

Time Response, Stability, and Steady State Error

1 Time Response

- The mathematical representation of a system (*Transfer function or State space*) is used to analyze its transient and steady-state responses to see if these characteristics yield the desired behavior.
- Performance of controlled systems can be tested and compared by their responses to certain test signals (*Step* functions, *impulse* functions, *ramp* functions, *sinusoidal* functions, etc.).
- A response of a dynamic system can be analyzed in two parts:
 - *Steady-state response*: The behavior of the output as $t \rightarrow \infty$
 - *Transient response*: The behavior of the output as it goes from an initial state to a final state.
- This chapter is devoted to the analysis of *system transient response*.

Time Response

Poles, Zeros, and System Response

- The output response of a system is the sum of two responses:
 - the forced response* (steady-state response or particular solution),
 - the natural response* (the homogeneous solution).

Output response = forced response (e.g. constant) + natural response (e.g. exponential)

- Poles** of a Transfer Function (TF): The values of s that cause $TF \rightarrow \infty$
- Zeros** of a TF: the values of s that cause $TF = 0$.

roots of the denominator (characteristic polynomial) of the transfer function

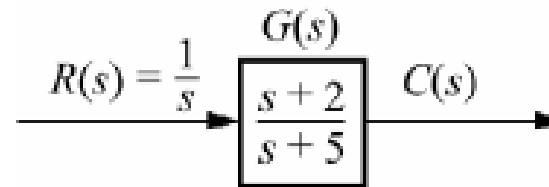
roots of the numerator of the transfer function

Example: Poles and Zeros of a First-Order System

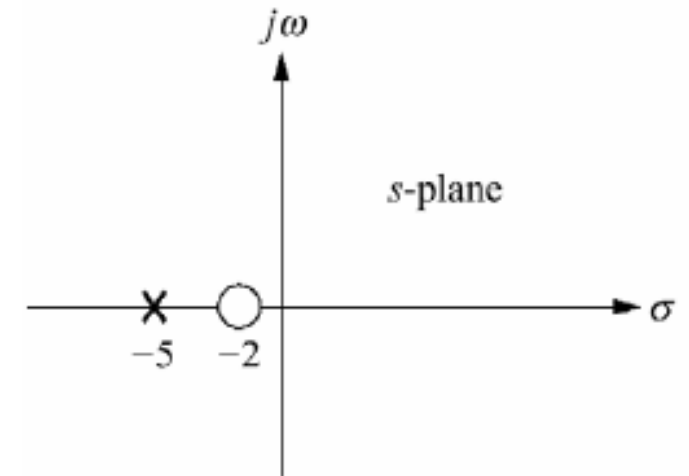
$$C(s) = \frac{(s+2)}{s(s+5)} = \frac{A}{s} + \frac{B}{s+5} = \frac{2/5}{s} + \frac{3/5}{s+5}$$

System Output (unit step response)

where $A = \left. \frac{(s+2)}{(s+5)} \right|_{s=0} = \frac{2}{5}$ and $B = \left. \frac{(s+2)}{s} \right|_{s=-5} = \frac{3}{5}$



System showing input and output



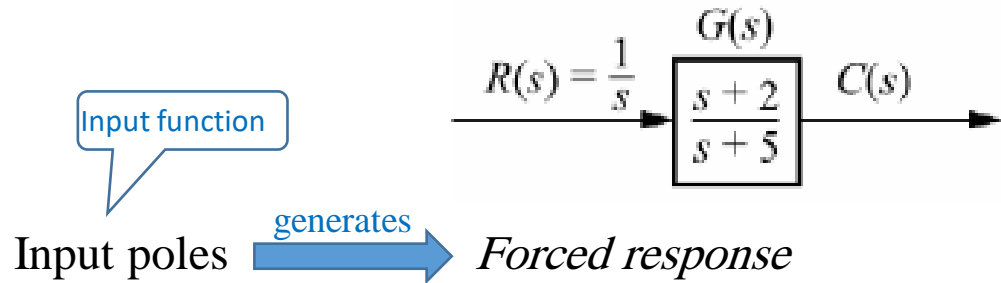
pole-zero plot of the system

Inverse Laplace transform:

$$c(t) = \frac{2}{5} + \frac{3}{5} e^{-5t}$$

System Output in time domain (time response)

Example: Poles and Zeros of First Order System

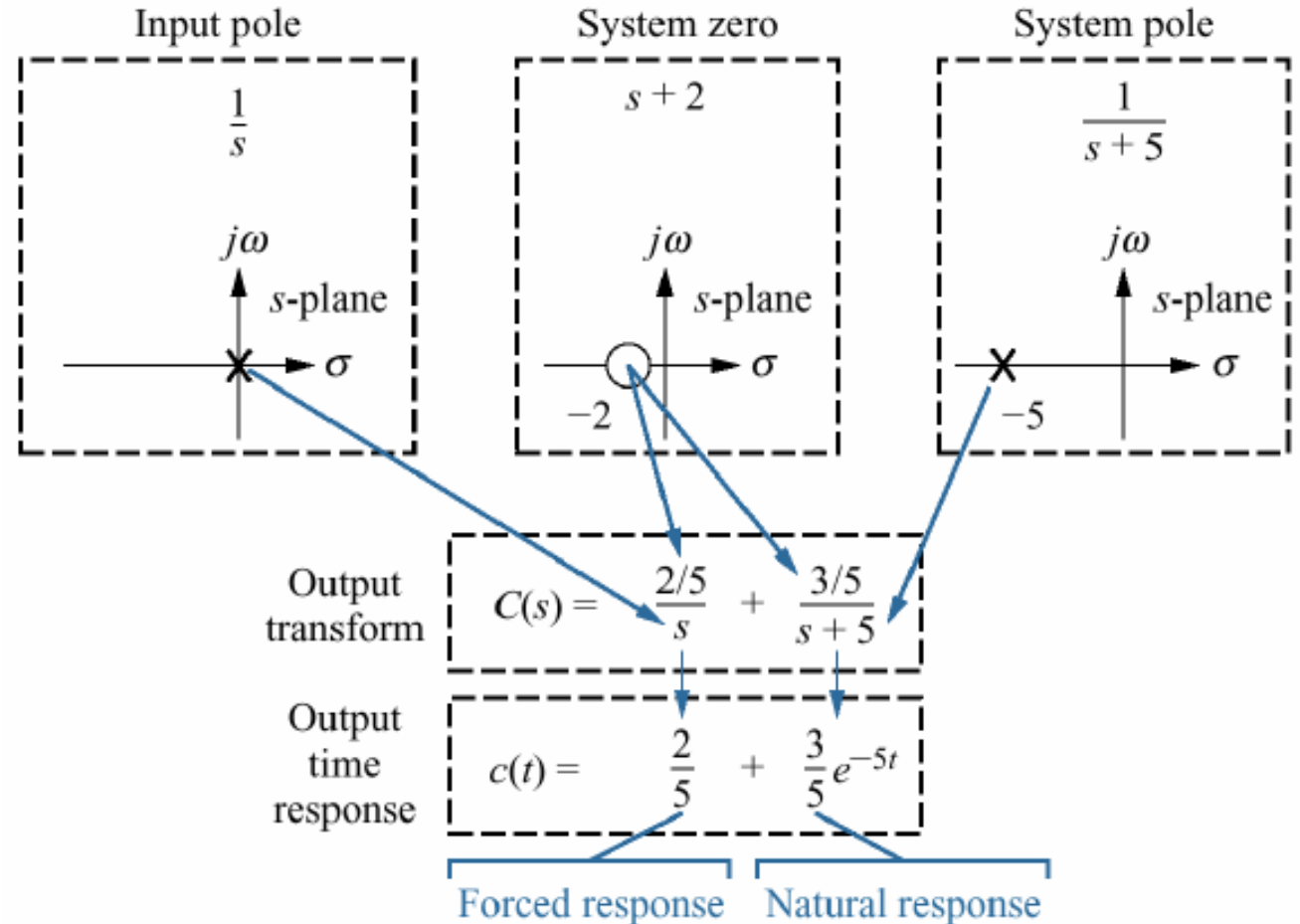


(pole at the origin generated a step function at the output)



Transfer function

(A pole α on the real axis generates an exponential response of the form $e^{-\alpha t}$ that will decay to zero).

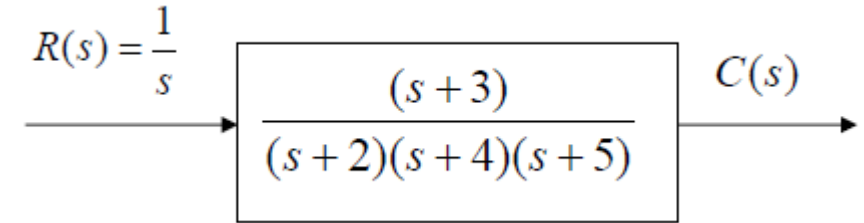


Evolution of a system response.

Evaluating Response using Poles

Problem:

Given the following system, write the output, $c(t)$, in general terms. Specify the forced and natural parts of the solution.



Solution:

$$C(s) = \underbrace{\frac{K_1}{s}}_{\text{Forced response}} + \underbrace{\frac{K_2}{(s+2)} + \frac{K_3}{(s+4)} + \frac{K_4}{(s+5)}}_{\text{Natural response}}$$

Taking inverse Laplace transform,

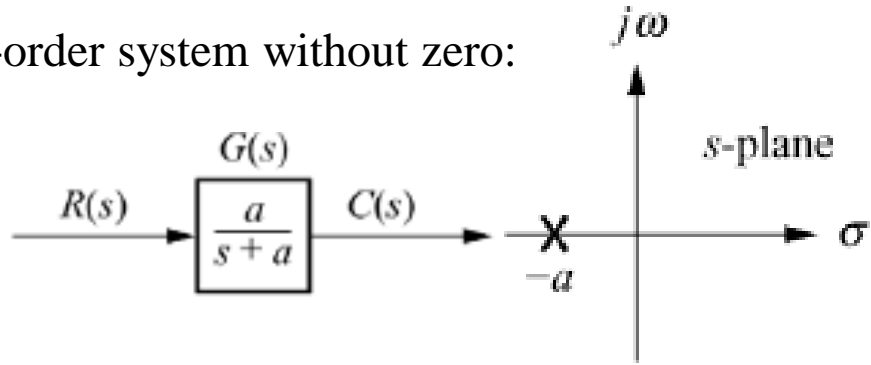
- Each system pole generates an exponential as part of the natural response.
- The input's pole generates the forced response.

$$c(t) = \underbrace{K_1}_{\text{Forced response}} + \underbrace{K_2 e^{-2t} + K_3 e^{-4t} + K_4 e^{-5t}}_{\text{Natural response}}$$

Poles of the system produce the Natural response that Will decay to zero

First-Order System: Time Constant

- A first-order system without zero:



- If the input is a unit step: $R(s) = \frac{1}{s}$ then the Laplace transform of the step response is :

$$C(s) = R(s)G(s) = \frac{a}{s(s+a)}$$

the input pole at the origin generated the forced response

- Taking the inverse transform

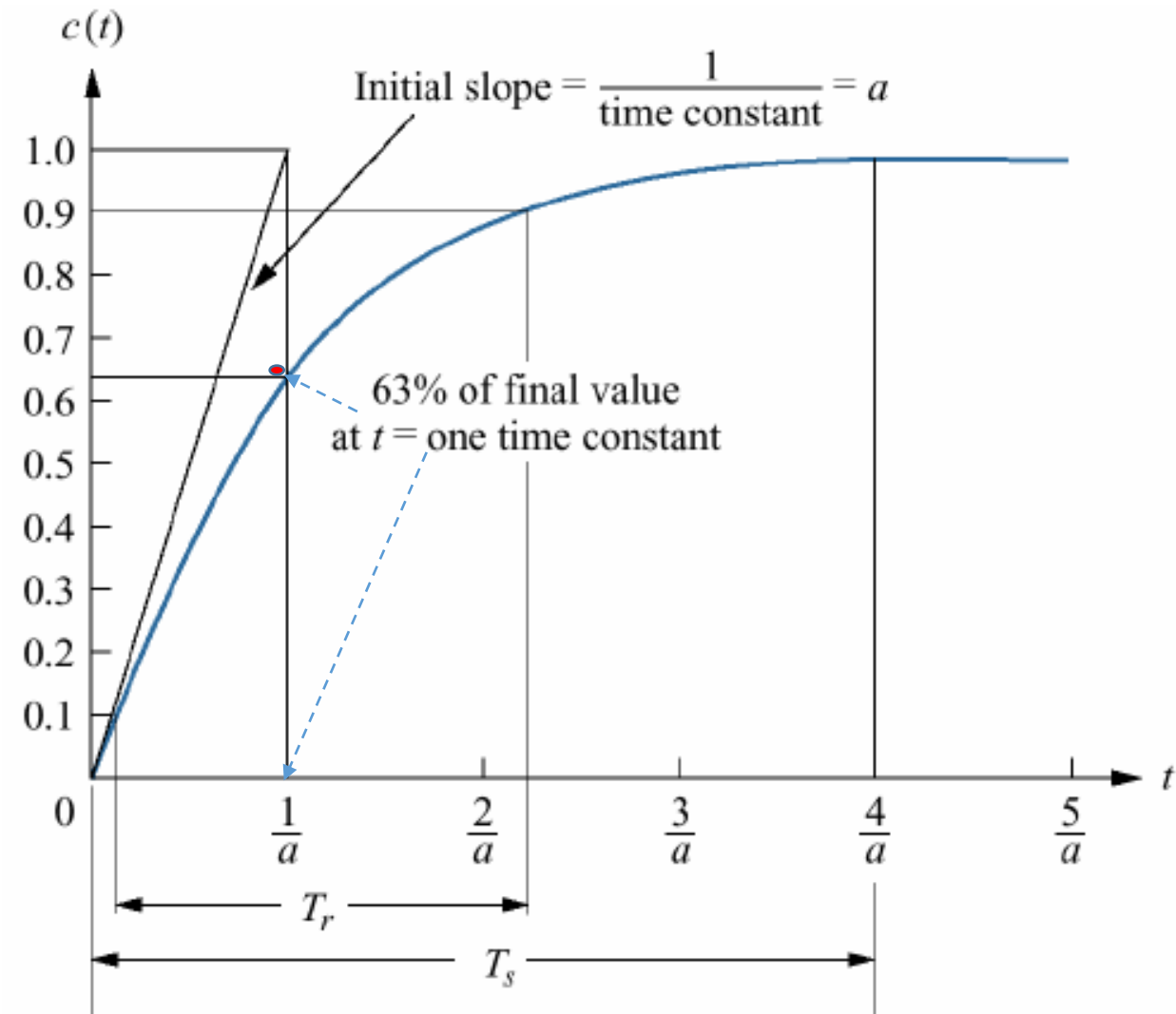
$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$$

The system pole at $-a$ generated the natural response

- Significance of parameter a (*system pole*) (only parameter needed to describe the transient response),

When $t = 1/a$, $e^{-at} \Big|_{t=1/a} = e^{-1} = 0.37$

Hence, $c(t) \Big|_{t=1/a} = 1 - 0.37 = 0.63$



Some Terminology

(three transient response performance specifications).

