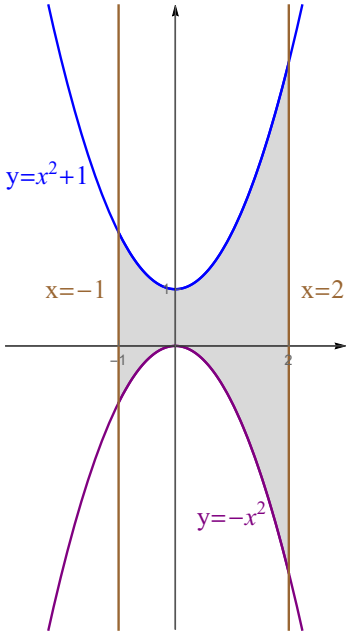
الحل :

المنحنى $y = -x^2$ يمثل قطع مكافئ رأسه $(0,0)$ وفتحته للأسفل .

المنحنى $y = x^2 + 1$ يمثل قطع مكافئ رأسه $(0,1)$ وفتحته للأعلى .

المنحنى $x = 2$ يمثل خط مستقيم يوازي محور y ويمر بالنقطة $(2,0)$.

المنحنى $x = -1$ يمثل خط مستقيم يوازي محور y ويمر بالنقطة $(-1,0)$.



$$\mathrm{A} = \int_{-1}^{2} \left[(x^2+1) - (-x^2)\right]\, dx = \int_{-1}^{2} \left(2x^2+1\right)\, dx = \left[2\,\frac{x^3}{3} + x\right]_{-1}^{2}$$
$$= \left(2\,\frac{(2)^3}{3} + 2\right) - \left(2\,\frac{(-1)^3}{3} - 1\right) = \left(\frac{16}{3} + 2\right) - \left(\frac{-2}{3} - 1\right)$$
$$= \frac{16}{3} + 2 + \frac{2}{3} + 1 = \frac{18}{3} + 3 = 6 + 3 = 9$$

(4) ارسم المنطقة المحصورة بين المنحنيين $y = 1 - x^2$ و $y = -x + 1$ ثم جد حجم الجسم الناتج من دوران هذه المنطقة حول محور x .

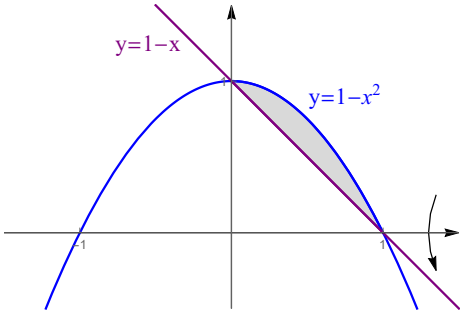
الحل :

$y = 1 - x^2$ يمثل قطع مكافئ رأسه $(0, 1)$ وفتحته للأسفل .

$y = -x + 1$ يمثل خط مستقيم يمر بالنقطة $(0, 1)$ وميله -1 .

إيجاد نقاط تقاطع المنحنيين $y = 1 - x^2$ و $y = -x + 1$:

$$-x + 1 = 1 - x^2 \implies x^2 - x = 0 \implies x(x-1) = 0 \implies x = 0\,,\ x = 1$$



باستخدام طريقة الوردات :

$$\mathrm{V} = \pi \int_0^1 \left[\left(1-x^2\right)^2 - (1-x)^2\right]\, dx = \pi \int_0^1 \left[\left(1 - 2x^2 + x^4\right) - \left(1 - 2x + x^2\right)\right]\, dx$$
$$= \pi \int_0^1 \left(1 - 2x^2 + x^4 - 1 + 2x - x^2\right)\, dx = \pi \int_0^1 \left(x^4 - 3x^2 + 2x\right)\, dx$$
$$= \pi \left[\frac{x^5}{5} - x^3 + x^2\right]_0^1 = \pi \left[\left(\frac{1}{5} - 1 + 1\right) - (0 + 0 + 0)\right] = \frac{\pi}{5}$$

(5) جد طول المنحنى $y = 1 + \frac{2}{3}\, x^{\frac{3}{2}}$ من $x = 0$ إلى $x = 8$.

الحل :

$$y' = 0 + \frac{2}{3}\left(\frac{3}{2}x^{\frac{1}{2}}\right) = x^{\frac{1}{2}} = \sqrt{x}$$

$$\mathrm{L} = \int_0^8 \sqrt{1+\left(\sqrt{x}\right)^2}\,dx = \int_0^8 \sqrt{1+x}\,dx = \int_0^8 (x+1)^{\frac{1}{2}}\,dx = \left[\frac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^8$$

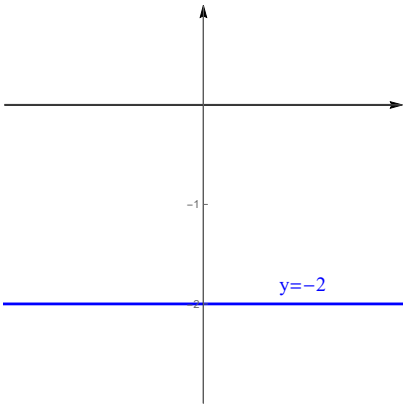
$$= \frac{2}{3}(8+1)^{\frac{3}{2}} - \frac{2}{3}(0+1)^{\frac{3}{2}} = \frac{2}{3}(9)^{\frac{3}{2}} - \frac{2}{3} = \frac{2}{3}(27-1) = \frac{52}{3}$$

(6) حول المعادلة القطبية $r = -2\csc\theta$ إلى معادلة كارتيزية وارسمها.

الحل :

$$r = -2\csc\theta \implies r = \frac{-2}{\sin\theta} \implies r\sin\theta = -2 \implies y = -2$$

المعادلة $y = -2$ تمثل خط مستقيم يوازي محور x ويمر بالنقطة $(0,-2)$.



(7) ارسم المنطقة الواقعة داخل المنحنى $r = 2+2\cos\theta$ وخارج المنحنى $r = 2$ وجد مساحتها.

الحل :

المنحنى $r = 2+2\cos\theta$ يمثل منحنى قلبي متناظر حول المحور القطبي .

المنحنى $r = 2$ يمثل دائرة مركزها القطب ، ونصف قطرها 2 .

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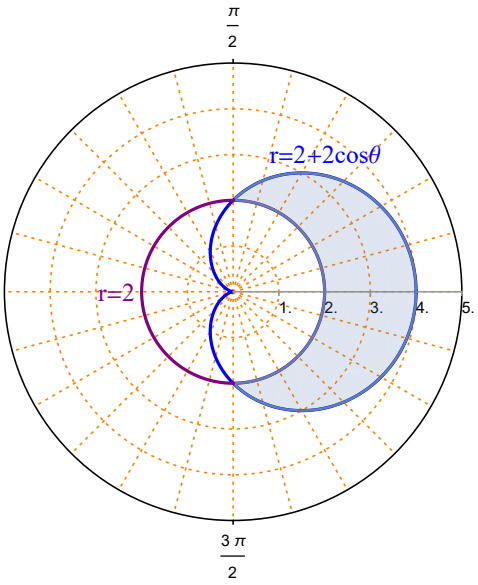

نقاط تقاطع المنحنى $r = 2 + 2\cos\theta$ مع المنحنى $r = 2$:

$$2 + 2\cos\theta = 2 \implies 2\cos\theta = 0 \implies \cos\theta = 0 \implies \theta = \frac{\pi}{2} \ , \ \theta = \frac{3\pi}{2}$$