1. Time constant T_c : Time it takes for the step response to rise to 63% of its final value.

$$T_c = \frac{1}{a}$$

- we can call the parameter a (*system pole*) the *exponential frequency* (The reciprocal of the time constant)
- T_c is related to the speed at which the system responds to a step input.

2. Rise Time T_r : Rise time is defined as the time for the response to go from *0.1* to *0.9* of its final value.

found by solving for the difference in time at $c(t_2) = 0.9$ and $c(t_1) = 0.1$

$$\begin{aligned} C(t_2) = 0.9 = 1 - e^{-at_2} &\Rightarrow t_2 = -\frac{\ln(0.1)}{a} = \frac{2.31}{a} \\ C(t_1) = 0.1 = 1 - e^{-at_1} &\Rightarrow t_1 = -\frac{\ln(0.9)}{a} = \frac{0.11}{a} \end{aligned} \Rightarrow T_r = t_2 - t_1 = \frac{2.13}{a} - \frac{0.11}{a} = \frac{2.2}{a}$$

time for $C(t) = 0.1$
time for $C(t) = 0.9$
Rise time: $T_r = \frac{2.2}{a}$

3. Settling time T_s : The time for the response to reach, and stay within, 2% of its final value.

Letting $C(T_s) = 0.98$ and solving $C(t)$ for time, $t = T_s$, we find the settling time to be

$$C(T_s) = 0.98 = 1 - e^{-aT_s} \Rightarrow T_s = -\frac{\ln(0.98)}{a} = \frac{4}{a}$$
$$T_s = \frac{4}{a}$$

First-Order Transfer Function via Testing

- With a step input, we can measure the time constant and the steady-state value, from which the transfer function can be calculated.

- A simple first order system has : $G(s) = K / (s+a)$,
and step response is: $C(s) = \frac{K}{s(s+a)} = \frac{K/a}{s} - \frac{K/a}{(s+a)}$

In the time domain (ILT): $c(t) = \frac{K}{a} - \frac{K}{a}e^{-at}$ (eq. 1)

- From *the response*, we identify K and a to obtain the transfer function.

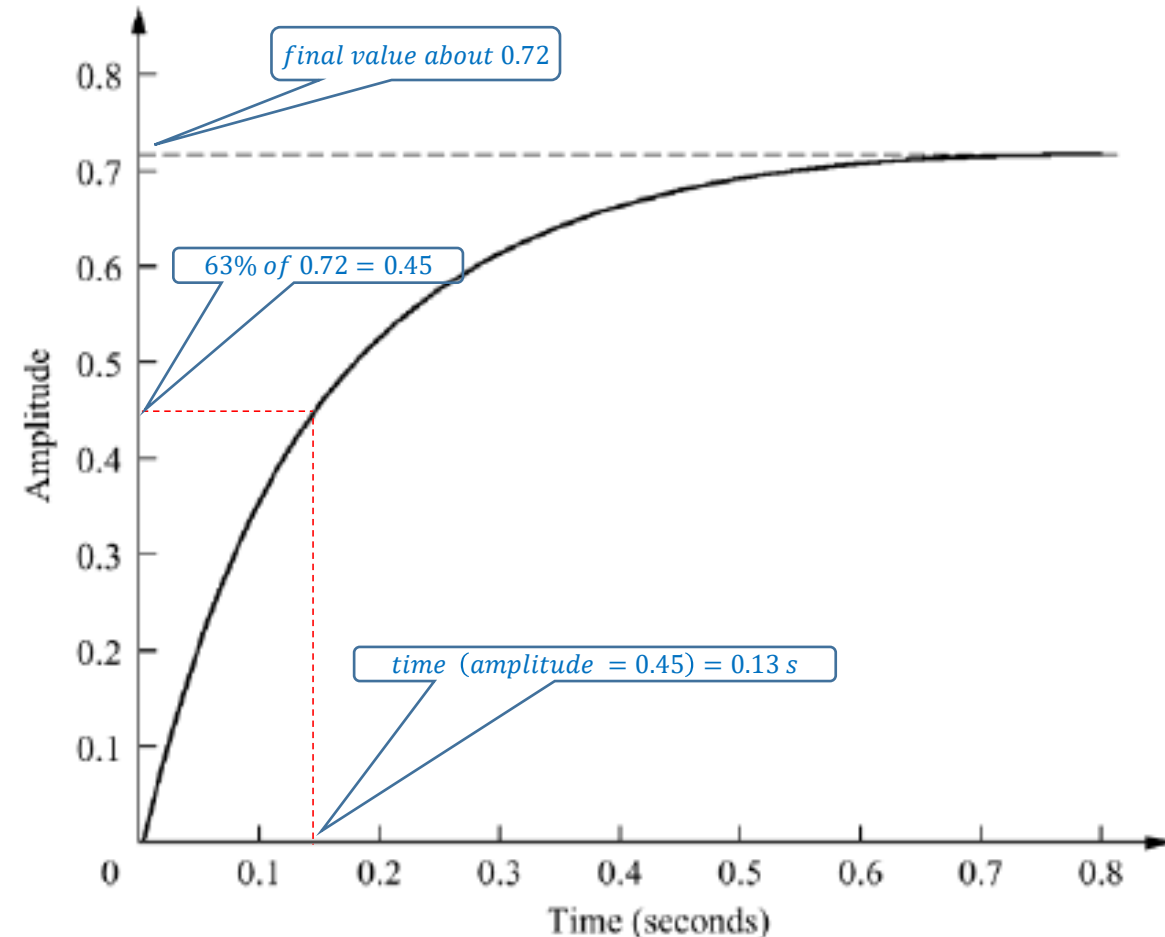
To find a :

Time constant = Time(0.63 × 0.72) 63% final value about 0.72

= Time(0.45) = 0.13 second from the graph

$a = 1/0.13 = 7.7$

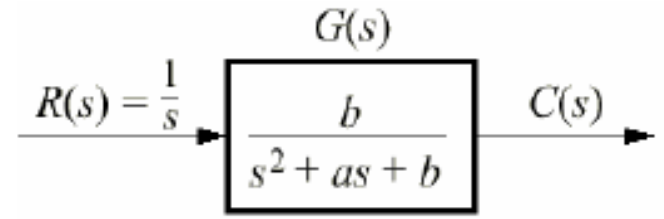
To find K : From (eq. 1), the forced response reaches a steady-state value of $K/a = 0.72$ → $K = 5.54$



$K = 5.54$ and $a = 7.7$ and $G(s) = \frac{K}{s+a}$ Transfer function, → $G(s) = \frac{5.54}{(s+7.7)}$

Second-Order System₁

- Parameters of First-order system determine the *speed* of the system.
- Parameters of Second-order system determine the *form* (*shape*) of the system.



General case of second order system

Consider the general system, $G(s) = \frac{b}{s^2 + as + b}$ $\xrightarrow{\text{System poles}}$ $s_{1,2} = -\frac{a}{2} \mp \frac{\sqrt{a^2 - 4b}}{2}$

- For *un-damped* (without damping) system, $a = 0$, and the poles are on $j\omega$ - axis at $\mp j\sqrt{b}$,

Natural Frequency



$$\therefore \omega_n = \sqrt{b},$$

Hence $b = \omega_n^2$

$\xrightarrow{\text{un-damped system}}$ $G(s) = \frac{b}{s^2 + b}$

- For an *under-damped* system, poles have *real part* $\sigma = -a/2$ (exponential decay),

Damping Ratio $\xrightarrow{\text{Exponential decay frequency}}$ $\xi = \frac{|\sigma|}{\omega_n} = \frac{a/2}{\omega_n} \Rightarrow a = 2\xi\omega_n$ $\left(\xi = \frac{\text{exponential decay}}{\text{natural frequency}} \right)$

ω_n : (Natural Frequency) the frequency of oscillation of the system without damping.

ξ : (Damping Ratio) dimensionless measure describing how oscillations in a system decay.

σ : (Attenuation) exponential decay, real part of the pole

$$G(s) = \frac{b}{s^2 + as + b}$$

$$b = \omega_n^2$$

$$a = 2\xi\omega_n$$

$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Poles



$$s_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

Canonical form

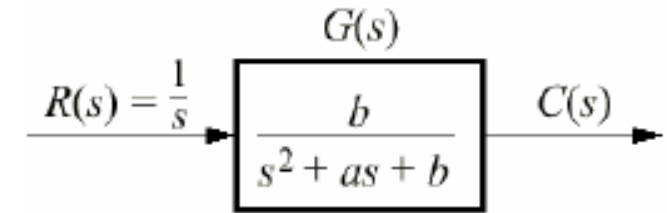
(two finite poles and no zeros)

Second-Order System₂

The sign of the discriminant of the denominator polynomial depends on the damping ratio ξ , three cases.

System poles $s_{1,2} = -\frac{a}{2} \mp \frac{\sqrt{a^2 - 4b}}{2}$ $a = 2 \xi \omega_n$ $b = \omega_n^2$

$s_{1,2} = -\xi \omega_n \pm \omega_n \sqrt{\xi^2 - 1}$



the general case
(two finite poles and no zeros)

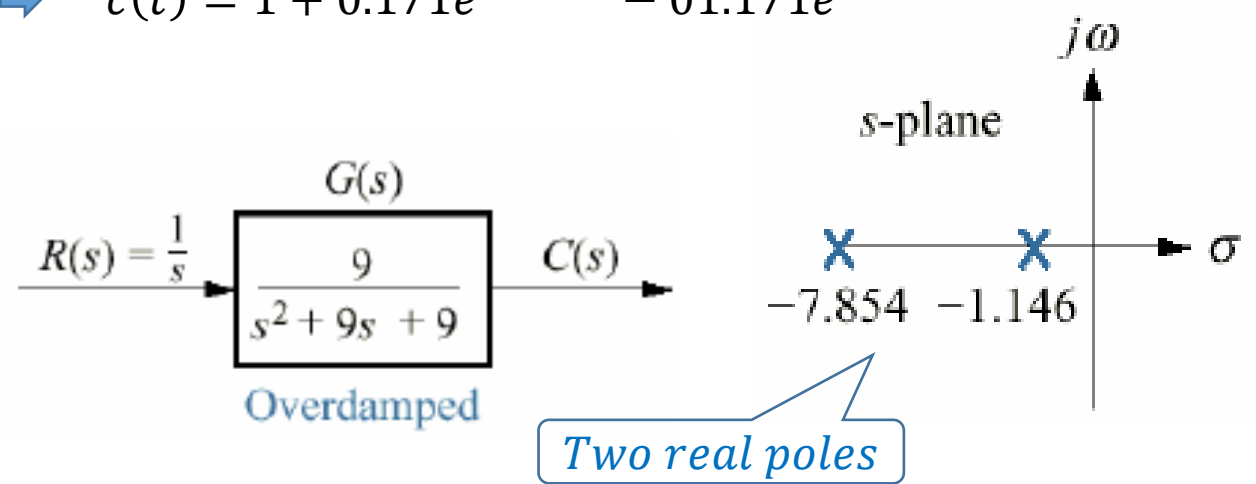
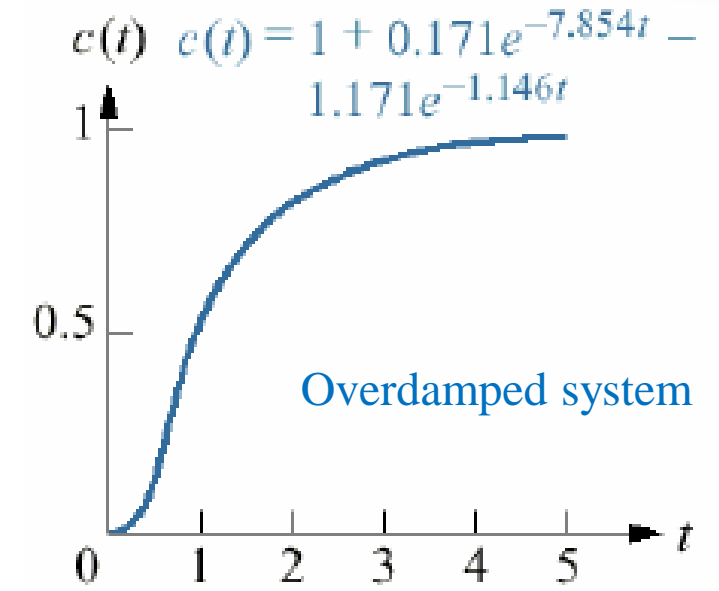
Case1: Overdamped system: ($\xi > 1$) Two real poles

$a = 9 \text{ and } b = 9$

$$C(s) = \frac{9}{s(s^2 + 9s + 9)} = \frac{9}{s(s + 7.854)(s + 1.146)}$$

→ $C(s) = \frac{1}{s} + \frac{0.171}{s + 7.854} - \frac{1.171}{s + 1.146}$

→ $c(t) = 1 + 0.171e^{-7.854t} - 0.171e^{-1.146t}$

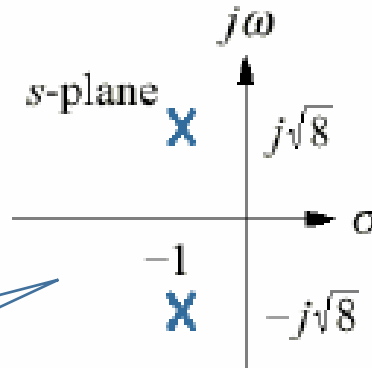
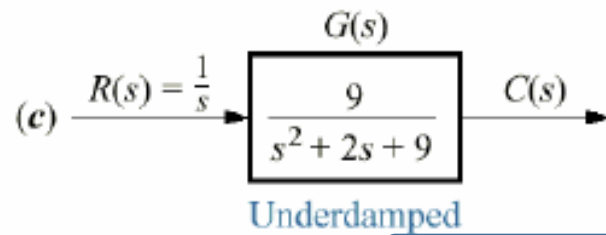


Second-Order System

Case 2: Under-damped Response ($0 < \xi < 1$): (*Two complex poles that come from the system*).

$$C(s) = \frac{9}{s(s^2 + 2s + 9)} = \frac{9}{s(s + [1 + j\sqrt{8}])(s + [1 - j\sqrt{8}])}$$

$a = 2$ and $b = 9$



Two complex poles

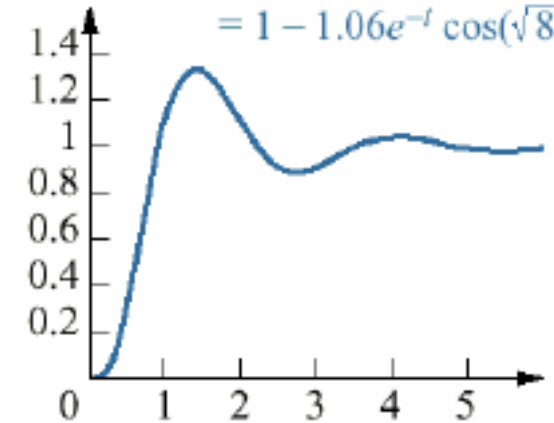
Poles from the system: $s = -1 \pm j\sqrt{8}$

exponential decay

frequency of the sinusoidal oscillation.

$$c(t) = 1 - e^{-t}(\cos\sqrt{8}t + \frac{\sqrt{8}}{8} \sin\sqrt{8}t)$$

$$= 1 - 1.06e^{-t} \cos(\sqrt{8}t - 19.47^\circ)$$



From, $K_4 \cos(\alpha + \beta) = K_4 \cos\alpha \cos\beta - K_4 \sin\alpha \sin\beta$
 $= K_2 \cos(\sqrt{8}t) + K_3 \sin(\sqrt{8}t)$

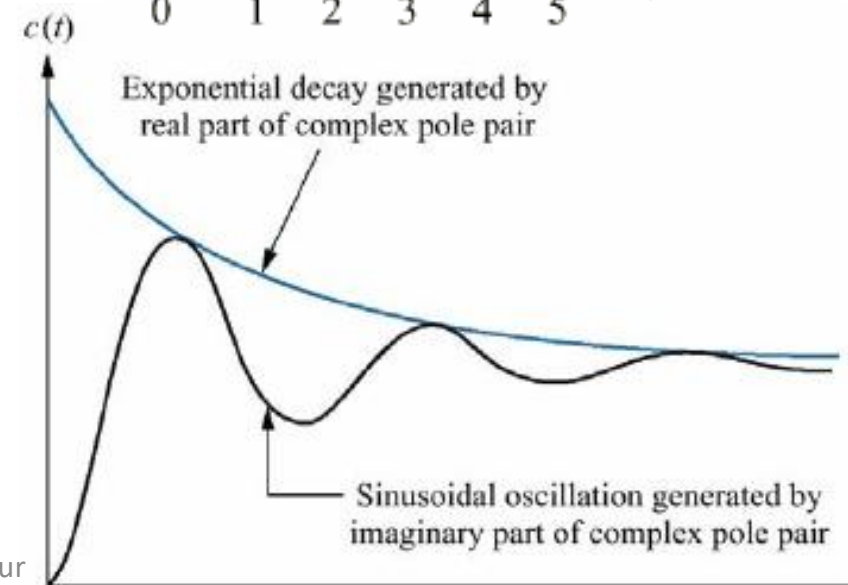
$$c(t) = K_1 + e^{-t}(K_2 \cos\sqrt{8}t + K_3 \sin\sqrt{8}t)$$

$$= K_1 + K_4 e^{-t}(\cos\sqrt{8}t - \varphi)$$

Where,

$$\varphi = \tan^{-1} \frac{K_3}{K_2}, \text{ and } K_4 = \sqrt{K_2^2 + K_3^2}$$

Second-order step response components generated by complex poles



Second-Order System

Case 3: Un-damped Response ($\xi = 0$) : pole at the origin that comes from the input and *two imaginary poles* that come from the system.

$$C(s) = \frac{9}{s(s^2 + 9)}$$

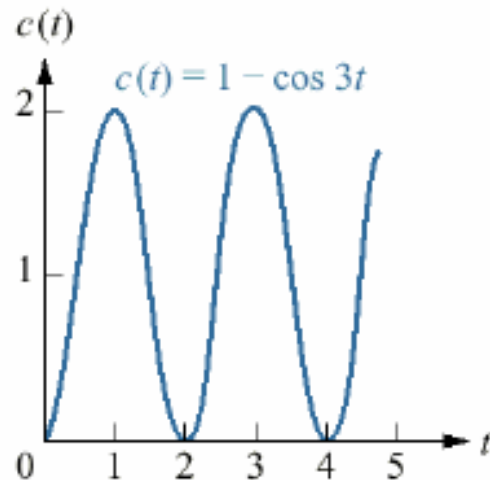
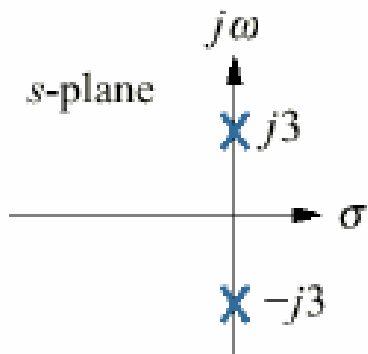
$a = 0$ and $b = 9$

two system poles on the imaginary axis

$$s = \pm j3$$

$$c(t) = K_1 + K_4 \cos(3t - \varphi)$$

There is no exponential term, so no decay.



Case 4: Critically Damped Response ($\xi = 1$) : pole at the origin that comes from the input and two multiple real poles that come from the system

$$C(s) = \frac{9}{s(s^2 + 6s + 9)} = \frac{9}{s(s + 3)^2}$$

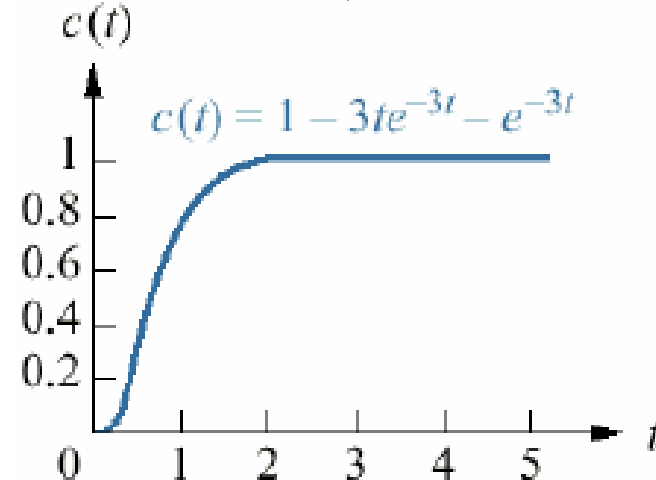
$a = 6$ and $b = 9$

input pole

system poles

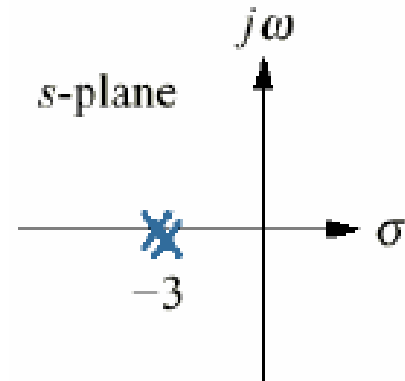
$$c(t) = K_1 + K_2 t e^{-3t} + K_3 e^{-3t}$$

There is no sinusoidal term, so no oscillation.



two poles (double) on the real axis at -3

$$s = -3, -3$$



Second-Order System

All Together

Over-damped responses

Two real poles at $-\sigma_1, -\sigma_2$
 $c(t) = K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$

$$\xi > 1$$

Under-damped responses

Two complex poles at $-\sigma_d \mp j\omega_d$
 $c(t) = A e^{-\sigma_d t} \cos(\omega_d t - \phi)$

$$0 < \xi < 1$$

Un-damped responses

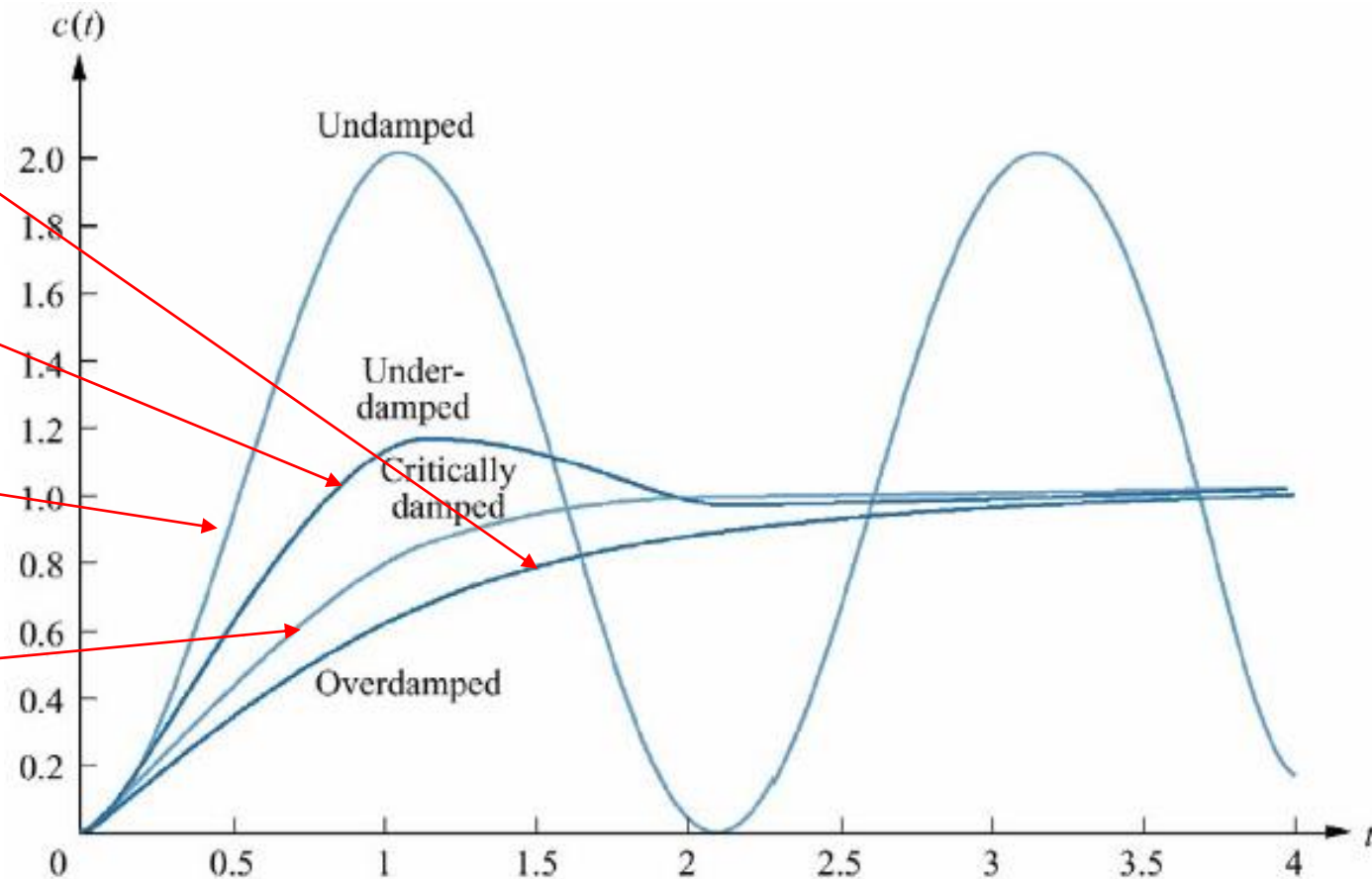
Two imaginary poles at $\mp j\omega_1$
 $c(t) = A \cos(\omega_1 t - \phi)$

$$\xi = 0$$

Critically damped responses

Two real poles at $-\sigma_1$
 $c(t) = K_1 e^{-\sigma_1 t} + K_2 t e^{-\sigma_1 t}$

$$\xi = 1$$



Step responses for second-order system damping cases

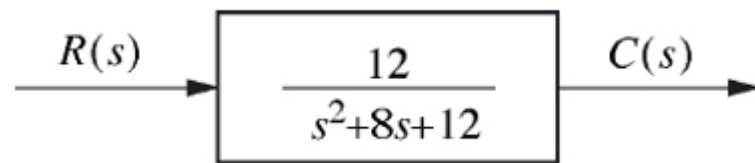
Second-Order System As a Function of Damping Ratio

- Relationship between the quantities ω_n and ξ and the pole location.

Solving for the poles of the transfer function

Example

For the system find the value of ξ and report the kind of response expected.