لاحظ أن المنطقة المطلوبة متناظرة حول المحور القطبي

$$A = 2\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} \left[(2+2\cos\theta)^2 - (2)^2\right] \, d\theta\right) = \int_0^{\frac{\pi}{2}} \left(4 + 8\cos\theta + 4\cos^2 - 4\right) \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[8\cos\theta + 4\cos^2\theta\right] \, d\theta = \int_0^{\frac{\pi}{2}} \left[8\cos\theta + 4\left(\frac{1+\cos 2\theta}{2}\right)\right] \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(8\cos\theta + 2 + 2\cos 2\theta\right) \, d\theta = \left[8\sin\theta + 2\theta + \sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \left(8\sin\left(\frac{\pi}{2}\right) + 2\left(\frac{\pi}{2}\right) + \sin(\pi)\right) - \left(8\sin(0) + 2(0) + \sin(0)\right)$$

$$= (8 + \pi + 0) - (0 + 0 + 0) = 8 + \pi$$

111 ريض - حساب التكامل
الفصل الدراسي الثاني 1443 هـ
حل الاختبار الفصلي
د. طارق عبدالرحمن مُحَمَّد الفاضل

السؤال الأول (12 درجة):

(1) استخدم مجموع ريمان لحساب التكامل المحدد $\int_0^2 (4x-3)\,dx$.

الحل : $[a,b]=[0,2]$ و $f(x)=4x-3$.

$$\Delta_x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$$

$$x_k = a + k\,\Delta_x = 0 + k\left(\frac{2}{n}\right) = \frac{2k}{n}$$

$$f\left(x_k\right) = 4\left(\frac{2}{n}\right) - 3 = \frac{8k}{n} - 3$$

$$R_n = \sum_{k=1}^{n} f\left(x_k\right)\ \Delta_x = \sum_{k=1}^{n}\left(\frac{8k}{n}-3\right)\ \left(\frac{2}{n}\right) = \sum_{k=1}^{n}\left(\frac{16k}{n^2}-\frac{6}{n}\right)$$

$$\sum_{k=1}^{n}\frac{16k}{n^2} - \sum_{k=1}^{n}\frac{6}{n} = \frac{16}{n^2}\sum_{k=1}^{n}k - \frac{6}{n}\sum_{k=1}^{n}1 = \frac{16}{n^2}\,\frac{n(n+1)}{2} - \frac{6}{n}\,(n) = 8\left(\frac{n+1}{n}\right) - 6$$

$$\int_0^2 (4x-3)\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[8\left(\frac{n+1}{n}\right)-6\right] = 8(1) - 6 = 2$$

(2) جد $F'(x)$ إذا كانت $F(x) = \int_{e^{2x}}^{2^{\cos x}} \frac{1}{\sqrt{2-t^2}}\,dt$

الحل :

$$F'(x) = \frac{d}{dx}\int_{e^{2x}}^{2^{\cos x}} \frac{1}{\sqrt{2-t^2}}\,dt$$

$$= \frac{1}{\sqrt{2-\left(2^{\cos x}\right)^2}}\,2^{\cos x}\,(-\sin x)\ln 2 - \frac{1}{\sqrt{2-\left(e^{2x}\right)^2}}\,e^{2x}(2)$$

$$= \frac{-2^{\cos x}\sin x\ \ln 2}{\sqrt{2-2^{2\cos x}}} - \frac{2\,e^{2x}}{\sqrt{2-e^{4x}}}$$

احسب $\frac{dy}{dx}$ فيمايلي :

$$y = 5^{2x}\ \cosh^{-1}\left(e^{\sqrt{x}}\right) \quad (3)$$

الحل :

$$\frac{dy}{dx} = \left(5^{2x}\ (2)\ln 5\right)\ \cosh^{-1}\left(e^{\sqrt{x}}\right) + 5^{2x}\left[\frac{1}{\sqrt{\left(e^{\sqrt{x}}\right)^2 - 1}}\ e^{\sqrt{x}}\ \frac{1}{2\sqrt{x}}\right]$$

$$= 5^{2x}\ 2\ln 5\ \cosh^{-1}\left(e^{\sqrt{x}}\right) + \frac{e^{\sqrt{x}}}{2\sqrt{x}\ \sqrt{e^{2\sqrt{x}} - 1}}$$

$$y = \sin^{-1} x\ \log_6 |8 + \tan 2x| \quad (4)$$

الحل :