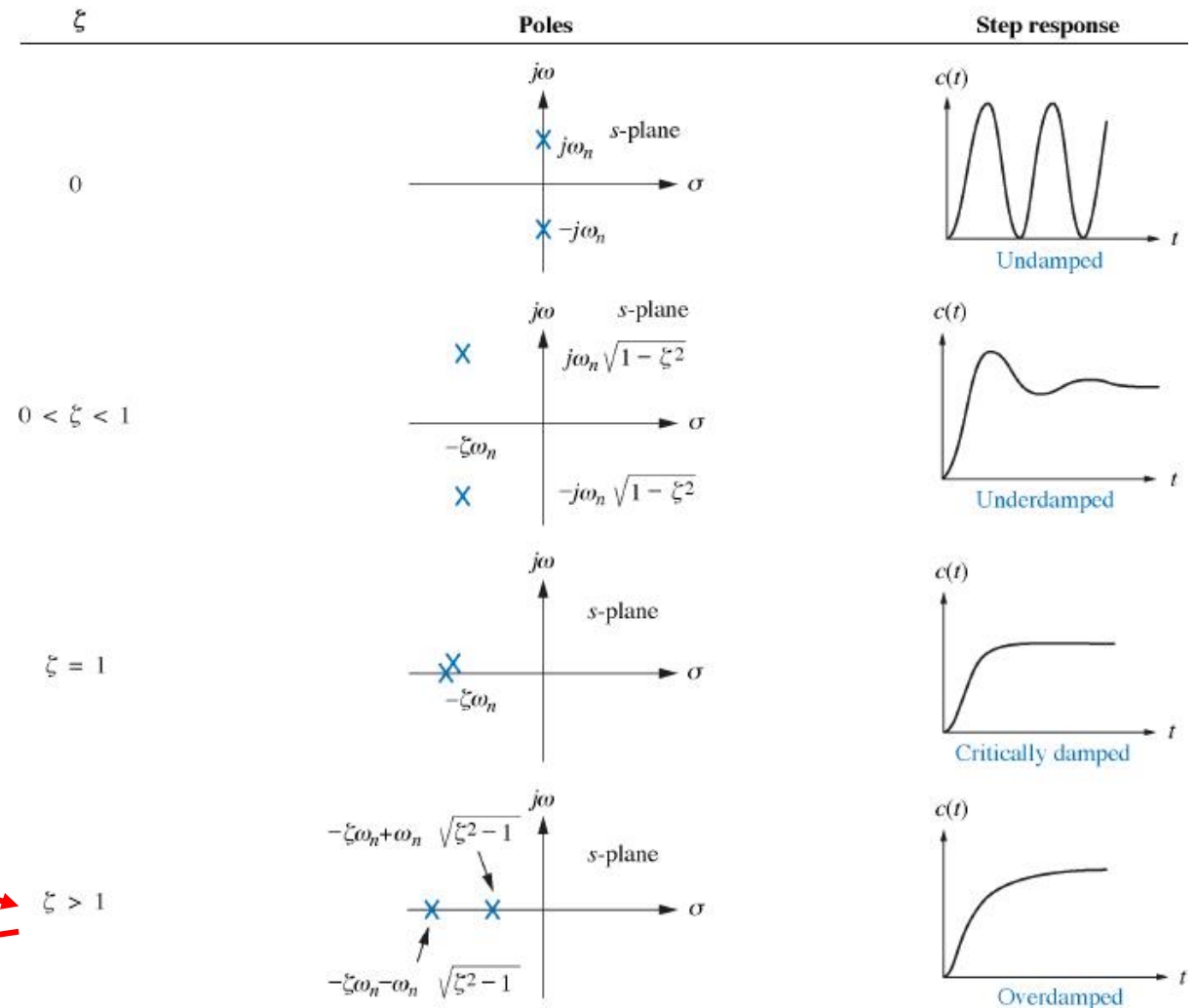


We have
$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{12}{s^2 + 8s + 12}$$

$\omega_n^2 = 12 \Rightarrow \omega_n = \sqrt{12}$

and $2\xi\omega_n = 8 \Rightarrow \xi = \frac{8}{2\sqrt{12}} = \frac{2}{\sqrt{3}} > 1$

System is over-damped.



Underdamped Second-Order Systems

- The nature of the response obtained is related to the value of the damping ratio ξ (over-damped, critically damped, underdamped, and un-damped responses.).
- Step response for the general second-order system,

$$C(s) = R(s)G(s) = \frac{1}{s} \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \rightarrow C(s) = \frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Expanding by partial fractions, ($\xi < 1$ the underdamped case)

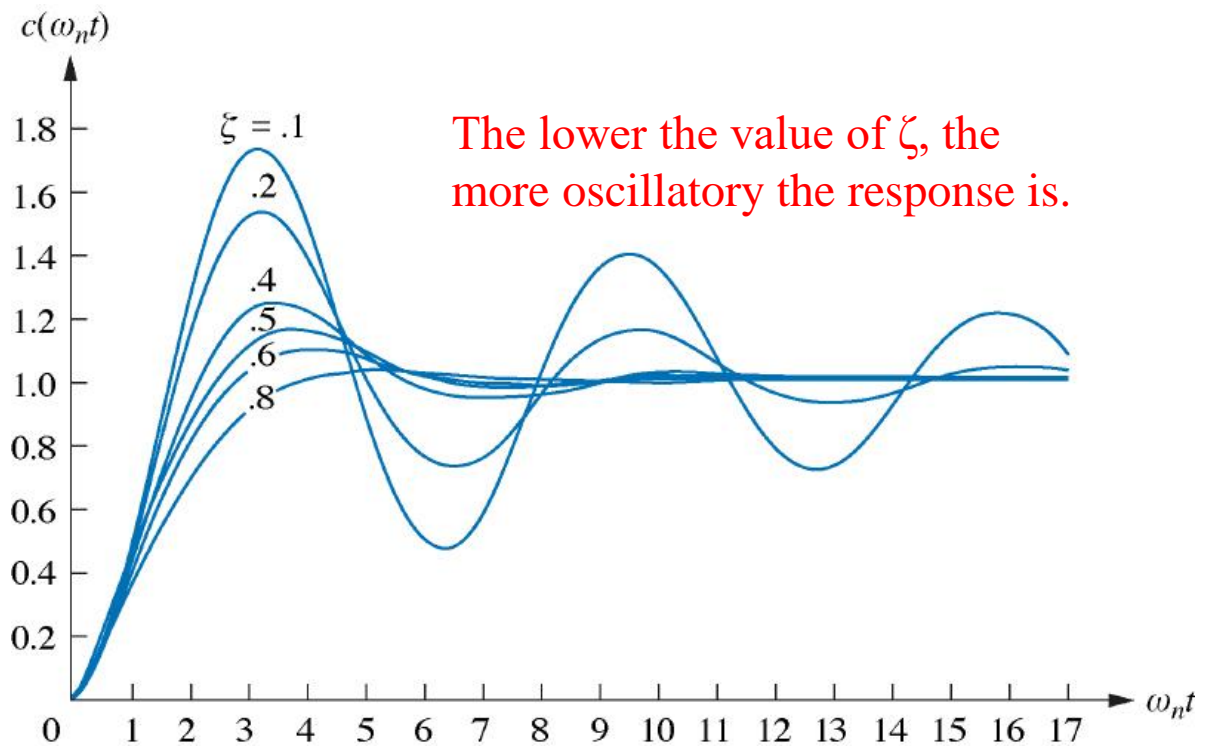
$$C(s) = \frac{1}{s} - \frac{(s + \zeta\omega_n) + \frac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s + \zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}$$

inverse
Laplace
transform

$$\rightarrow c(t) = 1 - e^{-\xi\omega_n t} \left(\cos \omega_n \sqrt{1-\xi^2} t + \frac{\xi}{\sqrt{1-\xi^2}} \sin \omega_n \sqrt{1-\xi^2} t \right)$$

$$= 1 - \frac{1}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \cos(\omega_n \sqrt{1-\xi^2} t - \varphi)$$

Where, $\varphi = \tan^{-1} \left(\frac{\xi}{\sqrt{1-\xi^2}} \right)$



Underdamped Second-Order Systems

Specifications

- Other parameters associated with the underdamped response are *rise time*, *peak time*, *percent overshoot*, and *settling time*.

Rise time
 T_r

The time required for the waveform to go from 0.1 of the final value to 0.9 of the final value.

Peak time
 T_p

The time required to reach the first, or maximum, peak.

$$T_p = \frac{\pi}{\omega_n \sqrt{1-\xi^2}}$$

percent overshoot
 $\%OS$

The amount that the waveform overshoots the steady-state, or final, value at the peak time, expressed as a percentage of the steady-state value.

$$\%OS = \frac{c_{\max} - c_{\text{final}}}{c_{\text{final}}} \times 100 \quad \left| \quad c_{\max} = c(T_p) = 1 + e^{-\left(\frac{\xi\pi}{\sqrt{1-\xi^2}}\right)} \right.$$

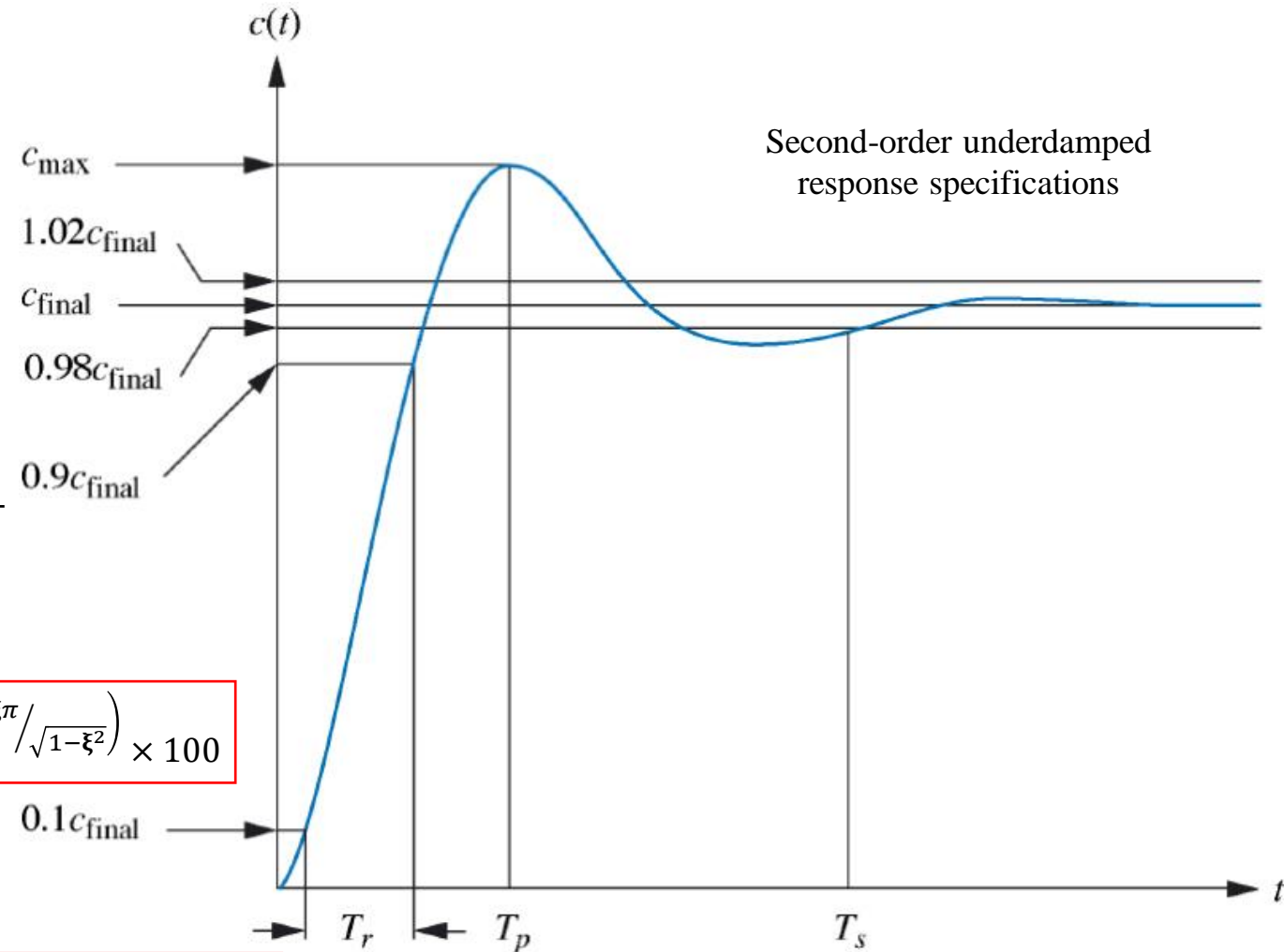
and $c_{\text{final}} = 1$ → $\%OS = e^{-\left(\frac{\xi\pi}{\sqrt{1-\xi^2}}\right)} \times 100$

settling time
 T_s

The time required for the transient's damped oscillations to reach and stay within $\pm 2\%$ of the steady-state value.

$$T_s = \frac{-\ln(0.02\sqrt{1-\xi^2})}{\xi\omega_n} \approx \frac{4}{\xi\omega_n}$$

Derivation: *self study*.



Under-damped Second-Order Systems

Specifications (continued)

from the Pythagorean theorem

$$\left. \begin{array}{l} \omega_d^2 + \sigma_d^2 = \omega_n^2 \\ \sigma_d = \xi \omega_n \end{array} \right\} \Rightarrow \omega_d = \sqrt{\omega_n^2 - \sigma_d^2} = \sqrt{\omega_n^2 - \xi^2 \omega_n^2} \rightarrow \omega_d = \omega_n \sqrt{1 - \xi^2}$$

Damped frequency of oscillation

Damping Ratio

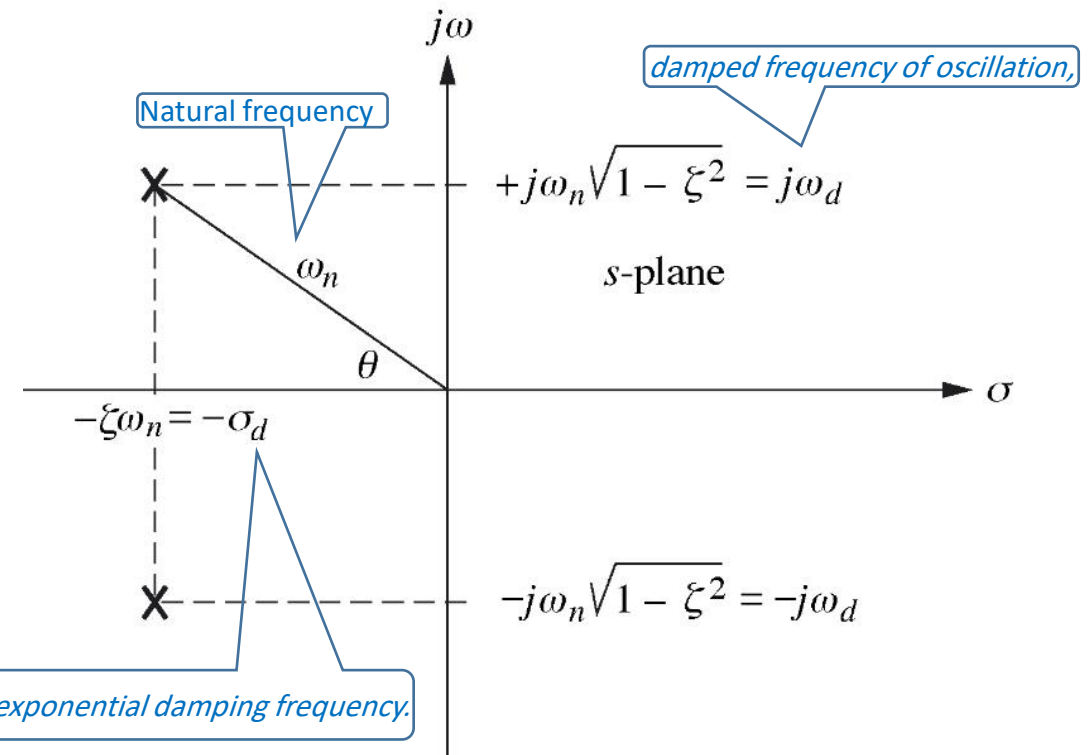
$$\cos(\theta) = \frac{\sigma_d}{\omega_n} = \frac{\xi \omega_n}{\omega_n} = \xi$$

inversely proportional to the imaginary part of the pole.