$$\frac{dy}{dx} = \left(\frac{1}{\sqrt{1-x^2}}\right)\log_6 |8 + \tan 2x| + \sin^{-1} x\left(\frac{\sec^2(2x)\ (2)}{8 + \tan 2x}\ \frac{1}{\ln 6}\right)$$

$$= \frac{\log_6 |8 + \tan 2x|}{\sqrt{1-x^2}} + \frac{2\sec^2(2x)\ \sin^{-1} x}{(8 + \tan 2x)\ \ln 6}$$

$$y = (\tan x)^{\sec x} \quad (5)$$

الحل :

$$y = (\tan x)^{\sec x} \implies \ln |y| = \ln \left|(\tan x)^{\sec x}\right| = \sec x\ \ln |\tan x|$$

باشتقاق الطرفين

$$\frac{y'}{y} = (\sec x \tan x)\ \ln |\tan x| + \sec x\ \left(\frac{\sec^2 x}{\tan x}\right)$$

$$y' = y\ \left[\sec x \tan x\ \ln |\tan x| + \frac{\sec^3 x}{\tan x}\right]$$

$$y' = (\tan x)^{\sec x}\ \left[\sec x \tan x\ \ln |\tan x| + \frac{\sec^3 x}{\tan x}\right]$$

$$y = \coth^{-1}\left(\frac{1}{x}\right) \quad (6)$$

الحل :

$$\frac{dy}{dx} = \frac{-1}{1 - \left(\frac{1}{x}\right)^2}\ \left(\frac{-1}{x^2}\right) = \frac{1}{x^2\left[1 - \frac{1}{x^2}\right]} = \frac{1}{x^2 - 1}$$

السؤال الثاني (18 درجة) : أحسب التكاملات التالية

(1) $\int \frac{\sec^2(\ln x)}{x}\, dx$

الحل :

$$\int \frac{\sec^2(\ln x)}{x}\, dx = \int \sec^2(\ln x)\ \left(\frac{1}{x}\right)\, dx = \tan(\ln x) + c$$

باستخدام القانون

$$\int \sec^2(f(x))\ f'(x)\, dx = \tan(f(x)) + c$$

(2) $\int \left(7 - x^2\right)^3\ x\, dx$

الحل :

$$\int \left(7 - x^2\right)^3\ x\, dx = \frac{1}{-2}\int \left(7 - x^2\right)^3\ (-2x)\, dx = \frac{1}{-2}\ \frac{(7 - x^2)^4}{4} + c$$

باستخدام القانون $+\, c$ $\int [f(x)]^n\, f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1}$ حيث $n \neq -1$.

(3) $\int_0^1 x\, 10^{2-x^2}\, dx$

الحل :

$$\int_0^1 x\, 10^{2-x^2}\, dx = \frac{1}{-2}\int_0^1 10^{2-x^2}(-2x)\, dx = \frac{1}{-2}\left[\frac{10^{2-x^2}}{\ln 10}\right]_0^1$$

$$= \frac{1}{-2}\left[\frac{10^{2-(1)^2}}{\ln 10} - \frac{10^{2-(0)^2}}{\ln 10}\right] = \frac{1}{-2}\left[\frac{10^1}{\ln 10} - \frac{10^2}{\ln 10}\right] = \frac{1}{-2}\left(\frac{-90}{\ln 10}\right) = \frac{45}{\ln 10}$$

باستخدام القانون $+\, c$ $\int a^{f(x)}\, f'(x)\, dx = \frac{a^{f(x)}}{\ln a}$.

(4) $\int \frac{x-1}{x+2}\, dx$

الحل :

$$\int \frac{x-1}{x+2}\, dx = \int \frac{(x+2)-3}{x+2}\, dx = \int \left(\frac{x+2}{x+2} - \frac{3}{x+3}\right)\, dx$$

$$= \int \left(1 - \frac{3}{x+2}\right)\, dx = \int 1\, dx - \int \frac{3}{x+2}\, dx = \int 1\, dx - 3\int \frac{1}{x+2}\, dx$$

$$= x - 3\ln|x+2| + c$$

باستخدام القانون $\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + c$.

(5) $\int \frac{e^{2x}}{\sqrt{9+e^{4x}}}\, dx$

الحل :

$$\int \frac{e^{2x}}{\sqrt{9+e^{4x}}}\, dx = \int \frac{e^{2x}}{\sqrt{(3)^2+(e^{2x})^2}}\, dx = \frac{1}{2}\int \frac{2\,e^{2x}}{\sqrt{(3)^2+(e^{2x})^2}}\, dx$$

$$= \frac{1}{2}\sinh^{-1}\left(\frac{e^{2x}}{3}\right) + c$$

باستخدام القانون

$$\int \frac{f'(x)}{\sqrt{a^2+[f(x)]^2}}\, dx = \sinh^{-1}\left(\frac{f(x)}{a}\right) + c$$

(6) $\int \frac{\tanh(\sqrt{x})}{\sqrt{x}}\, dx$

الحل :

$$\int \frac{\tanh(\sqrt{x})}{\sqrt{x}}\, dx = \int \tanh(\sqrt{x})\ \left(\frac{1}{\sqrt{x}}\right)\, dx = 2\int \tanh(\sqrt{x})\ \left(\frac{1}{2\sqrt{x}}\right)\, dx$$

$$= 2\ln\left|\cosh(\sqrt{x})\right| + c$$

باستخدام القانون

$$\int \tanh(f(x))\ f'(x)\, dx = \ln|\cosh(f(x))| + c$$

(7) $\int \frac{3x}{x^4+9}\, dx$

الحل :

$$\int \frac{3x}{x^4+9}\, dx = 3\int \frac{x}{(x^2)^2+(3)^2}\, dx = 3\ \left(\frac{1}{2}\right)\int \frac{2x}{(x^2)^2+(3)^2}\, dx$$

$$= \frac{3}{2}\,\frac{1}{3}\,\tan^{-1}\left(\frac{x^2}{3}\right) + c = \frac{1}{2}\,\tan^{-1}\left(\frac{x^2}{3}\right) + c$$

باستخدام القانون

$$\int \frac{f'(x)}{a^2 + [f(x)]^2}\,dx = \frac{1}{a}\,\tan^{-1}\left(\frac{f(x)}{a}\right) + c \;.$$

(8) $\displaystyle\int (x+1)\,\sqrt{x-1}\,dx$

الحل : بوضع $\sqrt{x-1} = u \implies x-1 = u^2 \implies x = u^2+1 \implies x+1 = u^2+2$

$$dx = 2u\,du$$

$$\int (x+1)\,\sqrt{x-1}\,dx = \int \left(u^2+2\right)\,u\,2u\,du = 2\int u^2\left(u^2+2\right)\,du$$

$$= 2\int \left(u^4+2u^2\right)\,du = 2\left(\frac{u^5}{5} + 2\,\frac{u^3}{3}\right) + c$$

$$= \frac{2}{5}\left(\sqrt{x-1}\right)^5 + \frac{4}{3}\left(\sqrt{x-1}\right)^3 + c = \frac{2}{5}\,(x-1)^{\frac{5}{2}} + \frac{4}{3}\,(x-1)^{\frac{3}{2}} + c$$

(9) $\displaystyle\int e^{-2x}\cosh x\,dx$

الحل :

$$\int e^{-2x}\cosh x\,dx = \int e^{-2x}\left(\frac{e^x + e^{-x}}{2}\right)\,dx = \int \left(\frac{e^{-x} + e^{-3x}}{2}\right)\,dx$$

$$= \int \left(\frac{e^{-x}}{2} + \frac{e^{-3x}}{2}\right)\,dx = \frac{1}{2}\,(-1)\int e^{-x}\,(-1)\,dx + \frac{1}{2}\,\frac{1}{-3}\int e^{-3x}(-3)\,dx$$

$$= \frac{-1}{2}\,e^{-x} - \frac{1}{6}\,e^{-3x} + c = -\frac{e^{-x}}{2} - \frac{e^{-3x}}{6} + c$$

باستخدام القانون $\displaystyle\int e^{f(x)}\,f'(x)\,dx = e^{f(x)} + c$