$$T_P = \frac{\pi}{\omega_n \sqrt{1 - \xi^2}} = \frac{\pi}{\omega_d}$$

inversely proportional to the real part of the pole.

$$T_s = \frac{-\ln(0.02 \sqrt{1 - \xi^2})}{\xi \omega_n} \approx \frac{4}{\xi \omega_n} \Rightarrow T_s = \frac{4}{\xi \omega_n} = \frac{4}{\sigma_d}$$



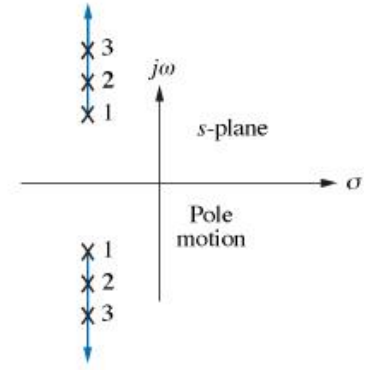
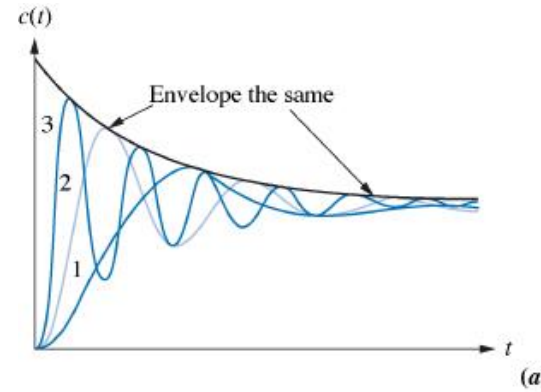
Under-damped Second-Order Systems

Step Response as Pole moves

poles move in a vertical direction (with constant real part)



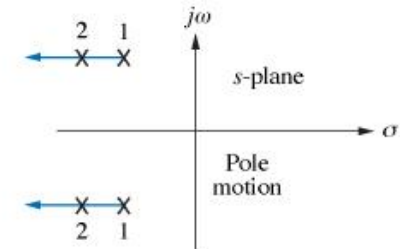
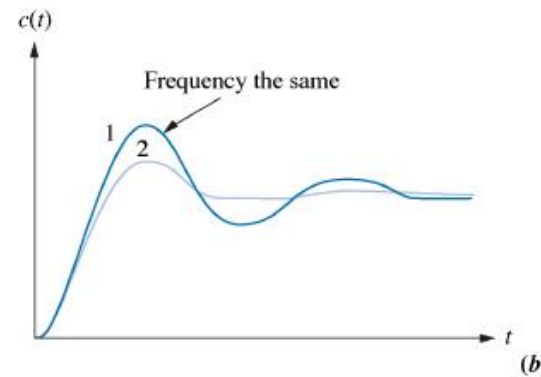
- frequency increases
- envelope remains the same (constant real part)
- settling time is virtually the same
- overshoot increases, the rise time decreases



poles move in a horizontal direction (with constant imaginary part)



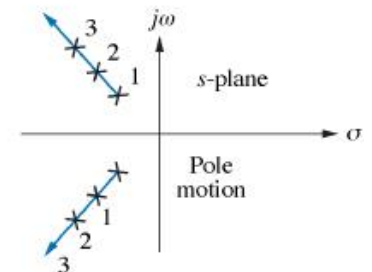
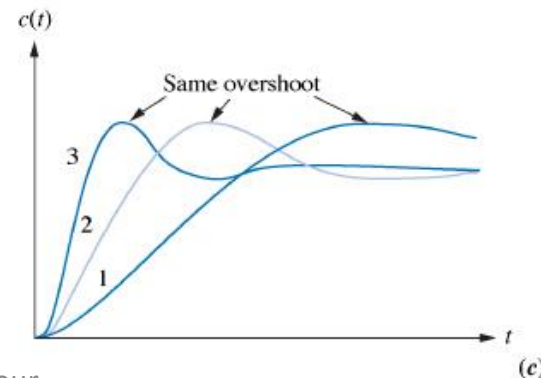
- As the poles move to the left, response damps out more rapidly.
- peak time is the same for all waveforms (constant imaginary part)



poles move in along a constant radial line direction



- The percent overshoot remains the same.
- The farther the poles are from the origin, the more rapid the response.



Finding TP, %OS, and TS From Pole Location

Problem: Given the pole plot find ξ , ω_n , T_p , %OS, and T_s .

Solution: _____

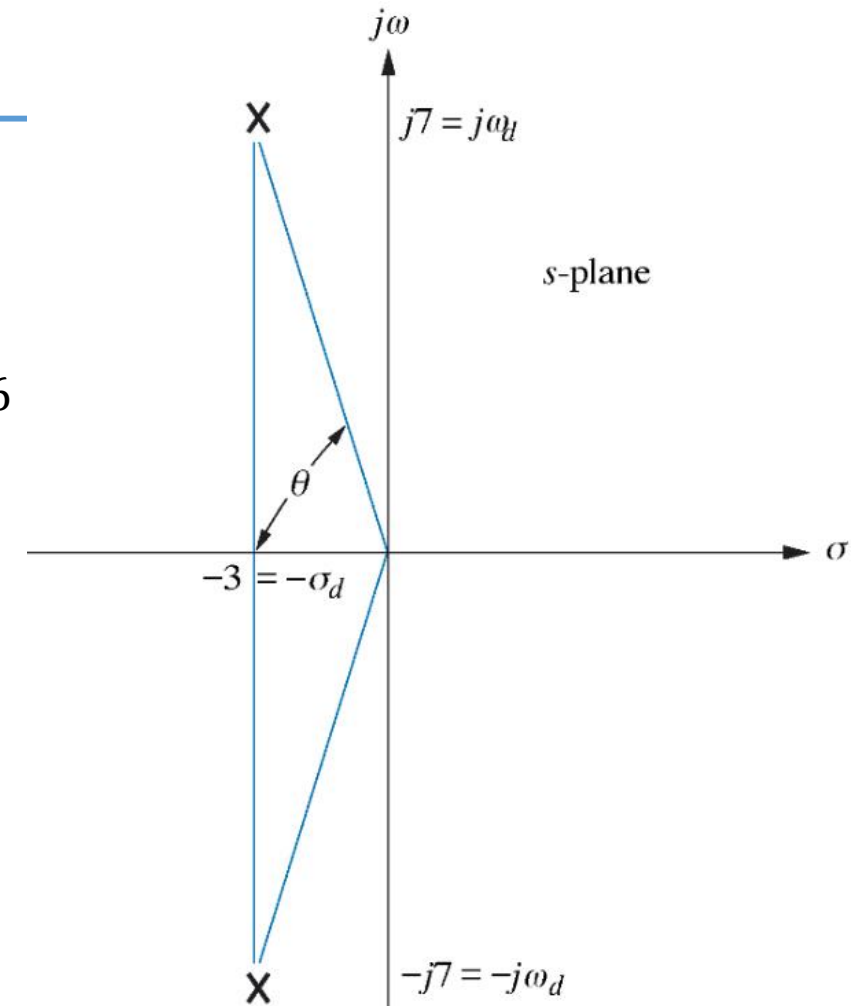
Damping ratio, $\zeta = \cos(\theta) = \cos[\arctan(7 / 3)] = 0.394$.

Natural frequency, $\omega_n^2 = \omega_d^2 + \sigma_d^2 \Rightarrow \omega_n = \sqrt{\omega_d^2 + \sigma_d^2} = \sqrt{7^2 + 3^2} = 7.616$

Peak time, $T_p = \frac{\pi}{\omega_d} = \frac{\pi}{7} = 0.449$ second

Percent overshoot, $\%OS = e^{-\left(\frac{\xi\pi}{\sqrt{1-\xi^2}}\right)} \times 100 = 26\%$

The approximate settling time, $T_s = \frac{4}{\sigma_d} = \frac{4}{3} = 1.333$ second



System Response with Additional Poles

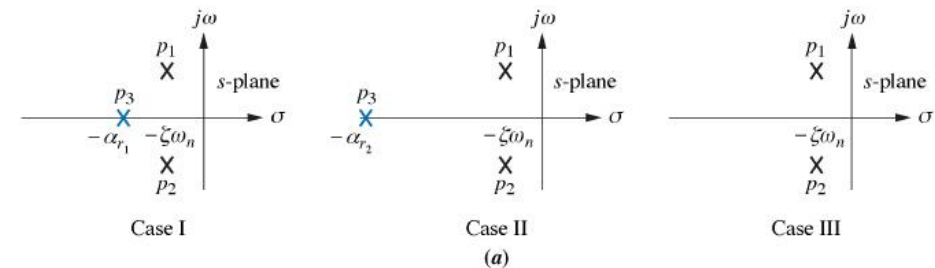
- If a system has more than two poles or has zeros, we cannot use the formulas to calculate the performance specifications that we derived.
- We need to approximate that system to a second-order system that has just two dominant complex poles.

Assuming two complex poles at $-\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$ And the real pole at $-\alpha_r$

Time domain step response, $c(t) = Au(t) + e^{-\xi\omega_n t} (B \cos \omega_d t + C \sin \omega_d t) + De^{-\alpha_r t}$

Case I: α_r is not much larger than $\zeta\omega_n$.

× Can't be approximated as 2nd-order system.

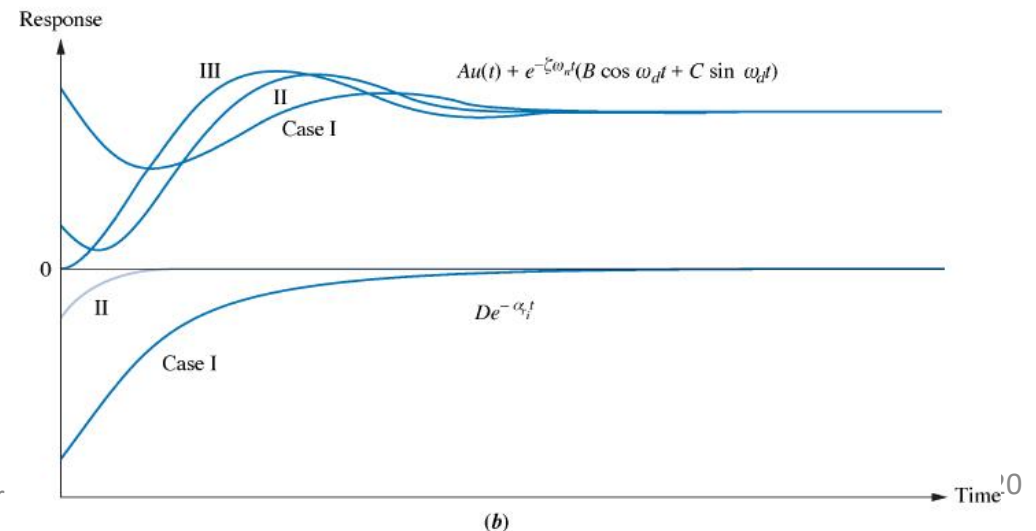


Case II: α_r is much larger than $\zeta\omega_n$.

Can be approximated as 2nd-order system.

Case III: $\alpha_r = \infty$.

Pure 2nd-order system.

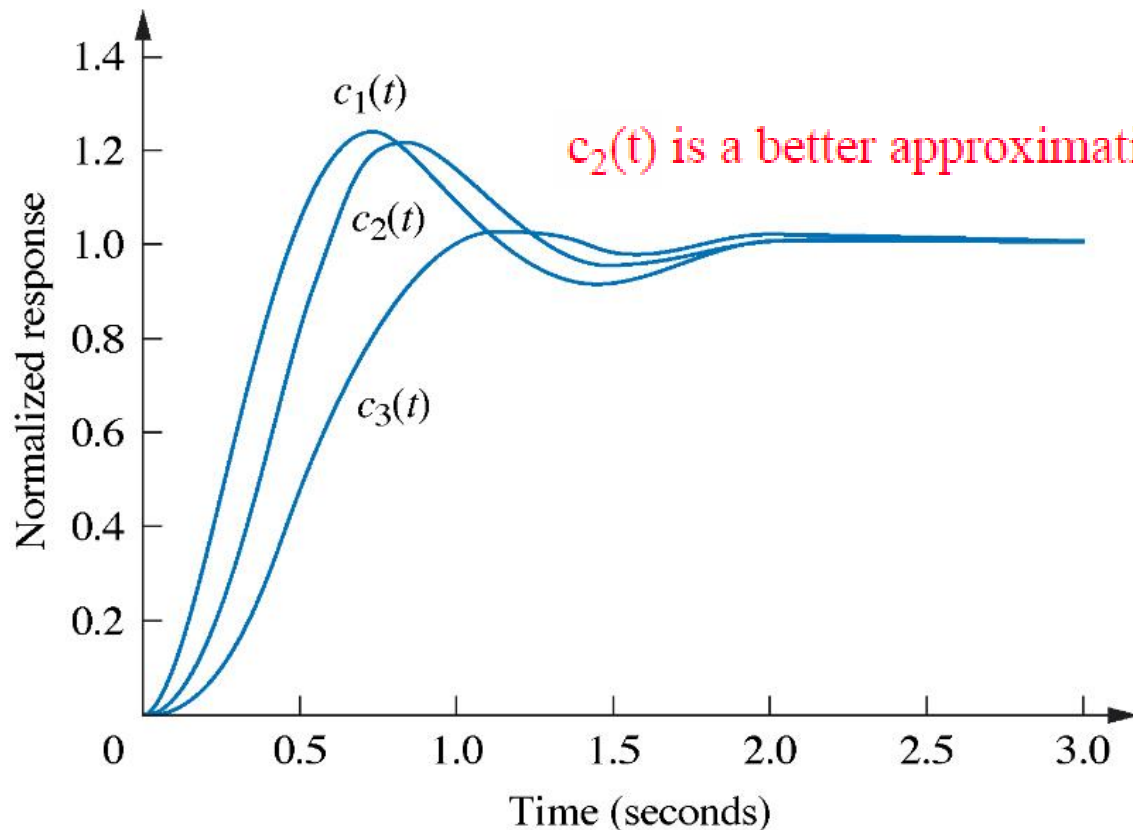


Comparing Responses of Three-Pole Systems

$$T_1(s) = \frac{24.542}{s^2 + 4s + 24.542} \longrightarrow 2^{\text{nd}}\text{-order system}$$

$$T_2(s) = \frac{24.542}{(s+10)(s^2 + 4s + 24.542)} \longrightarrow 3^{\text{rd}}\text{-order system, nondominant pole at -10}$$

$$T_3(s) = \frac{24.542}{(s+3)(s^2 + 4s + 24.542)} \longrightarrow 3^{\text{rd}}\text{-order system, nondominant pole at -3}$$



if the real pole is *five times*
farther to the left than the
dominant poles



system is represented by
its dominant second-
order pair of poles.

Evaluating Pole-Zero Cancellation

Effect of a zero on the system: A system with a zero consists of the derivative of the original response and the scaled version of the original response. If the zero is very large, the Laplace transform of the response is approximately the scaled version of the original response. As the zero becomes smaller, the derivative term contributes more to the response and has a greater effect.

$$(s + a)C(s) = sC(s) + aC(s)$$

derivative response
scaled response

pole-zero cancellation

Problem: For any function for which *pole-zero cancellation* is valid, find the approximate response.

$$C_1(s) = \frac{26.25(s+4)}{s(s+3.5)(s+5)(s+6)}$$

$$C_2(s) = \frac{26.25(s+4)}{s(s+4.01)(s+5)(s+6)}$$

Zero is at -4

Solution:

The partial-fraction expansion of $C_1(t)$ is

$$C_1(s) = \frac{1}{s} - \frac{3.5}{s+5} + \frac{3.5}{s+6} - \frac{1}{s+3.5}$$

Residue = 1

That residue (1) is not negligible. So a 2nd-order step response approximation cannot be made for $C_1(t)$.

Nearest pole = -3.5 close to zero

The partial-fraction expansion of $C_2(t)$ is

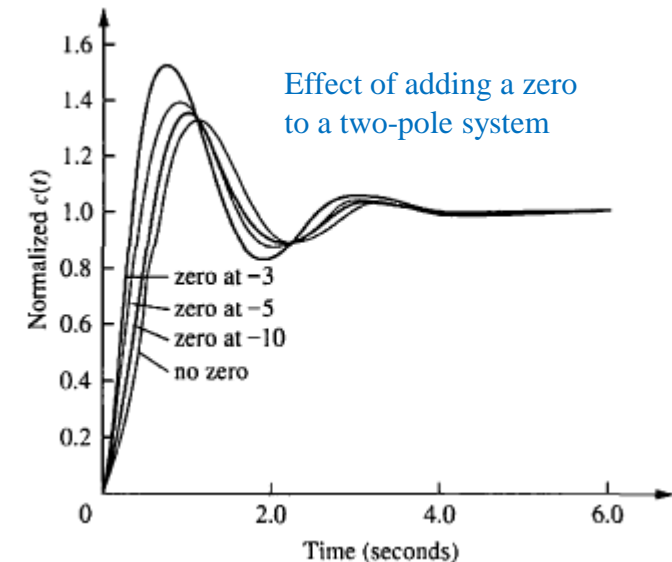
$$C_2(s) = \frac{0.87}{s} - \frac{5.3}{s+5} + \frac{4.4}{s+6} + \frac{0.033}{s+4.01}$$

Residue = 0.033

Nearest pole

That residue (0.033) is negligible, so cancel zero and that pole.

Hence, the approximate response, $c_2(t) = 0.87 - 5.3e^{-5t} + 4.4e^{-6t}$



2. Stability

Stability is the most important system specification.

$$c(t) = c_{forced}(t) + c_{natural}(t)$$

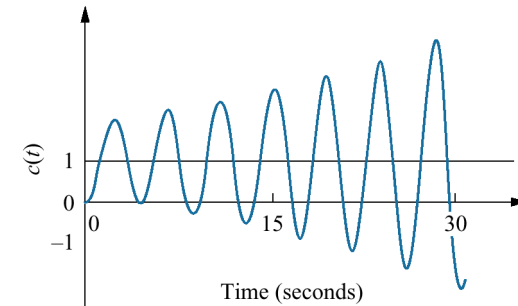
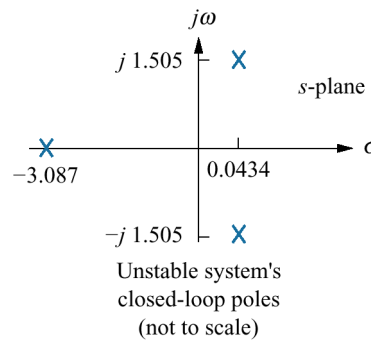
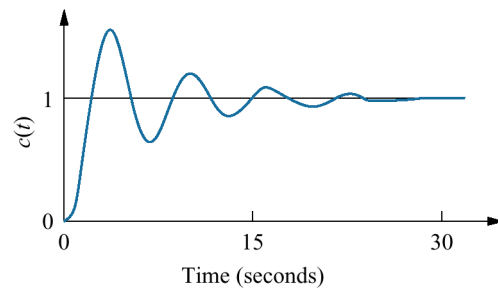
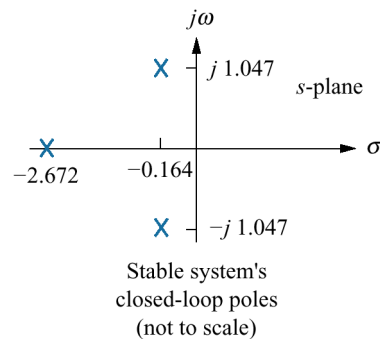
the total response of a system

Stable system: If natural response approaches zero as time approaches infinity (LTI System).

Marginally stable system: If natural response neither decays nor grows but remains constant or oscillates as time approaches infinity.

BIBO (Bounded Input, Bounded Output) yields stable system.

Stable systems have closed-loop transfer functions with poles only in the left-half plane.

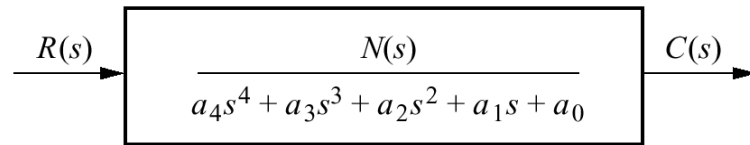


Routh-Hurwitz Criterion

This method Provides stability information with solving for system poles.

→ *How many* poles are in left / right plane or in jw axis, *not where*

Routh Table Generation:



Denominator: $a_4s^4 + a_3s^3 + a_2s^2 + a_1s + a_0$

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2			
s^1			
s^0			

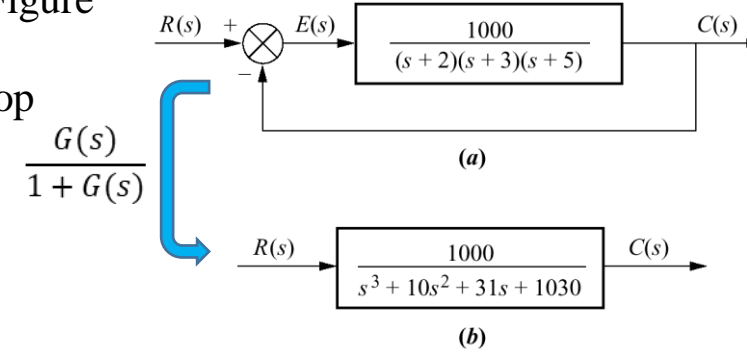
Completed Routh table

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$-\frac{\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$-\frac{\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$-\frac{\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$-\frac{\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$-\frac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$-\frac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$-\frac{\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$-\frac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$-\frac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Routh-Hurwitz Criterion: Example

PROBLEM: Make the Routh table for the system shown in Figure

SOLUTION: The first step is to find the equivalent closed-loop system



Interpreting the Basic Routh Table

How many *sign changes* in the first column \rightarrow
the number of poles in the right-half plane

Two such poles : unstable system.

s^3	1	31	0
s^2	10 1	1030 103	0
s^1	$-\frac{\begin{vmatrix} 1 & 31 \\ 1 & 103 \end{vmatrix}}{1} = -72$	$-\frac{\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$	$-\frac{\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$
s^0	$-\frac{\begin{vmatrix} 1 & 103 \\ -72 & 0 \end{vmatrix}}{-72} = 103$	$-\frac{\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$	$-\frac{\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$

the first column

Any row can be multiplied by a positive number. the row was multiplied by 1/10

the number of roots of the polynomial that are in the right half-plane is equal to the number of sign changes in the first column.

Routh-Hurwitz Criterion: Special Cases

1. Zero Only in the First Column

If the first element of a row is *zero*,
division by zero in the next row



replace the zero by *epsilon*, ϵ (very small quantity
 positive and negative) in the first column.

Example:

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

s^5	1	3	5
s^4	2	6	3
s^3	$\cancel{0} \ \epsilon$	$\frac{7}{2}$	0
s^2	$\frac{6\epsilon - 7}{\epsilon}$	3	0
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	0	0
s^0	3	0	0

Label	First Column	$\epsilon = +$	$\epsilon = -$
s^5	1	+	+
s^4	2	+	+
s^3	$\cancel{0} \ \epsilon$	+	-
s^2	$\frac{6\epsilon - 7}{\epsilon}$	-	+
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	+	+
s^0	3	+	+

Routh-Hurwitz Criterion: Special Cases

Zero Only in the First Column: reverse coefficients

The polynomial that has the reciprocal roots of the original, is another method that can be used when a zero appears only in the first column of a row.

reciprocal roots (s is replaced by 1/d), $s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0 = 0$

$$\begin{aligned} & \left(\frac{1}{d}\right)^n + a_{n-1}\left(\frac{1}{d}\right)^{n-1} + \dots + a_1\left(\frac{1}{d}\right) + a_0 = 0 \\ & \left(\frac{1}{d}\right)^n \left[1 + a_{n-1}\left(\frac{1}{d}\right)^{-1} + \dots + a_1\left(\frac{1}{d}\right)^{(1-n)} + a_0\left(\frac{1}{d}\right)^{-n} \right] \\ & = \left(\frac{1}{d}\right)^n [1 + a_{n-1}d + \dots + a_1d^{(n-1)} + a_0d^n] = 0 \end{aligned}$$

Since there are *two sign changes*, the system is *unstable and has two right-half-plane poles*

s^5	3	6	2
s^4	5	3	1
s^3	4.2	1.4	
s^2	1.33	1	
s^1	-1.75		
s^0	1		

Example:

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

Reverse coefficients:

$$D(s) = 3s^5 + 5s^4 + 6s^3 + 3s^2 + 2s + 1$$

Routh-Hurwitz Criterion: Special Cases

2. Entire Row is Zero

An *entire row* consists of *zeros* because there is an *even polynomial* that is a factor of the original polynomial (only even powers of s and have roots that are symmetrical about the origin.)

Example:

$$P(s) = s^4 + 6s^2 + 8 \qquad T(s) = \frac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56}$$

$$\frac{dP(s)}{ds} = 4s^3 + 12s + 0$$

Derivative of the polynomial of the row above the zeros row

s^5	1	6	8
s^4	7 1	42 6	56 8
s^3	0 4 1	0 12 3	0 0 0
s^2	3	8	0
s^1	$\frac{1}{3}$	0	0
s^0	8	0	0

the row was multiplied by 1/7

entire row consists of zeros

Stable system.

Pole Distribution via Routh Table with Row of Zeros

PROBLEM:

Tell *how many poles* are in the *right half-plane*, in the *left half-plane*, and on the *iw-axis*.

$$T(s) = \frac{20}{s^8 + s^7 + 12s^6 + 22s^5 + 39s^4 + 59s^3 + 48s^2 + 38s + 20}$$

$$P(s) = s^4 + 3s^2 + 2 \quad \text{:Even polynomial}$$

Taking the derivative

$$\frac{dP(s)}{ds} = 4s^3 + 6s + 0$$

the row was multiplied by 1/10

the row was multiplied by 1/20

entire row consists of zeros

the row was multiplied by 1/2

Two sign changes

interpretation

- No sign changes exist from the s^4 row down to the s^0 row (even polynomial does not have right-half-plane poles, all four of its poles are on the iw -axis by symmetry)
- The remaining roots (from s^8 row down to the s^4 row.) *two sign changes*, thus the other polynomial must have two roots in the right half-plane and two poles in the left half-plane.

No sign change

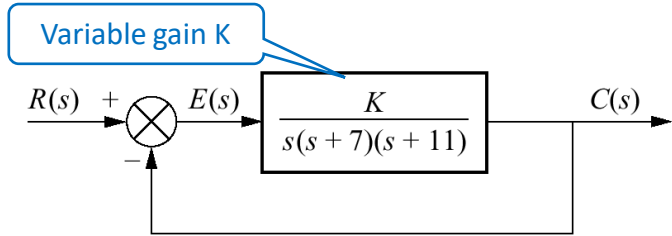
Polynomial			
Location	Even (fourth-order)	Other (fourth-order)	Total (eighth-order)
Right half-plane	0	2	2
Left half-plane	0	2	2
iw	4	0	4

s^8	1	12	39	48	20
s^7	1	22	59	38	0
s^6	10 -1	20 -2	10 1	20 2	0
s^5	20 1	60 3	40 2	0	0
s^4	1	3	2	0	0
s^3	2 2	6 3	0 0	0	0
s^2	3 $\frac{3}{2}$ 3	4 4	0	0	0
s^1	$\frac{1}{3}$	0	0	0	0
s^0	4	0	0	0	0

Stability Design via Routh-Hurwitz

Changes in the gain K of a feedback control system change the closed-loop pole locations (can move poles from region to another region on the S -plane).

PROBLEM: Find the range of gain, K , for the system that will cause the system to be stable, unstable, and marginally stable. Assume $K > 0$.



$$T(s) = \frac{K}{s^3 + 18s^2 + 77s + K}$$

s^3	1	77
s^2	18	K
s^1	$\frac{1386 - K}{18}$	
s^0	K	

can be positive, zero, or negative

1. If $K < 1386$, then *stable system*.
2. If $K > 1386$, then two sign changes; two right-half plane poles and one left-half plane pole. *Unstable system*.
3. If $K = 1386$, an entire row of zeros $\rightarrow j\omega$ poles.

replacing $K=1386$

$$P(s) = 18s^2 + 1386 \Rightarrow \frac{dP(s)}{ds} = 36s + 0$$

No sign change

s^3	1	77
s^2	18	1386
s^1	\emptyset 36	
s^0	1386	

Row of zeros

no sign changes from the even polynomial(s^2 to s^0)
no sign changes above the even polynomial

2 poles in $j\omega$ axis and one left-half plane pole
the system is marginally stable

Factoring via Routh-Hurwitz

The Routh-Hurwitz criterion is often used in limited applications to factor polynomials containing even factors.

PROBLEM: Factor the polynomial $s^4 + 3s^3 + 30s^2 + 30s + 200$ (1)

- Row s^1 is a row of Zeros

- Form the even polynomial at the s^2 row:

$$P(s) = s^2 + 10 \quad (2)$$

s^4	1	30	200
s^3	3 1	30 10	
s^2	20 1	200 10	
s^1	0 2	0 0	
s^0	10		

s^1 is a row of Zeros

- Dividing polynomial (1) by (2) yields:

$$s^4 + 3s^3 + 30s^2 + 30s + 200 = (s^2 + 10)(s^2 + 3s + 20)$$

$$= (s + j3.1623)(s - j3.1623) \times (s + 1.5 + j4.213)(s + 1.5 - j4.213)$$

$$\begin{array}{r}
 s^2 + 3s + 20 \\
 \hline
 s^2 + 10 \left| \begin{array}{l} s^4 + 3s^3 + 30s^2 + 30s + 200 \\ -s^4 - 10s^2 \\ \hline 3s^3 + 20s^2 + 30s + 200 \\ -3s^3 - 30s \\ \hline 20s^2 + 200 \\ -20s^2 - 200 \\ \hline 0 \end{array} \right. \\
 \hline
 \end{array}$$

Stability in State Space

- The values of the system's poles are equal to the *eigenvalues* of the system matrix, \mathbf{A} .
- *Eigenvalues* of the matrix \mathbf{A} are the solutions of the equation $\det(s\mathbf{I} - \mathbf{A}) = 0$

PROBLEM: find out how many poles are in the left half-plane, in the right half-plane, and on the $j\omega$ -axis.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 3 & 1 \\ 2 & 8 & 1 \\ -10 & -5 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0] \mathbf{x}$$

SOLUTION:

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 3 & 1 \\ 2 & 8 & 1 \\ -10 & -5 & -2 \end{bmatrix} = \begin{bmatrix} s & -3 & -1 \\ -2 & s-8 & -1 \\ 10 & 5 & s+2 \end{bmatrix} \Rightarrow \det(s\mathbf{I} - \mathbf{A}) = s^3 - 6s^2 - 7s - 52$$

Using this polynomial, form the Routh table

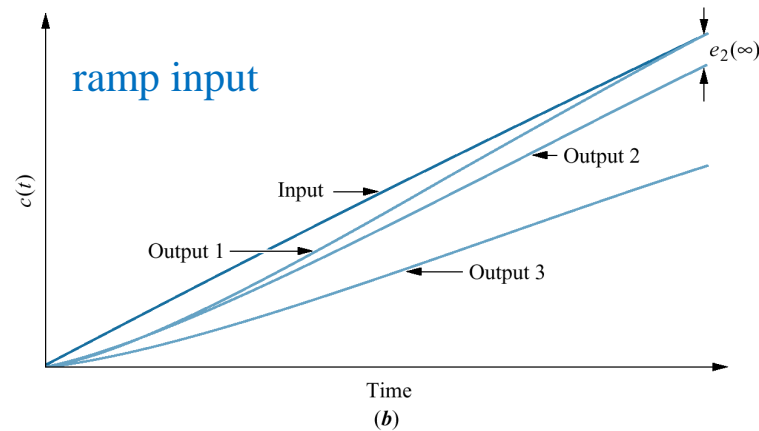
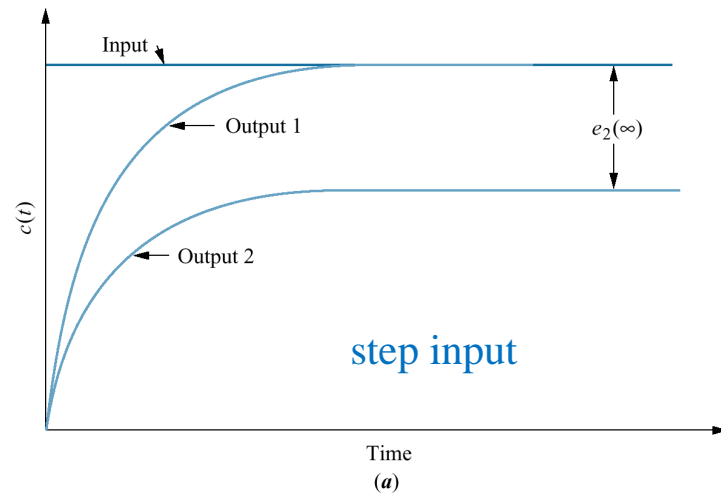
s^3	1	-7
s^2	-3 -3	-26 -26
s^1	$-\frac{47}{3}$ -1	0 0
s^0	-26	

one sign change: One right-half-plane pole and two left-half-plane poles.

 **Unstable system.**

3. Steady-State Errors

Steady-State Error: The difference between the input and the output for a prescribed test input (step / ramp / parabola) as $t \rightarrow \infty$



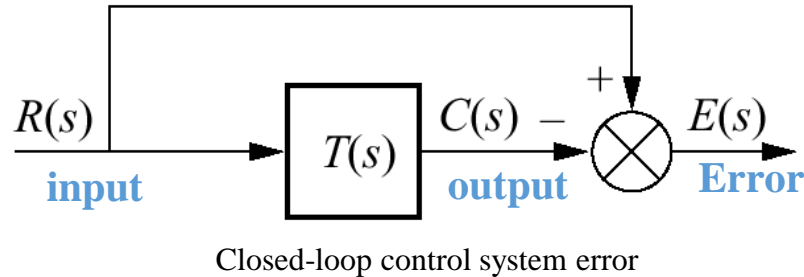
Steady-state error

Test waveforms for evaluating steady-state errors of position control systems

Waveform	Name	Physical interpretation	Time function	Laplace transform
	Step	Constant position	1	$\frac{1}{s}$
	Ramp	Constant velocity	t	$\frac{1}{s^2}$
	Parabola	Constant acceleration	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

Evaluating Steady-State Errors₁

- Many steady-state errors arise from nonlinear sources, such as backlash in gears.
- $T(s)$ is the closed-loop transfer function,



$$E(s) = R(s) - C(s) \quad \text{and} \quad C(s) = R(s)T(s)$$

$$\Rightarrow E(s) = R(s)[1 - T(s)]$$

Applying final value theorem $[f(\infty) = \lim_{s \rightarrow 0} sF(s)]$

$$e(\infty) = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} sR(s)[1 - T(s)]$$

Problem: Find the steady-state error for the following system with

$$T(s) = \frac{5}{s^2 + 7s + 10}$$

and input is step response.

Solution:

Step input $R(s) = \frac{1}{s}$

We have the error $E(s) = R(s)[1 - T(s)]$

$$\Rightarrow E(s) = \frac{s^2 + 7s + 5}{s(s^2 + 7s + 10)}$$

$T(s)$ is stable, hence $E(s)$ is stable.

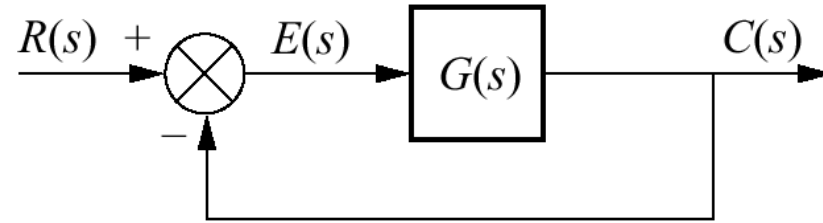
Applying final value theorem,

$$e(\infty) = \lim_{s \rightarrow 0} sR(s)[1 - T(s)] = \lim_{s \rightarrow 0} s \frac{s^2 + 7s + 5}{s(s^2 + 7s + 10)}$$

$$e(\infty) = 1/2$$

Evaluating Steady-State Errors₂

- Forward transfer function $G(s)$ with a unity feedback.



$$E(s) = R(s) - C(s), \quad \text{and} \quad C(s) = E(s)G(s)$$

$$\Rightarrow E(s) = \frac{R(s)}{1 + G(s)}$$

$$e(\infty) = \lim_{s \rightarrow 0} \frac{sR(s)}{1 + G(s)} \quad \text{[Final-value theorem]}$$

Ramp input: $t u(t)$

$$e(\infty) = e_{\text{ramp}}(\infty) = \lim_{s \rightarrow 0} \frac{s(1/s^2)}{1 + G(s)} = \lim_{s \rightarrow 0} \frac{1}{s + sG(s)} = \frac{1}{\lim_{s \rightarrow 0} sG(s)}$$

For zero steady-state error, $\lim_{s \rightarrow 0} sG(s) = \infty$

Step input: $u(t)$

$$e(\infty) = e_{\text{step}}(\infty) = \lim_{s \rightarrow 0} \frac{s(1/s)}{1 + G(s)} = \frac{1}{1 + \lim_{s \rightarrow 0} G(s)}$$

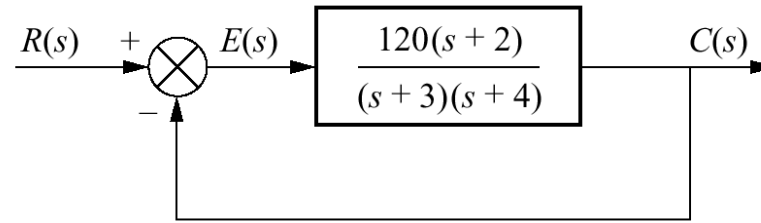
For zero steady-state error, $\lim_{s \rightarrow 0} G(s) = \infty$

Parabolic input: $(1/2)t^2 u(t)$

$$e(\infty) = e_{\text{parabolic}}(\infty) = \lim_{s \rightarrow 0} \frac{s(1/s^3)}{1 + G(s)} = \lim_{s \rightarrow 0} \frac{1}{s^2 + s^2G(s)} = \frac{1}{\lim_{s \rightarrow 0} s^2G(s)}$$

For zero steady-state error, $\lim_{s \rightarrow 0} s^2G(s) = \infty$

Evaluating Steady-State Errors example1



$u(t)$: unity step

Problem: Find steady-state errors for inputs $5u(t)$, $5tu(t)$, and $5t^2u(t)$ to the above system.

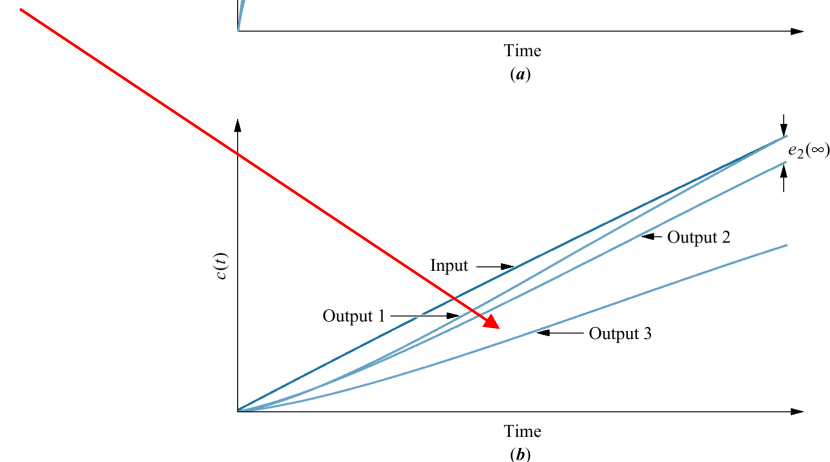
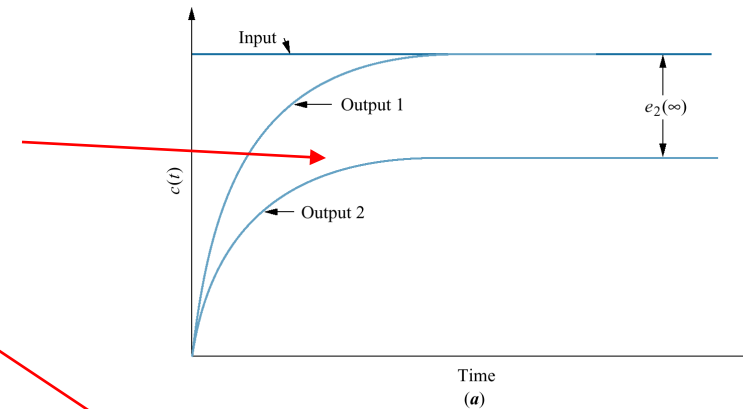
Solution: Let, the system is stable.

$$5u(t): e(\infty) = \frac{5}{1 + \lim_{s \rightarrow 0} G(s)} = \frac{5}{1 + 20} = \frac{5}{21}$$

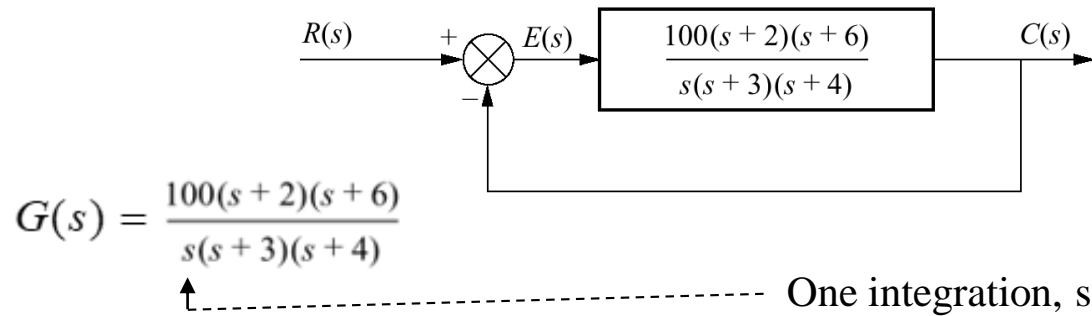
$$5tu(t): e(\infty) = \frac{5}{\lim_{s \rightarrow 0} sG(s)} = \frac{5}{0} = \infty$$

$$5t^2u(t): e(\infty) = \frac{10}{\lim_{s \rightarrow 0} s^2G(s)} = \frac{10}{0} = \infty$$

$$5t^2u(t) = 10 \left(\frac{1}{2} t^2 u(t) \right) \rightarrow 10 e_{parabolic}$$



Evaluating Steady-State Errors-example2



$5u(t): e(\infty) = \frac{5}{1 + \lim_{s \rightarrow 0} G(s)} = \frac{5}{\infty} = 0$
 ← No integration will make it constant, one integration makes it zero.

$5tu(t): e(\infty) = \frac{5}{\lim_{s \rightarrow 0} sG(s)} = \frac{5}{100} = \frac{1}{20}$
 ← No integration will make it infinity, one integration makes it constant.

$5t^2u(t): e(\infty) = \frac{10}{\lim_{s \rightarrow 0} s^2G(s)} = \frac{10}{0} = \infty$
 ← Two integrations will make it constant and 3 or more will make it zero.

Static Error Constants

The steady-state error *performance specifications* are called *static error constants*.

steady-state error.

$e_{step}(\infty) = \frac{1}{1 + \lim_{s \rightarrow 0} G(s)} = \frac{1}{1 + K_p}$	step input, $u(t)$
$e_{ramp}(\infty) = \frac{1}{\lim_{s \rightarrow 0} s G(s)} = \frac{1}{K_v}$	ramp input, $t u(t)$
$e_{parabola}(\infty) = \frac{1}{\lim_{s \rightarrow 0} s^2 G(s)} = \frac{1}{K_a}$	Parabolic input, $\frac{1}{2} t^2 u(t)$

Position constant, K_p :

$$K_p = \lim_{s \rightarrow 0} G(s) \quad e(\infty) = \frac{1}{1 + K_p}$$

Velocity constant, K_v :

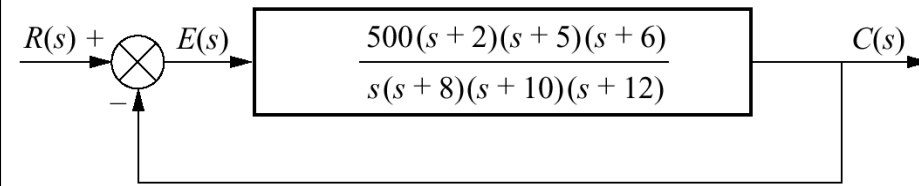
$$K_v = \lim_{s \rightarrow 0} sG(s) \quad e(\infty) = \frac{1}{K_v}$$

Acceleration constant, K_a :

$$K_a = \lim_{s \rightarrow 0} s^2 G(s) \quad e(\infty) = \frac{1}{K_a}$$

PROBLEM:

evaluate the static error constants and find the expected error for the standard step, ramp, and parabolic inputs.



$$K_p = \lim_{s \rightarrow 0} G(s) = \infty$$

$$K_v = \lim_{s \rightarrow 0} sG(s) = \frac{500 \times 2 \times 5 \times 6}{8 \times 10 \times 12} = 31.25$$

$$K_a = \lim_{s \rightarrow 0} s^2 G(s) = 0$$

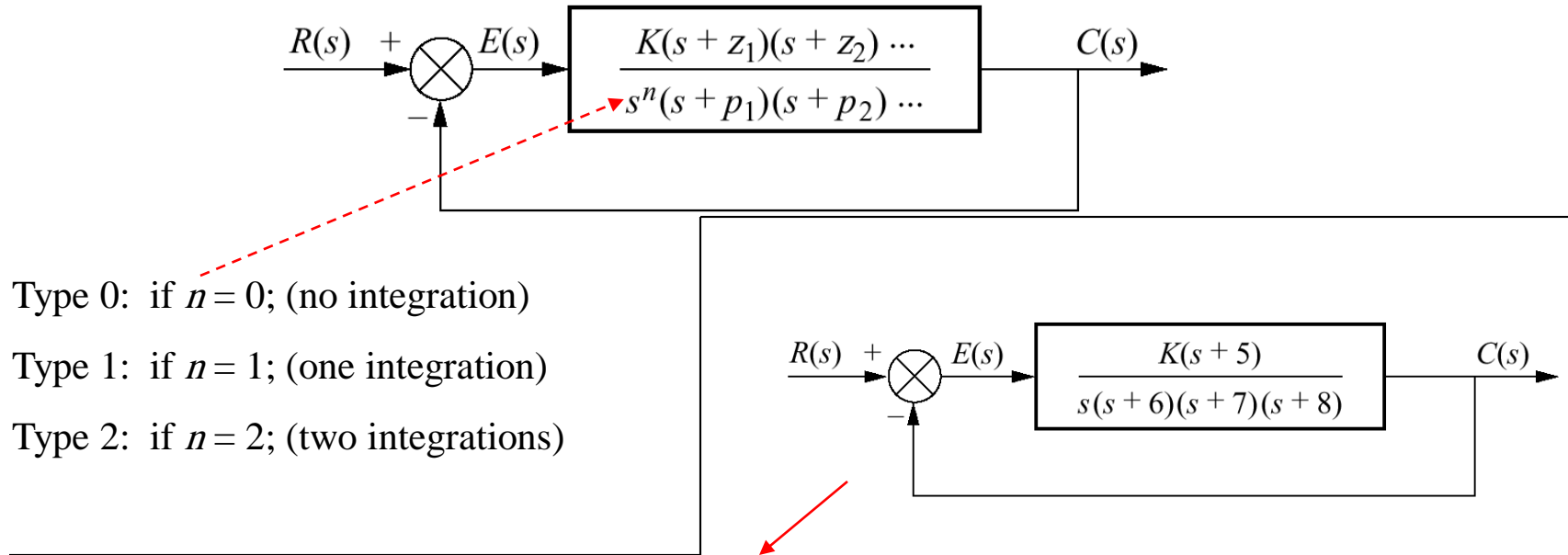
$$\text{For step input, } e(\infty) = \frac{1}{1 + K_p} = 0$$

$$\text{For ramp input, } e(\infty) = \frac{1}{K_v} = \frac{1}{31.25} = 0.032$$

$$\text{For parabolic input, } e(\infty) = \frac{1}{K_a} = \infty$$

System Type

we define *system type* to be the number of pure integrations (s^n) in the forward path.



Problem: Find the value of K so that there is 10% error in the steady state.

Solution: Type 1.

Input should be ramp, because only ramp yields a finite error in Type 1 system.

$$e(\infty) = \frac{1}{K_v} = 0.1$$

$$\Rightarrow K_v = 10 = \lim_{s \rightarrow 0} sG(s) = \frac{K \times 5}{6 \times 7 \times 8} = \frac{5 \times K}{336} \quad \Rightarrow K = 672$$

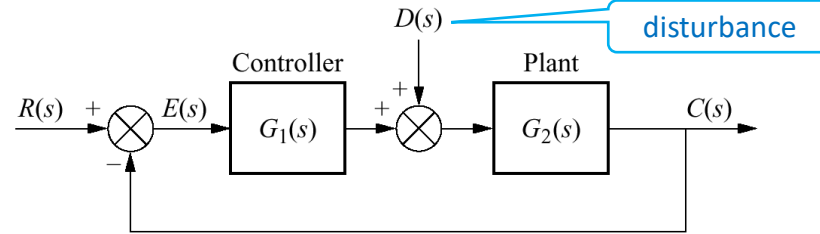
System Type

Relationships between input, system type, static error constants, and steady-state errors

Input	Steady-state error formula	Type 0		Type 1		Type 2	
		Static error constant	Error	Static error constant	Error	Static error constant	Error
Step, $u(t)$	$\frac{1}{1 + K_p}$	$K_p = \text{Constant}$	$\frac{1}{1 + K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v = \text{Constant}$	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a = \text{Constant}$	$\frac{1}{K_a}$

Steady-State Error for Disturbances

Feedback control systems are used to *compensate* for disturbances or unwanted inputs that enter a system.



$$C(s) = [E(s)G_1(s) + D(s)]G_2(s)$$

$$\rightarrow C(s) = E(s)G_1(s)G_2(s) + D(s)G_2(s)$$

$$\text{But } C(s) = R(s) - E(s)$$

$$\rightarrow R(s) - E(s) = E(s)G_1(s)G_2(s) + D(s)G_2(s)$$

$$\rightarrow R(s) = E(s) + E(s)G_1(s)G_2(s) + D(s)G_2(s)$$

$$\Rightarrow E(s) = \frac{1}{1 + G_1(s)G_2(s)} R(s) - \frac{G_2(s)}{1 + G_1(s)G_2(s)} D(s)$$

$$e(\infty) = \lim_{s \rightarrow 0} sE(s) = e_R(\infty) + e_D(\infty)$$

$$e_R(\infty) = \lim_{s \rightarrow 0} \frac{s}{1 + G_1(s)G_2(s)} R(s)$$

steady-state error due to $R(s)$,

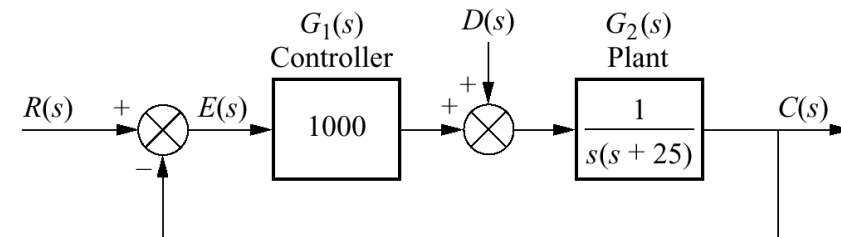
$$e_D(\infty) = - \lim_{s \rightarrow 0} \frac{sG_2(s)}{1 + G_1(s)G_2(s)} D(s)$$

steady-state error due to disturbance $D(s)$,

$$= - \frac{1}{\lim_{s \rightarrow 0} \frac{1}{G_2(s)} + \lim_{s \rightarrow 0} G_1(s)}, \text{ for step disturbance}$$

Problem:

Find the steady-state error component due to a step disturbance



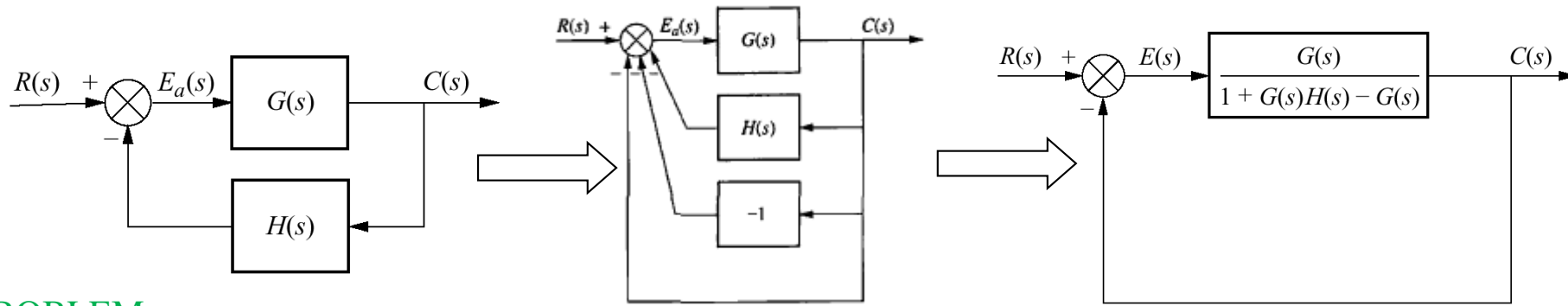
Solution:

$$e_D(\infty) = - \frac{1}{\lim_{s \rightarrow 0} \frac{1}{G_2(s)} + \lim_{s \rightarrow 0} G_1(s)} = - \frac{1}{0 + 1000} = -0.001$$

steady-state error due to step disturbance $D(s)=1/s$,

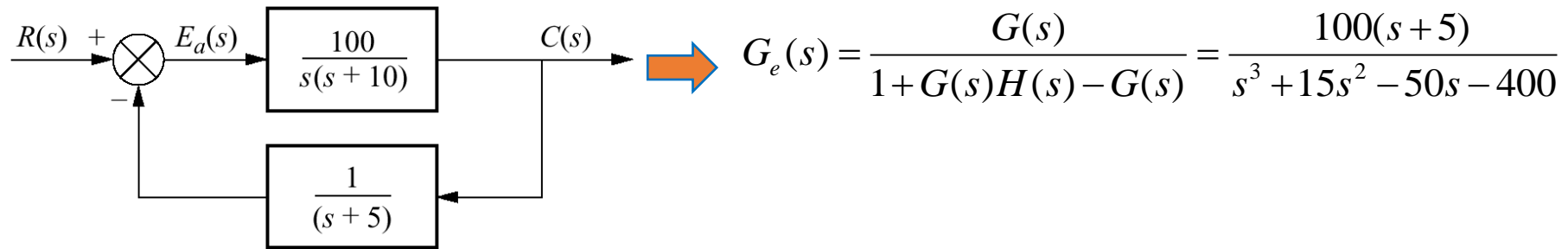
Steady-State Error for Nonunity Feedback System

Form a unity feedback system by adding and subtracting unity feedback paths (input and output units must be same.).



PROBLEM:

Find the system type, error constant, and the steady-state error for a unit step input.



Type 0 (as no integration).

For step input, static *error constant* is K_p .

$$\Rightarrow K_p = \lim_{s \rightarrow 0} G_e(s) = \frac{100 \times 5}{-400} = -\frac{5}{4}$$

Negative value means the output step is larger than the input step.

$$\leftarrow e(\infty) = \frac{1}{1 + K_p} = \frac{1}{1 - \frac{5}{4}} = -4$$